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Assymptotic behavior of non-linear inhomogeneous equations via non-standard analysis

Part 1. Second order equations

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Abstract. Techniques of non-standard analysis are used to prove some assymptotic properties of second order equations of the type $x''+c(t)g(x) = f(t)$, and of the type $\big((a(t)\varphi(x)x')'\big)+c(t)g(x) = f(t)$. Following various assumptions on $a(t)$, $c(t)$, $g(x)$ and $f(t)$ a number of theorems is proved concerning oscillation, or non-oscillation and boundedness. For example, an oscillation theorem resembling Wong's criterion is proved, however, with the integrability conditions applied to the inhomogeneous term. Also a novel application of energy arguments is introduced, and solutions are classified according to the assymptotic behavior of the total energy.

1. Introductory remarks. This paper proves some basic theorems of boundedness theory using non-standard techniques, in the framework of Robinson's theory [6], [7]. An expository monograph of Machover and Hirschfeld [5] outlines concisely the basic techniques of Robinson avoiding the use of the theory of types and relying more on algebraic arguments. A discussion of logical foundations of non-standard extensions is given in [6]. In particular a completeness theorem of Henkin [2] implies the existence of a (non-standard) extension of the real number system $^*\boldsymbol{R}$, which has the property that certain sentences suitably formulated for the real line $\boldsymbol{R}$ remain true for $^*\boldsymbol{R}$. (Since the sentences stated in the formal language $\mathscr{L}$, which are formulated in mathematical analysis usually imply relationships between elements of $\boldsymbol{R}$, of subsets of $\boldsymbol{R}$, etc., it is necessary to first establish some foundations within the framework of the theory of models).

In this article we shall assume that the reader is familiar with Robinson's theory, and we shall not attempt to offer another exposition. The higher order structure $^*\boldsymbol{R}$ discussed here is an ordered field (the usual algebraic operations in $\boldsymbol{R}$, the relations and their properties being expressible in the language $\mathscr{L}$, $^*\boldsymbol{R}$ properly containing the real number system $\boldsymbol{R}$. $^*\boldsymbol{R}$ contains elements which are larger in absolute value than any real number, which will be called *infinite numbers*, and their reciprocals which will be called *infinitesimals*. If x, y are elements of $^*\boldsymbol{R}$ such that $|x-y|$ is an infinitesimal, then we shall say that x is close to y (or infinitely close to y), and we shall denote it by writting $x \simeq y$. If x is a real number (i.e. $x \in \boldsymbol{R}$), x will be called a *standard number*, or a *standard element* of $^*\boldsymbol{R}$. Otherwise x shall be called *non-standard*. $^*\boldsymbol{R}_{+\infty}$, $^*\boldsymbol{R}_{-\infty}$ will denote respectively the infinite positive, and the infinite negative elements of $^*\boldsymbol{R}$. $^*\boldsymbol{R}_{bd}$ will denote elements of $^*\boldsymbol{R}$ bounded in absolute value by a standard number.

2. Statement of the problem. In this paper in part 1 we shall consider the assymptotic properties of equations of the form:

$$\big(a(t)\varphi(x)x'\big)' + c(t)g(x) = f(t) \tag{2}$$

on some infinite ray $I = [t_0, \infty)$, subject to the assumptions:

(i) $c(t), f(t) \in C[t_0, \infty)$,

(ii) $a(t) \in C^1[t_0, \infty)$,

(iii) $\varphi(x) \in C^1(-\infty, +\infty)$,

(iv) $g(x) \in C(-\infty, +\infty)$.

In part 1 we shall consider also the special cases of equation (2), i.e.

$$x'' + c(t)g(x) = f(t) \tag{1}$$

and the homogeneous equations (2^{H}), (1^{H}) obtained by putting $f(t) \equiv 0$ in (2) and (1) respectively. By a solution we shall always mean a classical solution, i.e. twice continuously differentiable solution of the corresponding equation. To prove our main theorems we shall require the following lemmas:

LEMMA 1.1. *A standard function $x(t)$ is oscillatory if and only if the function $x(t)$, $t \in {}^*\boldsymbol{R}$, vanishes for some $\tau \in {}^*\boldsymbol{R}_{+\infty}$.*

Proof. Assume that $x(t)$ is oscillatory. Then the sentence "$\tau \in \boldsymbol{R} \Rightarrow \exists\, t \in \boldsymbol{R}[t > \tau \,\&\, x(t) = 0]$" is a true sentence in our model $\boldsymbol{R}$. Hence it is true in $^*\boldsymbol{R}$. Choosing $\tau \in {}^*\boldsymbol{R}_{+\infty}$ completes the argument. Conversely let us assume that $x(\tau) = 0$ for some $\tau \in {}^*\boldsymbol{R}_{+\infty}$. Let $\bar{t}$ be a (fixed) standard number. The sentence "$\exists \tau \in {}^*\boldsymbol{R}[\tau > \bar{t}, \& \, x(\tau) = 0]$" is a sentence in our formal language $\mathscr{L}$ which is true in $^*\boldsymbol{R}$. Therefore it remains valid, when interpreted in $\boldsymbol{R}$. Since $\bar{t}$ was arbitrarily chosen standard number, we have shown that $x(t)$ is oscillatory.

LEMMA 1.2. *The standard function $x(t)$, $t \in [t_0, \infty)$ is unbounded if and only if $|x(\tau)| \in {}^*\boldsymbol{R}_{+\infty}$ for some $\tau \in {}^*\boldsymbol{R}_{+\infty}$.*

The proof is straighforward and shall be omitted.

Remark 1. It is know (see Robinson [6]) that given any standard function $g(x)$, such that $\int^{\infty} g(x)dx < \infty$, then $g(x)$ satisfies the following condition: for any $\xi, \eta \in {}^*\boldsymbol{R}_{+\infty}$ it is true that $\int_\xi^\eta g(x)\,dx \simeq 0$ ($s \simeq 0$ means s is an infinitesimal).

Remark 2. If $\lim_{\tau\to\infty} \int_{t_0}^{\tau} g(x)\,dx = +\infty$, then it is true that given any number $A \in {}^*\boldsymbol{R}$, and given any $\xi \in {}^*\boldsymbol{R}$, $\xi \geqslant t_0$, there exists η such that $\int_\xi^\eta g(x)\,dx > A$. Also it is true that given any $\tau \in {}^*\boldsymbol{R}_{bd}$ and $\xi \in {}^*\boldsymbol{R}_{+\infty}$, $\int_\tau^\xi g(x)\,dx \in {}^*\boldsymbol{R}_{+\infty}$.

Remark 3. If for some positive integer K, the K-th derivative $f^{(k)}(x)$ of a standard function $f(x)$ exists on $[t_0, \infty)$ and has the property $f^{(k)}(x) < a < 0$ for all x sufficiently large (i.e. for all x greater than some number $M > t_0$, $M \in {}^*\boldsymbol{R}$), then there exists $A \in {}^*\boldsymbol{R}_{+\infty}$ such that $f(x) < 0$ for all $x > A$. The proof is elementary.

Remark 4. If for any $t_1 \in {}^*\boldsymbol{R}_\infty$ there exist $t_2 > t_1$ such that for any $t > t_2$, $|x'(t)| - |x'(t_1)| \in {}^*\boldsymbol{R}_{-\infty}$, then $x(t) \in {}^*\boldsymbol{R}_{bd}$ for some $t \in {}^*\boldsymbol{R}_{+\infty}$.

3. Frequency and boundedness theorems for equations of the type.

$$x'' + c(t)g(x) = f(t), \tag{1}$$

$$x'' + c(t)g(x) = 0, \tag{1^H}$$

$$\big(a(t)\varphi(x)x'\big)' + c(t)g(x) = f(t), \tag{2}$$

$$\big(a(t)\varphi(x)x'\big)' + c(t)g(x) = 0, \tag{2^H}$$

$$x'' + b(t)x' + c(t)g(x) = f(t). \tag{1^a}$$

The first two theorems of this paper deal with the so-called frequency function

$$\hat{\pi}\big(\hat{x}(t)\big) = \pi(t).$$

We offer the following definition:

Let $\hat{x}(t)$ be the ray function continuous on $[t_0, \infty)$. (In our discussion $\hat{x}(t)$ will always denote a trajectory of a corresponding differential equation (1), (1^H), (2) or (2^H).) Let $\tilde{t}$ be any point of $[t_0, \infty)$. We define $\pi(\tilde{t}) = \hat{\pi}\big(\hat{x}(\tilde{t})\big)$ to be the maximum length of a half open (half closed) interval $[t_1, t_2)$ containing $\tilde{t}$, such that the open interval (t_1, t_2) contains no zeros of $\hat{x}(t)$. In the case when either $[t_0, \tilde{t})$ contains no zeros of $\hat{x}(t)$, or $[\tilde{t}, \infty)$ contains no zeros of $\hat{x}(t)$, we assign $\hat{\pi}\big(\hat{x}(\tilde{t})\big) = \pi(\tilde{t}) = +\infty$. Hence $\pi(t)$ has values in $\{R_+ \cup \infty\}$, is constant on any zero-free interval in $[t_0, \infty)$, and in general has a jump discontinuity at every zero of $\hat{x}(t)$. $\hat{\pi}\big(\hat{x}(t)\big)$ will be called the *frequency function* associated with the trajectory $\hat{x}(t)$. We shall prove theorems for equations (1), (1^H), (2), (2^H) concerning boundedness, and assymptotic behavior of the solutions.

LEMMA 1. *We consider equation* (1^{H}) *satisfying the following conditions:*

1) $\frac{g(x)}{x}$ *is a bounded function of* x, $x \neq 0$ (*and* $\limsup_{x\to 0}\left|\frac{g(x)}{x}\right| < +\infty$),

2) $\lim_{t\to\infty} c(t) = 0$.

Then $\lim_{t\to\infty} \pi(t) = +\infty$.

The proof of this lemma is elementary, and uses only standard arguments.

We have for $x \neq 0$

$$y'' + \left[\frac{c(t)g\big(x(t)\big)}{x(t)}\right] y = 0, \quad t \geqslant t_0,$$

with

$$\limsup_{x\to 0}\left|\frac{c(t)g\big(x(t)\big)}{x(t)}\right| < \infty, \quad t \geqslant t_0.$$

Assuming that $y(t)$ is oscillatory (otherwise there is nothing to prove) we apply the classical inequality due to Liapunov.

If $y(t)$ vanishes at $t = t_1$, and $t = t_2$, then

$$\int_{t_1}^{t_2}\left|\frac{c(t)g\big(x(t)\big)}{x(t)}\right| dt \geqslant \frac{4}{t_2 - t_1}.$$

Let M be any upper bound on $\left|\frac{g\big(x(t)\big)}{x(t)}\right|$ for all $x \neq 0$, $t \geqslant t_0$. Then $M\int_{t_1}^{t_2} |c(t)|\, dt \geqslant \frac{4}{t_2 - t_1}$, and since $c \to 0$, given $\varepsilon > 0$, we can choose t_1 sufficiently large so that $|c| < \varepsilon$, for all $t \in [t_1, t_2]$ and $(t_2 - t_1)^2 \geqslant 4/M\varepsilon$.

Hence $\lim_{t\to\infty}(t_2 - t_1) = +\infty$, as required. This method of proof fails to work for equation (1). However, a fairly straightforword non-standard argument shows this lemma to be valid for (1) if $\int^{\infty}\left|\frac{f(t)}{c(t)}\right| dt < \infty$, in adition to hypothesis (1) and (2). Instead of proving this lemma we shall offer the proof of the more general case of equation (2).

THEOREM 1. *We consider equation* (2), $t \geqslant t_0$, *subject to conditions*

(i) $\lim_{t\to\infty} c(t) = 0$,

(ii) $g(x)/x$ *is a bounded function of* x (*and in particular*

$$\limsup_{x\to 0}\left|\frac{g(x)}{x}\right| < \infty).$$

(iii) $f(t) > 0$ *for sufficiently large values of* t. (*Note that this assumption can be made whenever* $f(t)$ *is of constant sign.*)

(iv) $\liminf\limits_{x\to\infty} |c(x)| > 0$.

Suppose there exists a function $h(t) \in C^1[t_0, \infty)$ *such that*

(v) $h(t) > 0$ *for sufficiently large values of* t *and* $\lim\limits_{t\to\infty} h(t) = +\infty$.

(vi) $\lim\limits_{t\to\infty} \dfrac{h'(t)}{h(t)} = 0$.

(vii) $\lim\limits_{t\to\infty} a(t)h(t) = +\infty$.

(viii) $\lim\limits_{t\to\infty} |c(t)h(t)| = 0$.

Then $\lim\limits_{t\to\infty} \pi(x(t)) = +\infty$ *along any trajectory* $\hat{x}(t)$ *of* (2).

(We observe that condition (viii) implies condition (i), and that considerably weaker hypothesis are sufficient to the proof of this theorem if $a(t)$ and $c(t)$ differ in sign for all sufficiently large values of t.)

Proof. We study the behavior of the function $\psi(t) = \dfrac{h(t)a(t)\varphi(x)x'(t)}{x(t)}$ along a trajectory $\hat{x}(t)$ of equation (2). Assuming that $\hat{x}(t)$ is oscillatory we select points $t_1, t_2 \in {}^*\boldsymbol{R}_{+\infty}$, such that $\hat{x}(t_1) = \hat{x}(t_2) = 0$, and $\hat{x}(t) \neq 0$ on the open interval (t_1, t_2). (It is easy to show that such points exist in ${}^*\boldsymbol{R}_{+\infty}$.) The function $\psi(t)$ is defined, and is in fact continuously differentiable on (t_1, t_2). Let us assume that $\hat{x}(t) > 0$ on (t_1, t_2) (without any loss of generality since an identical argument follows if $x(t) < 0$. $\psi(t)$ is pseudo-unbounded on (t_1, t_2), and will assume all values in ${}^*\boldsymbol{R}$ on this open interval. In particular it is possible to select points τ_1, τ_2 $(\tau_2 > \tau_1)$ such that $\psi(\tau_1) \geqslant +1$, $\psi(\tau_2) \leqslant -1$ and $|\psi(t)| < M$ for some standard number M, and such that $\dfrac{\varphi(x)x'}{x} \in {}^*\boldsymbol{R}_{bd}$ on $[\tau_1, \tau_2]$. Clearly

$$\int_{\tau_1}^{\tau_2} \psi'(t)\,dt = \psi(\tau_2) - \psi(\tau_1) \leqslant -2,$$

$$\int_{\tau_1}^{\tau_2} \psi'(t)\,dt = \int_{\tau_1}^{\tau_2} \left[\left(\frac{h'}{h}\psi\right) + \left(\frac{hf}{\hat{x}}\right) - \left(c(t)h\frac{g(x)}{x}\right) - \frac{\psi^2}{h(t)a(t)\varphi(\hat{x})}\right] dt \leqslant -2.$$

But $\dfrac{h(t)f(t)}{\hat{x}(t)} > 0$ on (τ_1, τ_2), and therefore

$$\int_{\tau_1}^{\tau_2} \left[\frac{h'}{h}\psi - ch\frac{g(x)}{x} - \frac{\psi^2}{ha\varphi(x)}\right] dt < -2,$$

or

$$\left| \int_{\tau_1}^{\tau_2} \left[\frac{h'}{h} \psi - ch \frac{g(x)}{x} - \frac{\psi^2}{ha\varphi(x)} \right] dt \right| > 2.$$

However, our assumptions imply that every element of the sum in the integrand is an infinitessimal on (τ_1, τ_2). Hence $(\tau_2 - \tau_1) \epsilon {}^*\boldsymbol{R}_{+\infty}$. The equivalent "standard" statement is $\lim_{t\to\infty} \pi(\hat{x}(t)) = +\infty$, completing the proof.

Examples of application.

1. Consider the equation:

$$(\cos x \cdot x')' + \frac{1}{t} x \sin x = Ct^a, \quad t > 1,$$

where a is any real number, and C is a constant.

We observe that $h(t) = t^{1/2}$, $t > 1$ satisfies all the requirements of this theorem ($a(t) \equiv 1$ in this case).

2. A simple pendulum with variable mass: $x'' + c(t)\sin x = f(t)$, where $f(t) > 0$. Assume that $c(t) \approx t^a$, $a < 0$. Choose $h(t) = t^\beta$, where $0 < \beta$, and $a + \beta < 0$.

Theorem 2. *Suppose that*

(i) $\limsup_{x\to\infty} |\varphi(x)| = +\infty$, $\varphi(0) = 0$, $\varphi(x) \neq 0$ *if* $x \neq 0$.

(ii) $\limsup_{t\to\infty} |a(t)| < \infty$.

(iii) $\lim_{T\to\infty} \int^T c(t)\,dt = +\infty$, $c(t) > 0$.

(iv) $f(t) \geqslant 0$.

(v) $\lim_{T\to\infty} \left(\frac{\int^T f(t)\,dt}{\int^T c(t)\,dt} \right) = 0$.

(vi) $g(x) \neq 0$ *if* $x \neq 0$, $g(0) = 0$,

$$\lim_{|x|\to\infty} |g(x)| = +\infty.$$

Then any solution $\hat{x}(t)$ *of* (2) *will have the assymptotic behavior*: $\lim_{t\to\infty} |\hat{x}(t)| = +\infty$, *or* $\liminf_{t\to\infty} |\hat{x}(t)| = 0$.

Proof. Let us assume that there exists a solution $\hat{x}(t)$, such that for some $T > 0$, for some standard $m > 0$, $|g(\hat{x}(t))| > m$ for all $t > T$, i.e. $\liminf_{t\to\infty}|g(\hat{x}(t))| > m > 0$. Let t_2 be any point $\epsilon\ {}^*\boldsymbol{R}_{+\infty}$,

$$\int_T^{t_2}\Big[\big(a(t)\varphi(x(t))x'(t)\big)'+c(t)g(x(t))\Big]dt = \int_T^{t_2} f(t)dt,$$

$$\left|a(t)\varphi(x(t))x'(t)\Big|_T^{t_2} - \int_{t_1}^{t_2} f(t)dt\right| > m\int_T^{t_2} c(t)dt,$$

$$\left|\frac{a(t)\varphi(x(t))x'(t)}{\int_T^{t_2} c(t)dt} - \frac{\int_T^{t_2} f(t)dt}{\int_T^{t_2} c(t)dt}\right| > m.$$

But

$$\frac{\int_T^{t_2} f(t)dt}{\int_T^{t_2} c(t)dt} \approx 0 \qquad (\text{since } t_2 \in {}^*\boldsymbol{R}_{+\infty}).$$

Hence for any standard m', such that $m > m' > 0$

$$\left|\frac{a(t)\varphi(x(t))x'(t)}{\int_T^{t_2} c(t)dt}\right|_{t=T}^{t=t_2} > m'.$$

But $\int_T^{t_2} c(t)dt \in {}^*\boldsymbol{R}_{+\infty}$, which implies that

$$\varphi(x(t))x'(t) \in {}^*\boldsymbol{R}_\infty$$

for any $t_2 \in {}^*\boldsymbol{R}_\infty$. In particular $x'(t) \neq 0$ for any $t \in {}^*\boldsymbol{R}_\infty$, and now it is easy to show that there exists $\tilde{t} \in {}^*\boldsymbol{R}_{+\infty}$ such that for all $t > \tilde{t}$ we have $\varphi(\hat{x}(t)) \in {}^*\boldsymbol{R}_\infty$, which is possible only if $\hat{x}(t) \in {}^*\boldsymbol{R}_\infty$. But this implies that $\hat{x}(t) \in \boldsymbol{R}_\infty$ for all $t \in {}^*\boldsymbol{R}_\infty$. (Choose a standard number $\tilde{c} > 0$ and consider the statement in our language $\mathscr{L}$ "$\exists\ \tilde{t} \in {}^*\boldsymbol{R}$, such that for all $t^* > \tilde{t}$, $\hat{x}(t) > \tilde{c}$." This statement must be true in our model $\boldsymbol{R}$, hence "$\exists\ \tilde{t} \in \boldsymbol{R} \ni$ for all $t > \tilde{t}$ it is true that $\hat{x}(t) > \tilde{c}$".) Since $\tilde{c}$ is arbitrary this is equivalent to $\lim_{t\to\infty}|\hat{x}(t)| = \infty$.

COROLLARY 1. *If condition* (i) *is replaced by* (i^a) $\limsup_{x\to\infty}|\varphi(x)| < \infty$, *then every solution* $\hat{x}(t)$ *of* (2) *must have the property* $\liminf_{t\to\infty}|\hat{x}(t)| = 0$.

Proof. Observe that the conclusion of the entire argument, namely that for all $t \in {}^*\boldsymbol{R}_{+\infty}$ we have $\varphi(x(t)) \in {}^*\boldsymbol{R}_\infty$ is now impossible. The only remaining possibility is: $\liminf_{t\to\infty}|\tilde{x}(t)| = 0$. Part of the recent results of Hammett [1] is easily shown to be a corollary of this theorem.

We observe that the implication of Corollary 1: $\limsup_{t\to\infty}|\varphi(\hat{x}(t))| < \infty \Rightarrow \liminf_{t\to\infty}|\varphi(\hat{x}(t))| = 0$ can be proved under slightly weaker hypothesis. We really do not need the condition $\lim_{\tau\to\infty}\int^{\tau} c(t)\,dt = +\infty$ and no changes are required in the arguments if it is replaced by $\limsup_{t\to\infty}\int^{\tau} c(t)\,dt = +\infty$, since t_2 in the argument of Theorem 2 can be chosen so that $\int_T^{t_2} c(t)\,dt \in {}^*\boldsymbol{R}_{+\infty}$.

Definition. If given any $t_1 > t_0$, and given any $\varepsilon > 0$, $\delta > 0$, there exists $T_1 > t_1$, such that for any $T_2 > T_1$ the Lebesgue measure of the set S: $\mu(s)\{S\colon t\in[T_1, T_2]\in|\hat{x}(t)| > \varepsilon\}$ is less than δ, then we say that *essentially the limit of $\hat{x}(t)$ is zero*, and denote it by: $\operatorname{ess\,lim}_{t\to\infty}\hat{x}(t) = 0$.

A non-standard definition is much simpler. $\operatorname{ess\,lim}\hat{x}(t) = 0 \Leftrightarrow \hat{x}(t) = 0$ a.e. $t\in(T_1, T_2)$, $T_2 - T_1 \in {}^*\boldsymbol{R}_{bd}$ (where a.e. implies $\mu(S)\approx 0$, $S = \{t \mid \hat{x}(t) \neq 0\}$), $\mu\colon P({}^*\boldsymbol{R}) \to {}^*\boldsymbol{R}_+$. We observe that $T_1, T_2 \in {}^*\boldsymbol{R}_\infty$ must belong to the same galaxy (see [6]).

Theorem 3. *We assume*

(i) $\liminf_{t\to\infty}|c(t)| > 0$, (ii) $xg(x) > 0$ *if* $x \neq 0$,

(iii) $\varphi(0) = 0$ $(\varphi\in C^1(-\infty, +\infty))$, (iv) $\int^{\infty} f(t)\,dt < \infty$.

Then either the solutions $\hat{x}(t)$ of (2) *is non-oscillatory or else* $\operatorname{ess\,lim}\hat{x}(t) = 0$.

Proof. Assume that an oscillatory solution $\hat{x}(t)$ of (2) does exist. Let t_1, t_2 be two points $\in {}^*\boldsymbol{R}_{+\infty}$ such that $\hat{x}(t_1) = \hat{x}(t_2) = 0$. $x(t)\neq 0$ $\forall t\in(t_1, t_2)$

$$\int_{t_1}^{t_2}\big(a(t)\varphi(x)x'(t)\big)'\,dt + \int_{t_1}^{t_2} c(t)g\big(x(t)\big)\,dt \approx 0 \quad \text{(by (iv))}.$$

Hence

$$a(t)\varphi\big(x(t)\big)x'(t)\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} c(t)g\big(x(t)\big)\,dt \approx 0.$$

But

$$\varphi\big(x(t_2)\big) = \varphi\big(x(t_1)\big) = 0,$$

and therefore

$$\int_1^2 c(t)g\big(x(t)\big)\,dt \approx 0.$$

Since $|c(t)| > m > 0$ (for all $t\in \boldsymbol{R}_{+\infty}$) and $g\big(x(t)\big)$ is of constant sign, it follows that

$$\int_{t_1}^{t_2}\big|g\big(x(t)\big)\big|\,dt \approx 0.$$

If $g(x(t)) \not\approx 0$ on some subset of $[t_1, t_2]$ of measure $\mu > 0$, it follows that $\mu \approx 0$. Hence $x(t) \neq 0$ only on subset of $[t_1, t_2]$ on infinitessimal measure. But this implies that for any $t_1, t_2 \in {}^*\boldsymbol{R}_\infty$, such that $t_2 - t_1 \in {}^*\boldsymbol{R}_{bd}$, $\hat{x}(t) \not\approx 0$ at most on a subset of $[t_1, t_2]$ of infinitessimal measure.

Consider as an example of application any equation of the form

$$\left(a(t)x^m x'\right)' + x^{2k+1} = f(t),$$

where m, k are any non-negative integers, $a(t) > 0$, and $\int^\infty f(t)\,dt < \infty$.

In the next two theorems we offer a set of sufficient conditions for a bounded, or monotone behavior of solutions.

First let us introduce the following functions:

$$\Phi(x) = \operatorname{sign}(x)\cdot\int_0^x \varphi(\xi)\,d\xi, \qquad G(x) = \operatorname{sign}(x)\cdot\int_0^x \varphi(\xi)g(\xi)\,d\xi,$$

as before we shall assume that $\xi\varphi(\xi) > 0$ if $\xi \neq 0$ and $\xi\varphi(\xi)g(\xi) > 0$ if $\xi \neq 0$.

We observe that $\varphi(x)$ and $G(x)$ are monotone increasing functions of x.

THEOREM 4. *Suppose that*

(i) $\limsup |a(t)| < \infty$,

(ii) $\lim\limits_{t\to\infty}\left(a(t)f(t)\right) = 0$,

(iii) $\liminf\limits_{t\to\infty} a(t)c(t) = m > 0$.

Then either any continuable solution $\hat{x}(t)$ of (2) *is bounded, or* $\lim\limits_{t\to\infty}|\hat{x}(t)| = +\infty$.

Proof. Equation (2) implies the following equality

$$\text{(A)} \qquad \left[\left(a(t)(x)x'\right)^2\right]' + 2\{c(t)a(t)\varphi(x)g(x)x' - a(t)f(t)\varphi(x)x'\} \equiv 0,$$

along any trajectory $\hat{x}(t)$ of (2). We shall prove, using equality (A) that it is impossible for $\hat{x}(t)$ to be unbounded while $\liminf\limits_{t\to\infty}|x(t)| < \infty$.

Let us assume to the contrary that it is possible to find $t_1, t_2 \in {}^*\boldsymbol{R}_\infty$, $t_2 > t_1$ such that $\hat{x}(t_1) \in {}^*\boldsymbol{R}_{bd}$, $\hat{x}(t_2) \in {}^*\boldsymbol{R}_\infty$. Without any loss of generality we can assume that either $x(t)$ and $x'(t)$ have the same sign on $[t_1, t_2]$ (say positive) and that $x'(t_1) \approx 0$, or else we can choose t_1 so that $\hat{x}(t_1) = 0$ while $\hat{x}(t)$ and $x'(t)$ are of the same sign (say positive) on $[t_1, t_2]$. Hence $a(t_1)\varphi(\hat{x}(t_1))x'(t_1) \approx 0$. Choose $\tilde{T} \in (t_1, t_2)$. Then for some $\bar{t} \in [\bar{t}_1, \tilde{T}]$,

$$\int_{t_1}^{\tilde{T}} \left[a(t)f(t)\frac{d\Phi}{dx}x'\right]dt = a(\bar{t})f(\bar{t})\left[\Phi(x(\tilde{T})) - \Phi(x(t_1))\right].$$

Since $a(t)f(t) \approx 0 \ \forall\, t \in {}^*\boldsymbol{R}_{+\infty}$, it follows that

$$\int_{t_1}^{\tilde{T}} \big(a(t)f(t)\varphi(x)x'\big)dt \approx 0$$

for any $\tilde{T}$ such that $x(\tilde{T}) \in {}^*\boldsymbol{R}_{bd}$, $t_1 < \tilde{T} < t_2$. It is therefore true that for any $\tilde{t} \in [t_1, t_2]$ such that $\hat{x}(\tilde{t}) \in {}^*\boldsymbol{R}_{bd}$

$$\int_{t_1}^{\tilde{t}} \big(a(t)f(t)\varphi(x)x'\big)dt < K,$$

where

$$K = m\big[G\big(x(T)\big) - G\big(x(t_1)\big)\big],$$

$m = \big[\liminf\limits_{t\to\infty}\big(a(t)c(t)\big)\big]$, and where T is chosen so that $x(t) \in {}^*\boldsymbol{R}_{bd}$ and $G\big(x(T)\big) - G\big(x(t_1)\big) \not\approx 0$. Hence by a well-known lemma of non-standard analysis there exists $\hat{T} \in [t_1, t_2]$ such that $\hat{x}(\hat{T}) \in {}^*\boldsymbol{R}_{\infty}$. But

$$\int_{t_1}^{\hat{T}} \big(a(t)f(t)\varphi(x)x'\big)dt \leqslant k < m\cdot\big[G\big(x(\hat{T})\big) - G\big(x(t_1)\big)\big].$$

It follows that

$$\int_{t_1}^{T} \big[\big(af(t)\varphi(x)x'\big) - \big(ac(t)\varphi(x)g(x)x'\big)\big]dt < k_1 < 0,$$

where k_1 is a standard number. However, it follows from equation (A) that

$$\tfrac{1}{2}\big[a(\hat{T})\varphi\big(\hat{x}(\hat{T})\big)x'(\hat{T})\big]^2 = \int_{t_1}^{\hat{T}} \{\big[\big(af(t)\varphi(x)x'\big)\big] - [ac(t)\varphi(x)g(x)x']\}dt + \xi,$$

where ξ is an infinitessimal.

Hence

$$\tfrac{1}{2}\big[a(\hat{T})\varphi\big(\hat{x}(\hat{T})\big)\hat{x}'(\hat{T})\big]^2 < 0,$$

which is impossible.

We comment on the almost complete absence of hypothesis concerning the behavior of the non-linear functions $\varphi(x)$, $g(x)$, aside from the usual assumptions on respective differentiability and continuity.

THEOREM 5. *In addition to hypothesis* (i), (ii), *and* (iii) *of Theorem* 4 *let us also assume that*

(iv) $\lim\limits_{|x|\to\infty} \inf|\varphi(x)| > 0$,

(v) $\lim\limits_{|x|\to\infty} \inf g(x) > 0$.

Then $\lim_{t\to\infty}|x(t)| = +\infty$ *implies that* $|x(t)|$ *is a monotone increasing function for sufficiently large values of* t.

Proof. We only need to show that $\hat{x}'(t)$ is non-oscillatory. A non-standard characterization of non-oscillation is:

$$\hat{x}'(t) \neq 0 \ \forall\, t \in {}^*\boldsymbol{R}_{+\infty}.$$

Let us assume to the contrary that $\hat{x}'(t)$ is oscillatory. Then it is possible to choose $\tilde{t} \in {}^*\boldsymbol{R}_{+\infty}$ such that $\hat{x}'(\tilde{t}) = 0$, $\varphi(x)\hat{x}'(t) > 0$ on some interval $[\tilde{t}, T] \subset {}^*\boldsymbol{R}_{+\infty}$, since $\varphi(x)$ must be of constant sign. (Observe that this property can be stated as an appropriate sentence in our language in our model $\boldsymbol{R}$, hence it is true also in ${}^*\boldsymbol{R}$.) Then

$$\tfrac{1}{2}[a(T)\varphi(x)x'(T)]^2 = \int_{\tilde{t}}^{T} [(af)\varphi(x)x' - (ac)\varphi(x)g(x)x']\,dt$$
$$= \int_{\tilde{t}}^{T} \left[\big(af - acg(x)\big)\frac{d\varphi\big((x(t)\big)}{dt}\right]dt.$$

Since $af(t) \approx 0 \ \forall\, t \in {}^*\boldsymbol{R}_{+\infty}$, and $\exists k$ (a standard number) such that $ac(t)g\big(x(t)\big) > k > 0$, $\forall\, t \in {}^*\boldsymbol{R}_{+\infty}$, $\forall x \in {}^*\boldsymbol{R}_{\infty}$, it follows that $\varphi(x)x'[af - acg(x)]$ is negative on the interval $[\hat{t}, T]$, hence that $[a(T)\varphi(x)x'(T)]^2 < 0$ which is impossible.

This proves the monotone behavior of $\hat{x}(t)$ $(t \in {}^*\boldsymbol{R}_{+\infty})$.

THEOREM 6. *We assume hypothesis* (i)-(iv) *and also*

(v) $\lim_{t\to\infty} a(t) = \bar{a}$ *exists*,

(vi) *for sufficiently large values of* $|x|$, $\varphi(x)$ *is a monotone increasing function of* x.

(vii) $\lim_{|M|\to\infty} \int^{M} \varphi(\xi)\,d\xi = +\infty$.

Then all solutions of (2) *are bounded.*

Proof. Assume to the contrary that $\hat{x}(t) \in \boldsymbol{R}_{+\infty}$ for some $t \in {}^*\boldsymbol{R}_{+\infty}$. Hence it follows from Theorem 4 that $\hat{x}(t) \in {}^*\boldsymbol{R}_{+\infty}$ for all $t \in {}^*\boldsymbol{R}_{+\infty}$. Take any $t_1 \in {}^*\boldsymbol{R}_{+\infty}$, and choose $t_2 \in {}^*\boldsymbol{R}_{+\infty}$, $t_2 > t_1$ such that $t_2 - t_1 \not\approx 0$. It is easy to show that

$$\big[a(t_2)\varphi\big(x(t_2)\big)x'(t_2)\big]^2 - \big[a(t_1)\varphi\big(x(t_1)\big)x'(t_1)\big]^2$$
$$= 2\int_{t_1}^{t_2} \big(af\varphi(x)x' - ac\varphi(x)g(x)x'\big)\,dt \in {}^*\boldsymbol{R}_{-\infty}.$$

However, for all $t \in {}^*\boldsymbol{R}_{+\infty}$

$$a(t)\varphi\big(x(t)\big)x'(t) = (\bar{a} + \xi)\big[\varphi\big(x(t)\big)x'(t)\big],$$

where $\xi \approx 0$, and therefore for some $\xi_1, \xi_2 \approx 0$,

$$\big[(\bar{a}+\xi_2)\varphi(x(t_2))x'(t_2)\big]^2-\big[(\bar{a}+\xi_1)\varphi(x(t_1))x'(t_1)\big]^2 \in {}^*\boldsymbol{R}_{-\infty},$$

$$(\bar{a})^2\{[\varphi(x(t_2))x'(t_2)]^2-[\varphi(x(t_1))x'(t_1)]^2\}+2\bar{a}\big[\xi_2(\varphi(x(t_2))x'(t_2))^2-$$
$$-\xi_1(\varphi(x(t_1))x'(t_1))^2\big]+\big[\xi_2^2(\varphi(x(t_2))x'(t_2))^2-$$
$$-\xi_1^2(\varphi(x(t_1))x'(t_1))^2\big]+\big[\xi_2^2(\varphi(x(t_2))x'(t_2))^2-\xi_1^2(\varphi(x(t_1))x'(t_1))^2\big] \in {}^*\boldsymbol{R}_{-\infty}.$$

This is possible only if

$$[\varphi(x(t_2))x'(t_2)]^2-[\varphi(x(t_1))x'(t_1)]^2 \in {}^*\boldsymbol{R}_{-\infty}.$$

Since $\varphi(x(t_2)) \geqslant \varphi(x(t_1))$, it follows that $[x'(t_2)]^2-(x'(t_1))^2 \in {}^*\boldsymbol{R}_{-\infty}$. But by Remark 4 in the introductory discussion "$\exists\ T \in {}^*\boldsymbol{R}_{+\infty}$ such that $\hat{x}(T) \in {}^*\boldsymbol{R}_{bd}$" which is a contradiction.

TOTAL ENERGY ARGUMENTS. In all future discussion we shall assume that $a(t) \geqslant 0$ for all $t \geqslant t_0$. We shall denote by $p(t)$ the function $p(t) = -\sqrt{a}\varphi(x)x'$ (where $\sqrt{\ }$ denotes the positive square root). Then equation (2) can be written as a first order system:

$$\sqrt{a}\varphi(x)\cdot x' = -p(t), \qquad (\sqrt{a}\,p(t))' = c(t)g(x)-f(t).$$

The function $E(t)$ defined below will be called the *total energy*

$$E(t) = \tfrac{1}{2}a(t)p^2(t)+a(t)c(t)\int_0^x \varphi(\xi)g(\xi)d\xi$$

$$= a(t)\big[p^2(t)+c(t)G(x(t))\big],$$

where

$$G(x) = \int_0^x \varphi(\xi)g(\xi)d\xi.$$

We compute the total derivative of the total energy along a trajectory $\hat{x}(t)$ of (2),

$$\text{(B)}\qquad E'(t) = (\sqrt{a}\,p)(\sqrt{a}\,p)'+a(t)c(t)\varphi(\hat{x})g(\hat{x})\hat{x}'+[a(t)c(t)]'G(\hat{x}(t))$$
$$= [a(t)c(t)]'G(\hat{x})+f(t)\sqrt{a}\,\varphi(\hat{x})\hat{x}'.$$

DEFINITION. A solution $\hat{x}(t)$ of (2) or (2^{H}) is called *conservative* if $E(\hat{x}(t)) \equiv$ constant for all $t \geqslant t_0$, and is called *assymptotically conservative* if $\lim_{t\to\infty} E(\hat{x}(t))$ exists.

If every solution of (2^{H}) is (assymptotically) conservative, the equation is called (*assymptotically*) *conservative*. (Clearly (2^{H}) is conservative if $c(t) = k(a(t))^{-1}$, where k is some constant.)

The energy function is non-negative if

(vii) $G(x) > 0$ whenever $x \neq 0$,

(viii) $c(t) > 0$ for all $t \geqslant t_0$.

THEOREM 7. *Let us assume that* (vii), (viii) *are true, and* $[a(t)c(t)] \in C^1[t_0, \infty)$, $f(t) \neq 0$ *for sufficiently large values of* t,

$$\int^{\infty} |f(t)|\,dt < \infty, \quad \limsup_{t\to\infty} \left|\frac{[a(t)c(t)]'}{f(t)}\right| < \infty, \quad \limsup_{t\to\infty} a(t) < \infty.$$

If both $|\hat{x}(t)|$ *and* $|\hat{x}'(t)|$ *are bounded, then the system is assymptotically conservative.*

Proof. We need to show that $E(t_1) \approx E(t_2)$ for any $t_1, t_2 \in {}^*\boldsymbol{R}_{+\infty}$. From formula (B) we have

$$E(t_2) - E(t_1) = \int_{t_1}^{t_2} f(t)\left[\sqrt{a}\varphi(x)x' + \frac{[a(t)c(t)]'}{f(t)} G(x)\right]dt$$

$$\leqslant \sup_{t\in[t_1,t_2]} \left\{|\sqrt{a}\varphi(x)x'| + \left|\frac{[a(t)c(t)]'}{f(t)} G(x)\right|\right\} \int_{t_1}^{t_2} f(t)\,dt.$$

Since the quantity inside the brackets is bounded and $\int_{t_1}^{t_2} f(t)dt \approx 0$ $\forall\, t_1, t_2 \in {}^*\boldsymbol{R}_{+\infty}$, the conclusion of this theorem follows.

Frequently it is easier to check the behavior of total evergy function along an arbitrary trajectory of equation (2) than to check the assymptotic properties of the solution. It becomes important to deduce some particular properties of the solution from the corresponding behavior of the total energy function. Theorem 8 offers a set of sufficient conditions for $\lim_{t\to\infty}\hat{x}(t)$ to exist along arbitrary trajectory $\hat{x}(t)$ of equation (2).

THEOREM 8. *Assume that*

(i) $\limsup_{t\to\infty} (a(t)) < \infty$.

(ii) $\limsup_{t\to\infty} |a(t)c(t)| < \infty$.

(iii) $c(t) > 0$ *for sufficiently large values of* t.

(iv) $\lim_{x\to\infty} G(x) = +\infty$.

(v) *We assume the existence of a function* $\psi(t) \in C^1[t_0, \infty)$ *such that* $\lim_{t\to\infty}(\sqrt{a})f(t)\psi'(t)$ *exists, and is not zero.*

(vi) $\int^{\infty} |[a(t)c(t)]'\psi(t)|\,dt < \infty$.

(vii) $\lim_{t\to\infty} \psi(t)$ *exists.*

Then the existence of the limit $\lim_{t\to\infty} E(t)$ *implies that* $\lim_{t\to\infty} x'(t) = 0$.

Proof. (i), (ii), (iii) and (iv) imply that $x(t)$ and $x'(t)$ are bounded whenever $E(t)$ is bounded. Choosing $t_1, t_2 \in {}^*\boldsymbol{R}_{+\infty}$ we compute

$$\int_{t_1}^{t_2} E'(t)\psi(t)\,dt = \int_{t_1}^{t_2} \{[a(t)c(t)]'\psi(t)G(x(t)) + \sqrt{a}\,f(t)\psi(t)\varphi(x)x'(t)\}\,dt$$

$$= \Phi(x)\sqrt{a}f(t)\psi(t)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} [\sqrt{a}f(t)\psi(t)]'\Phi(x(t))\,dt + \int_{t_1}^{t_2} [a(t)c(t)]'\psi(t)G(x(t))\,dt.$$

Since $\Phi(x)$ is bounded and for any $t_1, t_2 \in {}^*\boldsymbol{R}_{+\infty}$ $(\Phi(x) = \int_0^x \varphi(s)ds)$,

$$\int_{t_1}^{t_2} \sqrt{a}(t)\psi(t)\,dt \approx 0,$$

it follows that

$$\int_{t_1}^{t_2} \sqrt{a}f(t)\psi(t)\Phi(x(t))\,dt \approx 0.$$

Condition (vi) implies

$$\int_{t_1}^{t_2} [a(t)c(t)]'\psi(t)\,dt \approx 0.$$

Hence

$$\int_{t_1}^{t_2} E(t)\psi'(t)\,dt \approx \Phi(x)\sqrt{a}f(t)\psi(t)\Big|_{t_1}^{t_2}.$$

Since $\lim_{t\to\infty} E(t)$ exists, and $\int_{t_1}^{t_2} \psi'(t)\,dt \approx 0$, it follows that

$$\Phi(x(t_2))\sqrt{a}(t_2)f(t_2)\psi(t_2) \approx \Phi(x(t_1))\sqrt{a}(t_1)f(t_1)\psi(t_1)$$

and because of (v)

$$\Phi(x(t_2)) \approx \Phi(x(t_1))$$

which implies

$$x(t_2) \approx x(t_1).$$

The equivalent standard statement is

$$\lim_{t\to\infty} x(t) \quad \text{exists.} \quad \text{QED.}$$

Consider as an example the equation

$$(x^\alpha x')' + \cos t x^\beta = t^2, \qquad \alpha+\beta > -1,$$

i.e.

$$a(t) \equiv 1, \quad c(t) = \cos t, \quad G(x) = \frac{1}{\alpha+\beta+1}\, x^{\alpha+\beta+1}.$$

We check that the function $\psi(t) = 1/t$ satisfies all conditions of this theorem.

As an example of this type of total evergy argument we shall offer a proof of assymptotic behavior of solutions of the non-linear equation

$$(*) \qquad x'' - h(t)(x')^m + ax^n = 0, \qquad \liminf_{t\to\infty} |h(t)| > 0,$$

where n, m are positive odd integers and $h(t)$ is positive for sufficiently large values of t, and a is a positive number. We intend to show that every bounded solution $\hat{x}(t)$ of this equation has the assymptotic behavior $\lim_{t\to\infty} \hat{x}'(t) = 0$, $\lim_{t\to\infty} |\hat{x}(t)| < \infty$. This will be done using total evergy arguments.

We first rewrite $(*)$ in the form

$$(**) \qquad x'' + ax^n = f(t),$$

where the "inhomogeneous term" $f(t)$ is in fact $f(t) = h(t)\big(\hat{x}'(t;\, t_0, x_0)\big)^m$. The total energy of the "modified" equation $(**)$ is:

$$E(t) = \frac{(x')^2}{2} + a\frac{x^{n+1}}{n+1}.$$

The rate of change of the total energy along a trajectory of a solution $\hat{x}(t)$ is

$$E'(t) = f(t)\hat{x}' = h(t)(\hat{x}')^{m+1}.$$

Hence $E'(t) \geqslant 0$. Since $E(t)$ is bounded, if $\hat{x}$ and $\hat{x}'$ are bounded $\lim_{t\to\infty} E(t)$ exists, and $\lim_{t\to\infty} E'(t) = 0$. However, this is possible only if $\lim_{t\to\infty} \hat{x}(t) < \infty$ and $\lim_{t\to\infty} \hat{x}'(t) = 0$.

Some additional examples of applications. Consider the equation of motion of a pendulum whose resorting force is decreasing with time

$$x'' + ct^{-a}\sin x = 0, \qquad a > 0.$$

(The restoring force comes from a magnet rather than gravity and the demagnetizing effect is of the form Ct^{-a}.)

By Lemma 1 we have $\lim x(t) = +\infty$, which can be easily predicted by purely physical arguments.

A more complex motion where the "apparent mass" depends on the position in the magnetic field would result in equation of motion of the form

$$\big(\varphi(x)x'\big)' + ct^{-a}\sin x = f(t).$$

Assume that $\varphi(x)$ is of the form

$$\varphi(x) = 1+\psi(x), \quad \psi(x) > 0,$$

and

$$f(t) \approx t^{-\beta}, \quad \beta > 0.$$

Then conditions of Theorem 1 are satisfied and $\hat{\pi}(\hat{x}(t)) = \pi(t)$ has the property $\lim_{t\to\infty}(\pi(t)) = +\infty$.

A non-linear equation of motion studied by Poincaré (*Ouvres,* Chapter 7) was represented in the usual polar coördinates by a simultaneous system of first order equations

$$p_\theta = a(\psi(\theta))^2\theta', \quad p_\theta' = c\varphi(\psi(\theta), \theta),$$

where a and c are constant. However, assuming that a, c are functions of time, and rewriting this system as a single equation we obtain a particular form of equation (2) of this paper:

$$[a(t)(\psi(\theta))^2\theta']' + c(t)g(\theta) = 0, \quad \text{where } g(\theta) = \Phi(\psi(\theta), \theta).$$

Consider as a special case the equation

$$[C_1(1+C_2e^{-t})\theta^2\theta']' + C_3\left(1+\frac{C_4}{t}\right)(\theta+C_5\sin\theta) = \frac{1}{t^\alpha},$$

$C_1, C_2, C_3, C_4, C_5 \geqslant 0$, $\alpha \geqslant 1$, $t > 1$. Then $\lim_{t\to\infty}|\theta(t)| = +\infty$ by Theorem 2.

We finally comment that the problems of existence and continuability of solutions for the initial value problem of equation (2) can be handled by existing standard arguments and will be discussed in a separate paper [4]. Estimates concerning the sup norm of solutions of (1) and (2) are given in [3] and [4].

4. Equations of the type (1) or (1^H). The boundedness and oscillation theorems for equation (2) can be clearly restated with much simpler hypothesis for the equation (1). However, some important results can be obtained for (1) which run into difficulties when an attempt is made to generalize them to equation (2). In the next theorem we shall consider equation (1^H), and prove a theorem which is curiously resembling a well-known result of Wong [10], except that the integrability condition appears to be applied to "the wrong function."

THEOREM 9. *Suppose that* $c(t)$, $f(t)$ *satisfy conditions* (i) *and* (ii) *of Lemma* 1 (*we do not assume* $c(t) > 0$), *and that* (a) $\lim_{t\to\infty}\frac{c(t)}{f(t)} = 0$. *Then for any bounded solution of* (1) $\lim_{t\to\infty} x'(t)$ *exists only if* $\int^{\infty} f(t)\,dt < \infty$, *and* $\lim_{t\to\infty} x(t)$ *exists only if* $\int^{\infty} tf(t)\,dt < \infty$.

Proof. We choose $\tau, \gamma \in {}^*\boldsymbol{R}_{+\infty}$ and compute:

$$x'(\gamma)-x'(\tau) = \int_\tau^\gamma f(t)\left[1-\frac{c}{f}\cdot g\big(x(t)\big)\right]dt.$$

Condition (a) implies $\dfrac{c(t)}{f(t)} \approx 0$ for all $t \in {}^*\boldsymbol{R}_{+\infty}$, while the assumption $\lim_{t\to\infty} x'(t) = 0$ is equivalent to the statement $x'(\gamma)-x'(\tau) \approx 0$ for all γ, $\tau \in {}^*\boldsymbol{R}_{+\infty}$. Hence

$$0 \approx \int_\tau^\gamma f(t)\left[1-\frac{c}{f}g\big(x(t)\big)\right]dt.$$

If $g\big(x(t)\big)$ is bounded in $\boldsymbol{R}$ this is easily shown to be equivalent to the condition $0 \approx \int_\tau^\gamma f(t)\,dt$, which is equivalent to

$$\lim_{b\to\infty}\int_a^b f(t)\,dt \quad \text{exists}.$$

An identical argument concerning the relationship

$$\int_\tau^\gamma (\tau-s)f(s)\left[1-\frac{c(s)}{f(s)}g\big(x(s)\big)\right]ds \approx 0,$$

shows that a necessary condition for $\lim_{t\to\infty} x(t)$ to exist is the convergence of the improper integral $\int_a^\infty tf(t)\,dt$. The more general cases of equations (2), (1) or (1^a) can be easily handled by similar techniques, provided assumptions are made to imply that roughly speaking for large values of t they behave like (1). Equations of the type

$$x''+h(t,x) = f(t)$$

can also be handled similarly, with additional assumptions.

We shall offer without proof the following oscillation theorems. The non-standard proofs are easy, and only use the fundamental theorem of calculus. The sequential version of these proofs is somewhat more difficult.

THEOREM 10. *If*

(a) $\displaystyle\lim_{\tau\to\infty}\left(\frac{\int^\tau f(t)\,dt}{\int^\tau c(t)\,dt}\right) = 0,$

(b) $\int^\tau c(t)\,dt = +\infty,\ c(t) > 0,$

(c) $xg(x) > 0$ *if* $x \neq 0,$

then the solutions of (1) *can not be bounded away from zero.*

Remark. The hypothesis of Theorem 10 also imply that either $x'(t)$ is oscillatory of $\lim_{t\to\infty} x(t) = 0$.

THEOREM 11. *Assume conditions* (b) *and* (c) *of Theorem* 10, *and suppose that there exists* $\tau \in [t_0, \infty)$ *such that* $\int_\tau^\infty f(t)\,dt = 0$. *Then all solutions of* (1) *are oscillatory.*

The technique of proof offered here is particularly well suited to boundedness and stability theorems. To demonstrate its application we shall prove the following novel boundedness theorem for equations of type (1).

THEOREM 12. *We consider equation* (1) *and assume that*

(a) $c(t) \in C[t_0, \infty)$, $c(t) > 0$ *for all sufficiently large values of* t.

(b) *There exist constants* $\varepsilon > 0$, $\delta > 0$, *such that for any* $\tilde{t} \in R$ *there can be found an open interval* (t_1, t_2), $t_1 > \tilde{t}$, *of length greater than* δ, *such that* $c(t) > \varepsilon$ *for all* $t \in (t_1, t_2)$ (*i.e. the lengths of intervals on which* $c(t)$ *is uniformly bounded away from zero do not converge to zero*).

(c) $\lim_{t\to\infty} f(t)/c(t) = +\infty$.

Then all solutions of (1) *are unbounded.* (*Observe that no conditions were given concerning the function* $g(x)$, *except that it is continuous.*)

Proof. Assume to the contrary that there exists a solution $\hat{x}(t)$ of (1^b) satisfying conditions (a), (b), (c), which is bounded. Hence we claim that $x''(t)$ must be of constant sign for all $t \in {}^*\boldsymbol{R}_{+\infty}$. Assuming that $x''(\tilde{t}) = 0$ for some $\tilde{t} \in {}^*\boldsymbol{R}_{+\infty}$, we see that $g(x(\tilde{t})) = f(\tilde{t})/c(\tilde{t})$, which is infinite because of condition (c). Since $x(\tilde{t})$ is near standard, and $g(x)$ is continuous function of x, we have obtained a contradiction, proving our claim. Since $c(t) > 0$ for sufficiently large values of t, we have for all $t \in {}^*\boldsymbol{R}_{+\infty}$,

$$\frac{x''}{c(t)} = -g(x(t)) + \frac{f(t)}{c(t)} \in {}^*\boldsymbol{R}_{+\infty}.$$

Condition (a) implies the existence of an interval (t_1, t_2) of length greater than some standard number $\delta > 0$, such that $c(t) \not\approx 0$ for all $\tau \in (t_1, t_2)$. Hence on the interval (t_1, t_2) $x''(t)$ is an infinite positive number. Since $x(t_1)$ was near standard, it follows easily that $x(t_2)$ is a positive infinite number in contradiction of the hypothesis. This completes the proof of the theorem. As an easy example of application we note that all solutions of the equation $x'' + (\log t)^\alpha g(x(t)) = t^\beta$, are unbounded, if α, $\beta \geqslant 0$ and if $g(x)$ is continuous.

5. Concluding remarks. Other applications of non-standard analysis will no doubt be forthcoming in almost any area of differential equations or control theory in the near future. The applications selected in this paper only reflected the personal taste of one of the authors. Theorem 11 is known in the "folklore" and can be proved using Liapunov's arguments. A more general theorem can perhaps be proved using the techniques of non-standard analysis, by examining in greater detail the particular cases of possible behavior of $c(t)$ and of $f(t)$ for infinite values of t. The authors would like to comment on the relative simplicity of the proofs of Theorems 1-5, and the insight the non-standard arguments give into such problems. Theorems 10-14 can be easily rewritten with $\varepsilon - \delta$ method of proof substituted for non-standard arguments. However, even here the non-standard analysis offered clues to the meaning of our hypothesis, and more or less fixed the hypothesis of these theorems. In Theorems 2 and 4 it would be difficult to translate the non-standard arguments into the $\varepsilon - \delta$ arguments without lengthening and complicating the arguments.

APPENDIX 1. So far in the discussion of oscillatory behavior of solutions of equations of type (1) we have stipulated that the inhomogenious term $f(t)$ is such that zeros of solutions are distinct. There are of course easy examples of pathological behavior of even linear equation

$$x'' + c(t)x = f(t), \tag{4}$$

if $f(t)$ is only assumed to be continuous. In fact it is easy to construct examples of pathological behavior of solutions of (4) even if $c(t)$ and $f(t)$ are of class $C^\infty[t_0, \infty)$. For example consider the function

$$\psi_n(t) = \begin{cases} 0 & \text{if } t \leqslant n\pi, \text{ or } t \geqslant (n+1)\pi, \\ \sin\left(\dfrac{t^2}{t-(n+1)\pi}\right) \cdot \exp\left(\dfrac{\pi/2}{\left[t - \dfrac{2n+1}{2}\pi\right]^2 - \dfrac{\pi^2}{4}}\right) & \\ & \text{if } n\pi < t < (n+1)\pi, \; n = 1, 2, \ldots, \end{cases}$$

and the function $x(t) = \sum_{n=1}^{\infty} \psi_n$; $x(t)$ is a bounded C^∞ function for all $t \geqslant 0$. The zeros of $x(t)$ have an accumulation point on every interval of length π, and the distance between consecutive zeros of $x(t)$ approaches zero for large values of t. Differentiating $x(t)$ twice and computing $f(t) = x(t) + x''(t)$, we see that $x(t)$ solves this differential equation (1) (with $c(t) \equiv 1$, and $f(t) \in C^\infty[0, \infty)$).

Below we offer sufficient conditions insuring that the zeros of solutions of equation of type (1) do not have accumulation points. We consider the equation

$$x'' + c(t)g(x, x') = f(t), \quad g(0, 0) = 0. \tag{1^c}$$

Suppose $f(t)$, $c(t)$ belong to the class $C^k[t_0, \infty)$, and $g(\xi, \eta)$ is k-times continuously differentiable function of the variables ξ, η.

Let us assume that $f(t)$ and all derivatives of $f(t)$ of order $1, 2, \ldots, k$ do not vanish simultaneously at some point $\bar{t} > t_0$. Then $\bar{t}$ can not be the accumulation point of the zeros of any solution $\hat{x}(t)$ of equation (1^c).

Proof. It is clear from equation (1^c) that any solution $\hat{x}(t)$ is of the class $C^{k+2}(t_0, \infty)$. If $\bar{t} \in (t_0, \infty)$ is an accumulation point of zeros of $x(t)$, then $\bar{t}$ is also an accumulation point of zeros of $x'(t), x''(t), \ldots, x^{(k+2)}(t)$. Hence by continuity $\bar{x}^{(j)}(t) = 0$ for every $j \leqslant k+2$. However, differentiating all terms of equation (1), we have

$$x''(t) + c'(t)g(x, x') + c(t)x'\frac{g(x, x')}{x} + c(t)x''\frac{g(x, x')}{x} = f'(t),$$

$$\vdots$$

$$x^{(k+2)}(t) + \mathfrak{L}(c, g, x', x'', \ldots, x^{(k)}) = f^{(k)}(t),$$

where $\mathfrak{L}(c, g, x', x'', \ldots, x^{(k)})$ is a linear combination of $g(x, x')$, and terms containing the derivatives of $x(t)$ of orders $1, 2, \ldots, k$. Substituting $t = \bar{t}$, we see that the left-hand sides of this system of equations are all equal to zero, and therefore $f(\bar{t}) = f'(\bar{t}) = \ldots = f^{(k)}(\bar{t}) = 0$. in contradiction of our hypothesis. This proves that the zeros of $\hat{x}(t)$ cannot have an accumulation point at $\hat{t}$. We also make the following observation: If both $x(t)$ and $x'(t)$ vanish simultaneously at some point $\hat{t} \in [t_0, \infty)$, then $f(\hat{t}) = 0$, $\lim\limits_{t\to\hat{t}} \frac{x''(t)}{f(t)} = 0$. To prove it, let us assume that j is the smallest natural number such that $f^{(j)}(\hat{t}) \neq 0$. We observe that $f(\hat{t}) = 0$ implies $x''(\hat{t}) = 0$. Differentiating again both sides of equation (1^c) j-times we obtain $x^{(i)}(\hat{t}) = 0$, for all $i \leqslant j+2$ consequently $\lim\limits_{t\to\hat{t}} \frac{x^{(j+2)}(t)}{f^{(j)}(t)} = 0$. Hence $\lim\limits_{t\to\hat{t}} \frac{x''(t)}{f(t)} = 0$, as can be checked by a repeated application of l'Hôspital's rule.

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