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ON THE CHARACTERISTIC FUNCTION OF SPACES OF CONSTANT HOLOMORPHIC CURVATURE

BY

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It is H. S. Ruse who introduced first the notion of a harmonic Riemannian space, and such spaces have been studied by H. S. Ruse, A. G. Walker, T. J. Willmore, A Lichnerowicz and others. Its typical examples are the Euclidean space and the space of constant curvature. An n-dimensional non-flat space of constant curvature is characterized as a harmonic Riemannian space with the characteristic function

$$f(\Omega) = 1+(n-1)\sqrt{2k\Omega}\cot\sqrt{2k\Omega},$$

where $\Omega = (1/2)s^2$ and s means the geodesic distance (cf. [2], p. 30). Lichnerowicz [1] has obtained the following

THEOREM (cf. [1] and [3]). *In any harmonic Riemannian space H^n with positive definite metric, its characteristic function $f(\Omega)$ satisfies the inequality*

$$\dot{f}^2(0)+\tfrac{5}{2}(n-1)\ddot{f}(0) \leqslant 0.$$

The equality sign is valid if and only if H^n is of constant curvature.

On the other hand, it is known (cf. [3], p. 141-147) that the complex projective space CP^m of real dimension $2m$ can admit a Kählerian metric by which CP^m becomes a harmonic Riemannian space. CP^m with such a metric is a space of constant holomorphic curvature as a Kählerian space.

In spite of these facts, the characteristic function of spaces of constant holomorphic curvature has not been known. In this paper we shall give it and also theorems for harmonic Kählerian spaces corresponding to Lichnerowicz' one cited above.

In Sections 1-3 we follow Yano-Bochner's notation. The summation convention will be assumed for Greek indices.

1. The Fubinian space. Let C^m be the complex number space of origin O with coordinates $\{z^a\}$. Denoting by $z^{a^*} = \bar{z}^a$ the complex conjugate of z^a, we introduce real-valued functions u and S by

$$u = \sum_{a=1}^{m} z^a z^{a^*} \quad \text{and} \quad S = 1 + 2ku,$$

where k is a non-zero real number. We consider the maximal connected domain F^m where $S > 0$, and define a Kählerian metric by

$$ds^2 = 2g_{\alpha\beta^*} dz^\alpha dz^{\beta^*} \quad \text{and} \quad g_{\alpha\beta^*} = \frac{1}{2k} \frac{\partial^2 \log S}{\partial z^\alpha \partial z^{\beta^*}}.$$

Such an $\{F^m, g\}$ will be called a *Fubinian space* of complex dimension m and of real dimension $n = 2m$.

If we denote by $f_\alpha, f_{\alpha\beta^*}, \dots$ the successive partial derivatives of a function f with respect to z^α, z^{β^*}, $\dots$, then it follows that $u_\alpha = z^{\alpha^*}, u_{\alpha^*} = \bar{u}_\alpha = z^\alpha$, $S_\alpha = 2kz^{\alpha^*}$ and $S_{\alpha^*} = 2kz^\alpha$. Thus the Fubinian metric is given by

$$g_{\alpha\beta^*} = \frac{1}{S^2}(S\delta_{\alpha\beta} - 2kz^{\alpha^*} z^\beta),$$

and hence we have

$$g^{\alpha\beta^*} = S(\delta^{\alpha\beta} + 2kz^\alpha z^{\beta^*}).$$

The Christoffel symbols are all zero except

$$\Gamma^\alpha_{\beta\gamma} = -\frac{2k}{S}(\delta^\alpha_\gamma z^{\beta^*} + \delta^\alpha_\beta z^{\gamma^*})$$

and their complex conjugates. The curvature tensor, the Ricci tensor and the scalar curvature are

$$R^\alpha_{\beta\gamma\varepsilon^*} = \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial z^{\varepsilon^*}} = -2k(g_{\beta\varepsilon^*}\delta^\alpha_\gamma + g_{\gamma\varepsilon^*}\delta^\alpha_\beta),$$

$$R_{\beta\varepsilon^*} = R^\alpha_{\beta\varepsilon^*\alpha} = 2(m+1)kg_{\beta\varepsilon^*} \quad \text{and} \quad R = 2g^{\beta\varepsilon^*}R_{\beta\varepsilon^*} = 4m(m+1)k,$$

respectively. Therefore, we have

$$k = \frac{R}{4m(m+1)} = \frac{R}{n(n+2)}. \tag{1.1}$$

2. Geodesics through O. Consider a Fubinian space $\{F^m, g\}$. We shall find the equation of geodesics which go through the origin O.

The differential equation of geodesic is

$$z''^\alpha + \Gamma^\alpha_{\beta\gamma} z'^\beta z'^\gamma = 0, \tag{2.1}$$

where and throughout the paper we denote by $'$ the derivative with respect to the arc length s. Our space being Fubinian, (2.1) becomes

$$z''^{\alpha} = \frac{4k}{S}\left(\sum z'^{\beta} z^{\beta*}\right) z^{\alpha}. \tag{2.2}$$

Now, let C^{α} be constant and $t(s)$ a real-valued function of s satisfying $t(0) = 0$ and $t'(s) > 0$. If we substitute $z^{\alpha} = C^{\alpha} t(s)$ into (2.2) and take account of $2g_{\alpha\beta*} z'^{\alpha} z'^{\beta*} = 1$, then it can be seen that

(i) for the case of $k > 0$, $z^{\lambda} = A^{\lambda} \tan(\sqrt{k}s)$ satisfy (2.2), and A^{λ} are constant such that $2k\sum A^{\lambda}\bar{A}^{\lambda} = 1$,

(ii) for the case of $k < 0$, $z^{\lambda} = A^{\lambda} \tanh(\sqrt{|k|}s)$ satisfy (2.2), and A^{λ} are constant such that $2k\sum A^{\lambda}\bar{A}^{\lambda} = -1$.

As any point of F^m is represented in the form of (i) or (ii), we know that the equation of any geodesic through O is (i) or (ii) according to $k > 0$ or $k < 0$.

Remark. If we write for $k < 0$ as $\tan(\sqrt{k}s) = i \tanh(\sqrt{|k|}s)$, then the geodesic of (ii) becomes

$$z^{\lambda} = A^{\lambda} \tan(\sqrt{k}s), \qquad \sum (\sqrt{2k}A^{\lambda})\overline{(\sqrt{2k}A^{\lambda})} = 1$$

which coincides with one of (i) in appearance.

3. $\Delta_2 s$ in $\{F^m, g\}$. It will be seen in this section that any Fubinian space is harmonic getting its characteristic function. For this purpose we calculate the Laplacian $\Delta_2 s$ in $\{F^m, g\}$.

Putting

$$k = \begin{cases} l^2 & \text{if } k > 0, \\ -l^2 & \text{if } k < 0, \end{cases}$$

we give an outline of the calculation for the case of $k > 0$, where l is a positive constant.

Any point z^{α} in F^m being on a unique geodesic through O, z^{α} is written as $z^{\alpha} = A^{\alpha} \tan(ls)$, $2k\sum A^{\alpha}\bar{A}^{\alpha} = 1$. Hence it follows that

$$2ku = 2k \sum z^{\alpha} z^{\alpha*} = \tan^2(ls), \tag{3.1}$$

$$S = 1 + 2ku = \sec^2(ls). \tag{3.2}$$

Differentiating (3.1) by z^{α}, we have

$$l z^{\alpha*} = \tan(ls)\sec^2(ls) s_{\alpha}, \tag{3.3}$$

from which it follows that

$$g^{\alpha\beta*} s_{\alpha} s_{\beta*} = l^2 u \cot^2(ls). \tag{3.4}$$

If we differentiate (3.3) by z^{β^*}, then

$$l\delta_{\alpha\beta} = lS(3S-2)s_\alpha s_{\beta^*} + S\tan(ls)s_{\alpha\beta^*}$$

follows. Multiplying this equation with $g^{\alpha\beta^*}$ and taking account of (3.4) and $\Delta_2 s = 2g^{\alpha\beta^*}s_{\alpha\beta^*}$, we have

$$(m+2ku)l = (3S-2)l^3 u \cot^2(ls) + \tfrac{1}{2}\tan(ls)\Delta_2 s.$$

Therefore, it follows that

$$\Delta_2 s = (2m-1)l\cot(ls) - l\tan(ls) \tag{3.5}$$

by virtue of (3.1) and (3.2).

Let M^n be an n-dimensional analytic Riemannian space and O a point of M^n. We denote by s the geodesic distance from O to the point in a neighbourhood of O. If $\Delta_2 s$ is a function of s only, then M^n is called to be *harmonic at* O. When M^n is harmonic at any point, it is called *harmonic* and denoted by H^n. For a harmonic Riemannian space H^n, if we put $\Omega = (1/2)s^2$, then it is known that $\Delta_2\Omega = f(\Omega)$ is a function of Ω only and does not depend on the reference point O. $f(\Omega)$ is called the *characteristic function of* H^n.

As the right-hand member of (3.5) is a function of s only, $\{F^m, g\}$ is harmonic (we notice that $\{F^m, g\}$ admits a holomorphic free mobility). Its characteristic function is

$$\Delta_2\Omega = f(\Omega) = 1 + (2m-1)ls\cot(ls) - ls\tan(ls) \tag{3.6}$$

by virtue of the identity $\Delta_2\Omega = 1 + s\Delta_2 s$.

Similarly, for $k = -l^2$, we can get

$$\Delta_2\Omega = f(\Omega) = 1 + (2m-1)ls\coth(ls) + ls\tanh(ls). \tag{3.7}$$

4. The calculations of $\dot{f}(0)$ and $\ddot{f}(0)$. Consider the function $f(\Omega)$ of $\Omega = (1/2)s^2$ given by (3.6), i.e.,

$$f(\Omega) = 1 + (2m-1)ls\cot(ls) - ls\tan(ls), \qquad k = l^2. \tag{4.1}$$

Evidently, $f(\Omega)$ satisfies $f(0) = 2m$. If we develop $f(\Omega)$ in the power series of Ω and s, respectively, then it holds that

$$\begin{aligned} f(\Omega) &= 2m + \dot{f}(0)\Omega + \frac{1}{2}\ddot{f}(0)\Omega^2 + \dots \\ &= 2m + f'(0)s + \frac{1}{2}f''(0)s^2 + \frac{1}{3!}f'''(0)s^3 + \frac{1}{4!}f''''(0)s^4 + \dots, \end{aligned} \tag{4.2}$$

where $\cdot$ means the operator taking the derivative with respect to Ω. Thus we know that

$$f'(0) = f'''(0) = 0, \quad \dot{f}(0) = f''(0), \quad \ddot{f}(0) = \tfrac{1}{3}f''''(0). \tag{4.3}$$

Now, if we write (4.1) as

$$(f(\Omega)-1)\tan(ls) = ls(2m-1-\tan^2(ls))$$

and develop the both sides in the power series of s taking account of (4.2), (4.3) and

$$\tan x = x+\frac{1}{3}x^3+\frac{2}{15}x^5+\dots,$$

then we can get

$$\dot{f}(0) = -\frac{4(m+1)}{3}k, \quad \ddot{f}(0) = -\frac{16(m+7)}{45}k^2.$$

Therefore, $f(\Omega)$ satisfies

$$\dot{f}^2(0)+\frac{5(m+1)^2}{m+7}\ddot{f}(0) = 0. \tag{4.4}$$

Similarly, it can be seen that (4.4) is valid for the $f(\Omega)$ of (3.7) in the case of $k = -l^2$.

5. Harmonic Kählerian spaces. In what follows Latin indices $i, j, k, h, \dots$ run through 1 to n (or $2m$), and the summation convention is assumed. Let H^n be an n-dimensional harmonic Riemannian space (with positive definite metric) and $f(\Omega)$ its characteristic function. It is known that H^n is an Einstein space and

$$R = -\frac{3n}{2}\dot{f}(0) \tag{5.1}$$

holds good (cf. [1], 9·2). On the other hand, it is easy to get the identity

$$R_{ijkh}R^{ijkh} = -\frac{3n}{2}\left\{\dot{f}^2(0)+\frac{5(n+2)}{2}\ddot{f}(0)\right\} \tag{5.2}$$

from equation (11·1) in [1].

Now, consider a $2m$-dimensional Kählerian space K^{2m}, and let g_{ij} and F_i^h be the (positive definite) Kählerian metric and the complex structure with respect to a real coordinate, respectively. These tensors satisfy

$$g_{kh}F_i^kF_j^h = g_{ij}, \quad F_i^hF_h^j = -\delta_i^j,$$

$$F_{ij}(=g_{jh}F_i^h) = -F_{ji}.$$

If K^{2m} satisfies

$$R_{ijkh} = k(g_{ih}g_{jk}-g_{ik}g_{jh}+F_{ih}F_{jk}-F_{ik}F_{jh}-2F_{ij}F_{kh}),$$

then it is called a *space of constant holomorphic curvature*, where k is a (real) constant given by (1.1). It is known that such a space with non-zero k is locally regarded as a Fubinian space. Hence a space of constant holomorphic curvature ($k \neq 0$) is harmonic, and its characteristic function is $f(\Omega)$ of (3.6) or (3.7) and satisfies (4.4).

We consider the converse problem. Let hK^{2m} be a harmonic Kählerian space (with positive definite metric) and $A = T_{ijkh}T^{ijkh}$ the square of the tensor T_{ijkh} defined by

$$T_{ijkh} = R_{ijkh} - \lambda(g_{ih}g_{jk} - g_{ik}g_{jh} + F_{ih}F_{jk} - F_{ik}F_{jh} - 2F_{ij}F_{kh}),$$

where λ is real constant. If A vanishes identically, then hK^{2m} is of constant holomorphic curvature.

We calculate A by taking account of (5.1), (5.2) and the identities (see, for example, Yano [4])

$$F^{ij}R_{ijkh} = -2F^{j}_{k}R_{jh}, \qquad F^{ij}F^{kh}R_{ijkh} = -2R,$$
$$F^{jk}R_{ijkh} = F^{j}_{i}R_{jh}, \qquad F^{ih}F^{jk}R_{ijkh} = R,$$

then the following equation is obtained:

$$32(m+1)\lambda^2 + 48\dot{f}(0)\lambda - 3\{\dot{f}^2(0) + 5(m+1)\ddot{f}(0)\} = \frac{A}{m} \geqslant 0. \tag{5.3}$$

Taking the discriminant, we obtain

$$\dot{f}^2(0) + \frac{5(m+1)^2}{m+7}\ddot{f}(0) \leqslant 0, \tag{5.4}$$

from which it follows

THEOREM 1. *In any 2m-dimensional harmonic Kählerian space, the inequality*

$$\dot{f}^2(0) + \frac{5(m+1)^2}{m+7}\ddot{f}(0) \leqslant 0$$

holds good.

When the equality sign holds in (5.4), there exists a real λ satisfying $A = 0$ and hence our hK^{2m} is of constant holomorphic curvature. Thus we have

THEOREM 2. *A 2m-dimensional harmonic Kählerian space is of constant holomorphic curvature if and only if its characteristic function satisfies*

$$\dot{f}^2(0) + \frac{5(m+1)^2}{m+7}\ddot{f}(0) = 0.$$

THEOREM 3. *A 2m-dimensional space of constant holomorphic curvature ($k \neq 0$) is characterized as a harmonic Kählerian space with charac-*

teristic function given by

$$f(\Omega) = 1 + (2m-1)ls\cot(ls) - ls\tan(ls),$$

or

$$f(\Omega) = 1 + (2m-1)ls\coth(ls) + ls\tanh(ls)$$

according to $k = l^2$ *or* $k = -l^2$.

REFERENCES

[1] A. Lichnerowicz, *Sur les espaces Riemanniens complètement harmoniques*, Bulletin de la Société Mathématique de France 72 (1944), p. 146-168.

[2] H. S. Ruse, A. G. Walker and T. J. Willmore, *Harmonic spaces*, Roma 1961.

[3] T. J. Willmore, *Some properties of harmonic Riemannian manifold*, Convegno di Geometria Differenziale, Venezia 1953.

[4] K. Yano, *Differential geometry on complex and almost complex spaces*, 1965.

[5] — and S. Bochner, *Curvature and Betti numbers*, Annals of Mathematics Studies 32, Princeton 1953.

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