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Trigonometric interpolation (0, 2, 3) case

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Introduction. The object of this paper is to discuss the simple case of (0, 2, 3) interpolation by trigonometric polynomials at the points $x_k = \frac{2k\pi}{n}$, $k = 0, 1, 2, \ldots, n-1$. By (0, 2, 3) interpolation we mean the problem of finding a trigonometric polynomial whose values, second and third derivatives are prescribed at some given points. In view of our earlier work [5] on $(0, M)$ interpolation it seems plausible to consider the $(0, M, N)$ case, $M < N$; however, a justification for considering the very special case (0, 2, 3) lies in its simplicity and in the fact that the method used in this paper can be applied to the general problem of $(0, M, N)$ case and that it brings out most of the salient feature of the general $(0, M, N)$ case.

An obvious difference between the (0, 2) case studied by Kiš [4] and (0, 2, 3) case studied here is that in our case we consider trigonometric polynomials of higher order. Another special difference is that in the (0, 2, 3) case interpolatory polynomials exist and are unique for both n even or n odd, provided the polynomials are of the form given by (2.2) and (2.3) according as n is even or odd, whereas in (0, 2) case existence and uniqueness hold only for n odd. A still more interesting distinction between these two cases is that the sequence of interpolatory polynomials in our case converges uniformly to the given function only if $f(x)$ is periodic, continuous and $f(x) \in \mathrm{Lip}\,\alpha$, $\alpha > 0$, whereas Kiš [4] requires the Zygmund condition in the (0, 2) case.

For earlier reference to (0, 2) interpolation, we refer to the works of Balázs and Turán [1] and a survey paper by Balázs [2].

2. Statement of the main theorems. We are interested in the trigonometric polynomial $R_n(x)$ of suitable order such that

$$(2.1)\qquad R_n(x_k) = a_k\,,\quad R_n''(x_k) = b_k\,,\quad R_n'''(x_k) = c_k\,,$$

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where $x_k = \frac{2\pi k}{n}$, $k = 0, 1, 2, \ldots, n-1$. We shall call this case $(0, 2, 3)$ *interpolation.* When n is even $(= 2m)$ we require the trigonometric polynomial $R_n(x)$ to have the form

$$d_0 + \sum_{k=1}^{3m-1} (d_k \cos kx + e_k \sin kx) + d_{3m} \cos 3mx \tag{2.2}$$

but when n is odd $(= 2m+1)$ we require it to have the form

$$d_0 + \sum_{k=1}^{3m+1} (d_k \cos kx + e_k \sin kx)\,. \tag{2.3}$$

We shall prove the following

THEOREM 2.1. *Let n be even $(= 2m)$. The trigonometric polynomial $R_n(x)$ satisfying* (2.1) *having form* (2.2) *is given by*

$$R_n(x) = \sum_{k=0}^{n-1} a_k U(x-x_k) + \sum_{k=0}^{n-1} b_k V(x-x_k) + \sum_{k=0}^{n-1} c_k W(x-x_k) \tag{2.4}$$

where

$$W(x) = -\frac{1}{n^3}\left[\sum_{j=1}^{m} \frac{4j \sin jx}{n^2-3j^2} + \sum_{j=m+1}^{3m-1} \frac{(3n-2j)\sin jx}{n^2-3(n-j)^2}\right], \tag{2.5}$$

$$V(x) = \frac{1}{n^3}\left[1 + 2\sum_{j=1}^{m-1} \frac{n^2+3j^2}{n^2-3j^2}\cos jx - \right.$$
$$\left. -\frac{1}{4}\sum_{j=m+1}^{3m-1} \frac{n^2+3(3n-2j)^2}{n^2-3(n-j)^2}\cos jx + \frac{1}{2}(\cos mx - \cos 3mx)\right], \tag{2.6}$$

$$U(x) = \frac{1}{n}\left[1 + \frac{2}{n^2}\sum_{j=1}^{m-1} \frac{(n^2-j^2)^2}{(n^2-3j^2)}\cos jx - \right.$$
$$\left. -\frac{1}{n^2}\sum_{j=m+1}^{3m-1} \frac{(n-j)^2(2n-j)^2}{(n^2-3(n-j)^2)}\cos jx + \frac{1}{8}(9\cos mx - \cos 3mx)\right]. \tag{2.7}$$

For n odd $(= 2m+1)$ $R_n(x)$ satisfying (2.1) *and of form* (2.3) *is given by* (2.4) *where*

$$W(x) = -\frac{1}{n^3}\left[\sum_{j=1}^{m} \frac{4j}{n^2-3j^2}\sin jx + \sum_{j=m+1}^{3m+1} \frac{(3n-2j)\sin jx}{n^2-3(n-j)^2}\right], \tag{2.8}$$

$$(2.9)\qquad V(x) = \frac{1}{n^3}\left[1 + 2\sum_{j=1}^{m} \frac{n^2+3j^2}{n^2-3j^2}\cos jx - \right.$$
$$\left. -\frac{1}{4}\sum_{j=m+1}^{3m+1} \frac{n^2+3(3n-2j)^2}{n^2-3(n-j)^2}\cos jx\right],$$

$$(2.10)\qquad U(x) = \frac{1}{n}\left[1 + \frac{2}{n^2}\sum_{j=1}^{m} \frac{(n^2-j^2)\cos jx}{n^2-3j^2} - \right.$$
$$\left. -\frac{1}{n^2}\sum_{j=m+1}^{3m+1} \frac{(n-j)^2(2n-j)^2\cos jx}{n^2-3(n-j)^2}\right].$$

Let us consider

$$(2.11)\qquad R_n(x) = \sum_{k=0}^{n-1} f(x_k)\,U(x-x_k) + \sum_{k=0}^{n-1} b_k V(x-x_k) + \sum_{k=1}^{n-1} c_k W(x-x_k),$$

where $f(x)$ is a given 2π-periodic continuous function and b_k, c_k are arbitrary numbers. We prove the following

THEOREM 2.2. *Let $f(x)$ be 2π-periodic continuous function with $f(x) \in \operatorname{Lip}\alpha$, $\alpha > 0$ and if*

$$(2.12)\qquad |b_k| = o\left(\frac{n^2}{\log n}\right), \quad |c_k| = o\left(\frac{n^3}{\log n}\right), \quad k = 0, 1, 2, \ldots, n-1,$$

then $R_n(x)$ as given by (2.11) *converges uniformly to $f(x)$ on every closed finite interval on the x-axis.*

3. Proof of Theorem 2.1. To avoid repetition we shall give in full the method for obtaining $W(x)$; the proof for $V(x)$ and $U(x)$ are similar and so we omit the details. Further we shall prove the theorem for n even. The proof when n is odd is similar.

Since $W(x)$ satisfies the conditions

$$(3.1)\qquad W(x_k) = W''(x_k) = 0, \quad W'''(x_k) = \begin{cases} 1 & \text{for} \quad k = 0, \\ 0 & \text{for} \quad 1 \leqslant |k| \leqslant n-1 \end{cases}$$

we set $W(x) = \sin mx\, g(x)$, $n = 2m$, where $g(x)$ is a trigonometric polynomial of order $\leqslant 2m$. Then $W''(x_k) = 0$ gives $g'(x_k) = 0$, $k = 0, 1, 2, \ldots$ $\ldots, n-1$. Whence

$$(3.2)\qquad g'(x) = \sin mx\, r(x),$$

where $r(x)$ is a trigonometric polynomial of order $\leqslant m$. Also the last condition given in (3.1) gives

$$(3.3)\qquad (-1)^k m[3g''(x_k) - m^2 g(x_k)] = \begin{cases} 1 & \text{for} \quad k = 0, \\ 0 & \text{for} \quad 1 \leqslant |k| \leqslant n-1. \end{cases}$$

Hence

$$(3.4) \qquad m[3g''(x)-m^2g(x)] = \frac{1}{2m}\sin mx\cot\frac{x}{2}+\sin mx\, h(x)\,,$$

where $h(x)$ is a trigonometric polynomial of order $\leqslant m$. Applying the operator D to (3.4) and $m(3D^2-m^2)$ to (3.2) (here D denotes differentiation with respect to x), we have

$$(3.5) \qquad \frac{1}{2m}D\left(\sin\frac{nx}{2}\cot\frac{x}{2}\right)+\sin\frac{nx}{2}h'(x)+\frac{n}{2}\cos\frac{nx}{2}h(x)$$
$$= \frac{n}{2}\left[-n^2\sin\frac{nx}{2}r(x)+3\left\{n\cos\frac{nx}{2}r'(x)+\sin\frac{nx}{2}r''(x)\right\}\right].$$

Since

$$(3.6) \qquad \sin\frac{nx}{2}\cot\frac{x}{2} = 1+2\sum_{j=1}^{m-1}\cos jx+\cos mx\,, \qquad n = 2m$$

we have from (3.4) on putting $x = x_k$

$$3nr'(x_k)-h(x_k) = -\frac{4}{n^2}\cos mx_k\sum_{j=1}^{m-1}j\sin jx_k = \frac{4}{n^2}\sum_{j=1}^{m-1}(m-j)\sin jx_k$$

for $k = 0, 1, 2, \ldots, n-1$.

Hence

$$(3.7) \qquad 3nr'(x)-h(x) = \frac{4}{n^2}\sum_{j=1}^{m-1}(m-j)\sin jx+a\sin mx\,,$$

where a is arbitrary. Eliminating $h(x)$ from (3.5) and (3.7) we have

$$3nr''(x)+n^3r(x) = 1+\frac{2}{m^2}\sum_{j=1}^{m-1}(m^2-j^2)\cos jx+2ma\cos mx\,,$$

so that

$$(3.8) \qquad r(x) = \frac{1}{m}\left[\frac{1}{2n^2}+\frac{1}{m^2}\sum_{j=1}^{m-1}\frac{m^2-j^2}{n^2-3j^2}\cos jx\right]+\frac{2a\cos mx}{m^2}.$$

Using (3.2) we now get from (3.8)

$$g'(x) = \sin mx\, r(x)$$
$$= \frac{1}{m}\left[\frac{\sin mx}{2n^2}+\frac{1}{2m^2}\sum_{j=1}^{m-1}\frac{m^2-j^2}{n^2-3j^2}\{\sin(m+j)x+\sin(m-j)x\}\right]+\frac{2a}{m^2}\sin 2mx$$

which gives

$$g(x) = C - \frac{1}{m}\left[\frac{\cos mx}{8m^3} + \frac{1}{2m^2}\sum_{j=1}^{m-1}\frac{m^2-j^2}{n^2-3j^2}\left\{\frac{\cos(m+j)x}{m+j} + \frac{\cos(m-j)x}{m-j}\right\}\right] - \frac{a}{2m^3}\cos 2mx ,$$

whence $W(x) = \sin mx\, g(x)$. Since $W(x-x_k)$ contains no term containing $\sin 3mx$, we have $a = 0$. The constant $C = -1/2m^4$ follows from (3.3); on rearranging the terms we get (2.5).

4. In this section we shall determine the estimates of the fundamental polynomials.

LEMMA 4.1. *The following estimate is valid*

$$\sum_{k=0}^{n-1} |W(x-x_k)| \leqslant C_1 n^{-3}\log n , \tag{4.1}$$

where C_1 is a numerical constant.

In order to prove (4.1) we need the following known inequality (Jackson [3], page 120):

$$\sum_{k=0}^{n-1} \max_p \left|\sum_{j=0}^{p} \sin j(x-x_k)\right| \leqslant 4n\log n . \tag{4.2}$$

Since $0 \leqslant \frac{4j}{n^2-3j^2} \leqslant \frac{4}{m}$ for $0 \leqslant j \leqslant m$ and is increasing function of j for $1 \leqslant j \leqslant m$, we have by Abel's inequality

$$\left|\sum_{j=1}^{m} \frac{4j}{n^2-3j^2}\sin j(x-x_k)\right| \leqslant \frac{4}{m}\max_{1\leqslant p\leqslant m}\left|\sum_{j=1}^{p}\sin j(x-x_k)\right| .$$

Similarly, since $0 < \frac{3n-2j}{n^2-3(n-j)^2} \leqslant \frac{4}{m}$ for $m+1 \leqslant j \leqslant 3m-1$ and $\frac{3n-2j}{n^2-3(n-j)^2}$ is a decreasing function of j for $m+1 \leqslant j \leqslant 3m-1$ we can again use Abel's inequality. Then using (4.2) we get (4.1).

LEMMA 4.2. *For the estimates of $V(x-x_k)$ we have the following inequality*

$$\sum_{k=0}^{n-1} |V(x-x_k)| \leqslant C_2 n^{-2}\log n , \tag{4.3}$$

where C_2 is a numerical constant.

Proof. As above we take n to be even. We can rewrite (2.6) in the form

(4.4) $$V(x)=\frac{\sin mx}{4m^3}\left[\sum_{j=1}^{2m-1}a_j\sin jx+\tfrac{1}{2}\sin 2mx\right],$$

where

(4.5) $$a_j=\frac{\bullet m^2+3(n-j)^2}{n^2-3(m-j)^2}.$$

Since

$$\sum_{j=1}^{2m-1}a_j\sin j(x-x_k)=\sum_{j=1}^{m}+\sum_{j=m+1}^{2m-1}=S_1+S_2$$

we observe that a_j is decreasing for $1\leqslant j\leqslant m$ and so we have by Abel's inequality

$$|S_1|\leqslant 3\max_{1\leqslant p\leqslant m}\left|\sum_{j=1}^{p}\sin j(x-x_k)\right|.$$

It is easy to see that

$$a_j=\frac{4m^2+3(m-j)^2}{4m^2-3(j-m)^2}-\frac{6m(j-m)}{4m^2-3(j-m)^2}.$$

We have

$$|S_2|\leqslant\left|\sum_{j=m+1}^{2m-1}\frac{4m^2+3(m-j)^2}{4m^2-3(j-m)^2}\sin j(x-x_k)\right|+$$
$$+\left|\sum_{j=m+1}^{2m-1}\frac{6m(j-m)}{4m^2-3(j-m)^2}\sin j(x-x_k)\right|.$$

Now we use again Abel's inequality on each of the above series on the right which is possible, as the coefficients in the above sums are increasing function of j for $m+1\leqslant j\leqslant 2m-1$. Thus

$$|S_2|\leqslant 3\max_{m+1\leqslant p\leqslant 2m-1}\left|\sum_{j=m+1}^{p}\sin j(x-x_k)\right|.$$

Now combining the estimates for S_1 and S_2 and using (4.2) we have Lemma 4.2.

LEMMA 4.3. *For the fundamental polynomial $U(x-x_k)$ the following estimate is valid*

(4.6) $$\sum_{k=0}^{n-1}|U(x-x_k)|\leqslant C_3\log n,$$

where C_3 is a positive numerical constant.

Proof. Another representation of $U(x)$ for n even $(=2m)$ is given by

$$U(x)=\frac{\sin mx}{n}\left[\cot\frac{x}{2}+\frac{2}{n^2}\sum_{j=1}^{2m-1}\beta_j\sin jx+\frac{\sin 2mx}{4}\right], \tag{4.7}$$

where

$$\beta_j=\frac{(m-j)^2(3m-j)^2}{4m^2-3(m-j)^2}. \tag{4.8}$$

Since

$$\sum_{j=1}^{2m-1}\beta_j\sin j(x-x_k)=\sum_{j=1}^{m}+\sum_{j=m+1}^{2m-1},$$

from (4.8) it is easy to see that β_j, as a function of j is decreasing for $1\leqslant j\leqslant m$ while for $m+1\leqslant j\leqslant 2m-1$, β_j increases as j increases. Therefore using Abel's inequality in each part and using (4.2) we get the required result.

5. In order to prove our main theorem we require a lemma on approximating polynomials.

LEMMA 5.1. *If $f(x)$ is a continuous 2π-periodic function and satisfying $f(x)\in \operatorname{Lip}\alpha$, $0<\alpha\leqslant 1$, then there exists a trigonometric polyniomal $T_n(x)$ of order $\leqslant n$ such that*

$$|f(x)-T_n(x)|=O(n^{-\alpha}), \tag{5.1}$$

$$T_n^{(p)}(x)=O(n^{p-\alpha}),\qquad p=2,3,\ldots \tag{5.2}$$

The formula (5.1) is due to Jackson and is well-known. The proof of (5.2) is exactly similar to a corresponding lemma of Kiš ([4], (p. 270-271).

6. Proof of Theorem 2.2. By Lemma 5.1 there exists a trigonometric polynomial $T_n(x)$ of order n which satisfies (5.1) and (5.2). By the uniqueness theorem we have

$$\begin{aligned}T_n(x)-R_n(x)&=\sum_{k=0}^{n-1}\big(T_n(x_k)-f(x_k)\big)\,U(x-x_k)+\sum_{k=0}^{n-1}T_n''(x_k)V(x-x_k)-\\&\quad-\sum_{k=0}^{n-1}b_kV(x-x_k)+\sum_{k=0}^{n-1}T_n'''(x_k)W(x-x_k)-\sum_{k=0}^{n-1}c_kW(x-x_k)\\&=\sum_{r=1}^{5}I_r\,.\end{aligned}$$

By (5.1) and (4.6) we have

$$I_1=C_3\log n\,o(n^{-\alpha})=o(1)\qquad\text{as}\qquad 0<\alpha\leqslant +1\,.$$

On using (5.2), (4.3) and (2.12) we have

$$I_2 = C_2 n^{-2} \log n\, o(n^{2-\alpha}) = o(1)\,, \qquad 0 < \alpha \leq +1\,,$$

and

$$I_3 = o\left(\frac{n^2}{\log n}\right) C_2 n^{-2} \log n = o(1)\,.$$

Lastly using (2.12), (4.1) and (5.2) we get

$$I_4 = C_1 n^{-3} \log n\, o(n^{3-\alpha}) = C_4 \frac{\log n}{n^\alpha} = o(1)\,, \qquad \alpha > 0\,,$$

and

$$I_5 = C_1 n^{-3} \log n\, o\left(\frac{n^3}{\log n}\right) = o(1)\,.$$

Therefore $|T_n(x) - R_n(x)| = o(1)$. Now using (5.1) again we have

$$|f(x) - R_n(x)| = |f(x) - T_n(x) + T_n(x) - R_n(x)| = o(1)\,.$$

This completes the proof of our main theorem.

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