ANNALES
POLONICI MATHEMATICI
XXIII (1970)

On a certain property of solution of the equation $u_t = u_{xx} + f(x, t, u)$

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It is a classical result that if we are given two solutions u_1, u_2 on $[0, a]$ of the differential equation $u' = f(t, u)$ (f continuous) such that $u_1(0) = u_2(0) = u_0$, then for every $0 < t_0 \leqslant a$ and $v_0 \in [u_1(t_0), u_2(t_0)]$ there is a solution u on $[0, a]$ which satisies $u(0) = u_0$, $u(t_0) = v_0$. The situation is especially simple if u_1 is the maximum solution and u_2 is the minimum one. The present paper attempts to give a certain theorem concerning an analogous property for the non-linear parabolic equation $u_t = u_{xx} + f(x, t, u)$.

Let $R = \{(x, t): a \leqslant x \leqslant b, 0 \leqslant t \leqslant T\}$. The interior of R is denoted by R^0, the boundary by FR.

Γ stands for the plane set composed of points $(x, 0)$ with $a \leqslant x \leqslant b$ and $(a, t), (b, t)$ with $0 \leqslant t \leqslant T$.

By a regular function in R we mean a function u which is continuous on R, continuously differentiable in t to $\partial u/\partial t$ and twice in x to $\partial^2 u/\partial x^2$ for $0 < t \leqslant T, a < x < b$.

Suppose that the functions $u(x, t), g(x, t, z)$ and $\varphi(x, t)$ are continuous in R, $Q = \{(x, t, z): (x, t) \in R, z \text{ arbitrary}\}$ and in Γ, respectively.

Define the function $r(x, t)$ by means of the formula

$$r(x, t) = \frac{1}{2\sqrt{\pi}} \int_0^t \int_a^b \frac{\exp[-(x-\xi)^2/4(t-\zeta)]}{\sqrt{t-\zeta}} g\big(\xi, \zeta, u(\xi, \zeta)\big) d\xi d\zeta$$

and let $q(x, t)$ be the solution in R^0 of the equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}$$

such that $q(x, t) = \varphi(x, t) - r(x, t)$ for $(x, t) \in \Gamma$.

We put

$$v(x, t) = q(x, t) + r(x, t) \quad \text{for } (x, t) \in R.$$

Denote by $T(u, g, \varphi)$ the transformation $u \to v$:

$$v = T(u, g, \varphi).$$

One can prove [1] that if $u_n \underset{R}{\Rightarrow} u, g_n \underset{Q}{\Rightarrow} g, \varphi_n \underset{\Gamma}{\Rightarrow} \varphi$ (1), then $v_n = T(u_n, g_n \varphi_n) \Rightarrow v = T(u, g, \varphi)$ on R.

If u_n, g_n, φ_n are bounded in the sup norm, then $\{v_n\}$ is compact.

If $g(x, t, z)$ is continuous in (x, t, z) and Hölder continuous in x and z, then the solution z of the equation

$$z = T(z, g, \varphi)$$

is a regular solution of the problem:

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + g(x, t, z), \tag{1}$$

$$z(x, t) = \varphi(x, t) \quad \text{on } \Gamma \text{ (2)}. \tag{2}$$

Let us introduce the following condition:

CONDITION (H). *There exist functions $u_0(x, t), v_0(x, t)$ which are regular in R and satisfy the inequalities*

$$\frac{\partial u_0}{\partial t} < \frac{\partial^2 u_0}{\partial x^2} + f(x, t, u_0) \quad \textit{in } R^0 + (FR - \Gamma), \tag{3}$$

$$\frac{\partial v_0}{\partial t} > \frac{\partial^2 v_0}{\partial x^2} + f(x, t, v_0) \quad \textit{in } R^0 + (FR - \Gamma), \tag{4}$$

$$u_0(x, t) < \varphi(x, t) < v_0(x, t) \quad \textit{for } (x, t) \in \Gamma. \tag{5}$$

We say that the regular solution $u(x, t)$ of the problem (1) and (2), is the *maximum solution* (*minimum solution*) if for every other solution $v(x, t)$ of that problem the inequality $v(x, t) \leqslant u(x, t)$ $(v(x, t) \geqslant u(x, t))$ holds in R.

Following Mlak [2] we formulate the following

LEMMA. *Let assumption* (H) *be satisfied. Suppose that the functions $\varphi(x, t)$ and $f(x, t, z)$ are continuous in Γ and Q, respectively, and let $f(x, t, z)$ be Hölder continuous with regard to x and z. Then* (1) *and* (2) *has the maximum solution $\bar{u}(x, t)$ and the minimum solution $\underline{u}(x, t)$.*

Our basic theorem is the following one.

THEOREM. *Let the assumptions of the lemma be satisfied, and denote by S the set of points (x, t, u) such that $(x, t) \in R$ and $\underline{u}(x, t) \leqslant u \leqslant \bar{u}(x, t)$. If $(x^0, t^0, u^0) \in S$, then* (1) *and* (2) *has a solution $u(x, t)$ such that $u(x^0, t^0) = u^0$.*

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(1) $s_n \Rightarrow s$ means that s_n tends uniformly to s.

(2) For references, see [1].

Proof. Define the function $f^*(x, t, z)$ as follows:

$$f^*(x,t,z) = \begin{cases} f\big(x, t, u_0(x,t)\big) & \text{if } u_0(x,t) > z, \\ f(x,t,z) & \text{if } u_0(x,t) \leqslant z \leqslant v_0(x,t), \\ f\big(x,t,v_0(x,t)\big) & \text{if } \qquad z > v_0(x,t). \end{cases}$$

The function f^* is bounded. We choose constants M and K such that $\sup|f^*| < M$, $\sup|\varphi| < K$. Then the functions $\tilde{u}_0 = -Mt-K$, $\tilde{v}_0 = MT+K$ satisfy (H) with $\tilde{u}_0 = u_0$, $\tilde{v}_0 = v_0$, $f = f^*$. It is easy to check that f^* is Hölder continuous in x and z.

Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f^*(x, t, u), \tag{6}$$

$$u(x,t) = \varphi(x,t) \quad \text{on } \Gamma. \tag{7}$$

Every solution v of (6) and (7) satisfies $u_0(x,t) < v(x,t) < v_0(x,t)$ (see [2]).

Let $\varepsilon > 0$ and let $v(x,t)$ be an arbitrary solution of (6) and (7). There exists a continuous function $h_\varepsilon(x, t, u; v)$ defined on Q depending on ε and (the fixed) $v(x,t)$ such that:

(i) $|h_\varepsilon(x,t,u;v)| \leqslant M+\varepsilon$ on Q,

(ii) $|f^*(x,t,u) - h_\varepsilon(x,t,u;v)| \leqslant \varepsilon$ on Q,

(iii) $h_\varepsilon(x,t,u;v)$ is uniformly Lipschitz continuous with respect to u,

(iv) $v(x,t)$ is the unique solution of the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + h_\varepsilon(x,t,u;v), \qquad u(x,t) = \varphi(x,t) \quad \text{on } \Gamma.$$

In order to see this, let $\tilde{Q} = \{(x,t,u)\colon (x,t)\in R,\ u_0(x,t) \leqslant u \leqslant v_0(x,t)\}$ and $w(x,t,u)$ be a polynomial with the properties:

$$|w(x,t,u)| \leqslant M \quad \text{on } \tilde{Q},$$

$$|f^*(x,t,u) - w(x,t,u)| \leqslant \tfrac{1}{2}\varepsilon \quad \text{on } \tilde{Q}.$$

Define the function $w^*(x,t,u)$ as follows:

$$w^*(x,t,u) = \begin{cases} w\big(x,t,u_0(x,t)\big) & \text{if } u_0(x,t) > u, \\ w(x,t,u) & \text{if } u_0(x,t) \leqslant u \leqslant v_0(x,t), \\ w\big(x,t,v_0(x,t)\big) & \text{if } \qquad u > v_0(x,t). \end{cases}$$

Let

$$h_\varepsilon(x,t,u;v) = w^*(x,t,u) - w^*(x,t,v(x,t)) + f^*(x,t,v(x,t)) \quad \text{on } Q.$$

Then

$$|h_\varepsilon(x,t,u;v) - w^*(x,t,u)| \leqslant |f^*(x,t,v(x,t)) - w^*(x,t,v(x,t))| \leqslant \tfrac{1}{2}\varepsilon$$

and

$$|f^*(x,t,u) - h_\varepsilon(x,t,u;v)| \leqslant |f^*(x,t,u) - w^*(x,t,u)| + |w^*(x,t,u) - h_\varepsilon(x,t,u;v)| \leqslant \varepsilon \quad \text{on } Q,$$

so that conditions (i)-(iii) follow.

Condition (iv) follows from the equalities

$$h_\varepsilon(x,t,v(x,t);v) = w^*(x,t,v(x,t)) - w^*(x,t,v(x,t)) + f^*(x,t,v(x,t)) = f^*(x,t,v(x,t)) = \frac{\partial v(x,t)}{\partial t} - \frac{\partial^2 v(x,t)}{\partial x^2}.$$

Let $\bar{u}(x,t)$ be the maximum solution and $\underline{u}(x,t)$ the minimum solution of (6) and (7). For a given $\varepsilon > 0$, let $\bar{h}_\varepsilon(x,t,u;\bar{u})$ and $\underline{h}(x,t,u;\underline{u})$ be the functions with the properties (i)-(iv), when $v(x,t) = \bar{u}(x,t)$, $\underline{u}(x,t)$, respectively.

Consider the 1-parameter family of equations

$$\text{(8)} \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g_\lambda(x,t,u),$$

$$\text{(9)} \qquad u(x,t) = \varphi(x,t) \quad \text{on } \Gamma,$$

where $0 \leqslant \lambda \leqslant 1$ and

$$\text{(10)} \qquad g_\lambda(x,t,u) = \lambda \bar{h}_\varepsilon(x,t,u;\bar{u}) + (1-\lambda)\underline{h}_\varepsilon(x,t,u;\underline{u}).$$

The function $g_\lambda(x,t,u)$ is uniformly Lipschitz continuous with respect to u and

$$\text{(11)} \qquad |g_\lambda(x,t,u)| \leqslant \lambda|\bar{h}_\varepsilon(x,t,u;\bar{u})| + (1-\lambda)|\underline{h}_\varepsilon(x,t,u;\underline{u})| \leqslant M + \varepsilon.$$

Problem (8), (9) for a fixed λ has the unique solution $u(x,t,\lambda)$. One can prove [3] (p. 147) that if $\lambda \to \lambda^0$, then $u(x,t,\lambda) \underset{R}{\rightrightarrows} u(x,t,\lambda^0)$.

If (x^0,t^0) is fixed, $u(x^0,t^0,\lambda)$ is a continuous function of λ. Since $u(x^0,t^0,0) = \underline{u}(x^0,t^0)$, $u(x^0,t^0,1) = \bar{u}(x^0,t^0)$, so that $\underline{u}(x^0,t^0) \leqslant u^0 \leqslant \bar{u}(x^0,t^0)$, there exists a λ^0-value, $0 \leqslant \lambda^0 \leqslant 1$, such that $u(x^0,t^0,\lambda^0) = u^0$. The choice of an λ^0 depends on ε. Say $\lambda^0 = \lambda^0(\varepsilon)$. Let $\varepsilon = 1/n$, $n > 1$, and let $g^n(x,t,u) = g_\lambda(x,t,u)$, where $\lambda = \lambda^0(1/n)$. Thus (ii) and (10) show that

(12) $$|f^*(x, t, u) - g^n(x, t, u)| \leqslant 1/n \quad \text{on } Q$$

and by the choice of $\lambda = \lambda^0$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g^n(x, t, u),$$

$$u(x, t) = \varphi(x, t) \quad \text{on } \Gamma.$$

has a unique solution $u^n(x, t)$ such that $u^n(x^0, t^0) = u^0$. Note that $|g^n(x, t, u)| \leqslant M + 1/n$ on Q imply that $|u^n(x, t)| \leqslant (M + 1/n)t + K$ on R (see [1]).

Obviously

(13) $$u^n = T(u^n, g^n, \varphi).$$

Hence $\{u^n\}$ is compact. The sequence $u^1(x, t), u^2(x, t), \ldots$ has a subsequence $u^{a_1}(x, t), u^{a_2}(x, t), \ldots$ which is uniformly convergent, say to $u = u(x, t)$ on R and $u(x^0, t^0) = u^0$.

From (13)

(14) $$u^{a_n} = T(u^{a_n}, g^{a_n}, \varphi).$$

But by (12) $g^{a_n}(x, t, u) \Rightarrow f^*(x, t, u)$ on Q.

By a limit passage in (14) we get $u = T(u, f^*, \varphi)$. It follows that $u(x, t)$ is a solution of the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f^*(x, t, u),$$

$$u(x, t) = \varphi(x, t) \quad \text{on } \Gamma.$$

But $\underline{u}(x, t) \leqslant u(x, t) \leqslant \overline{u}(x, t)$ on R.

It follows from the definition of f^* that

$$f^*\big(x, t, u(x, t)\big) = f\big(x, t, u(x, t)\big).$$

This proves that u is a solution of (1), (2) and $u(x^0, t^0) = u^0$.

References

[1] W. Mlak, *The first boundary value problem for non-linear parabolic equation,* Ann. Polon. Math. 5 (1958), pp. 257-262.

[2] — *Parabolic differential inequalities and Chaplighin's method,* ibidem 8 (1960), pp. 139-153.

[3] J. Szarski, *Differential inequalities,* Warszawa 1965.

Reçu par la Rédaction le 27. 5. 1967