ANNALES
POLONICI MATHEMATICI
XXI (1969)

Regular solutions of a linear functional equation in the indeterminate case

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§ 1. Introduction. In the present note we continue the investigation of the regular solutions $\varphi(x)$ of the linear functional equation

$$\varphi[f(x)] = g(x)\varphi(x) + h(x) \tag{1}$$

started in our previous paper [1]. The functions occurring in equation (1) are real-valued functions of a real variable, defined and continuous in an interval $I = \langle 0, a)$, $0 < a \leqslant +\infty$; the function $f(x)$ is strictly increasing, $0 < f(x) < x$ in $(0, a)$, $f(0) = 0$.

The phrase "regular solution" will be used throughout this paper in the following sense (cf. [1]):

DEFINITION 1. A function $\varphi(x)$ will be called a *regular solution* of equation (1) in the interval I if it satisfies equation (1) in I, is continuous in I and has the right-sided derivative at the point $x = 0$.

A study of regular solutions makes thus a starting point for the investigation of C^1 solutions of equation (1) in I.

We shall also explain the exact meaning of the notion of a "solution depending on an arbitrary function".

DEFINITION 2. We say that equation (1) has in I a *regular* (resp. *continuous*) *solution depending on an arbitrary function* if there exists an interval $J \subset I$ such that every continuous function on J can be extended to a regular (resp. continuous) solution of equation (1) in I.

In the present paper we deal with the case where there exists a continuous solution of the equation

$$\varphi[f(x)] = g(x)\varphi(x) \tag{2}$$

depending on an arbitrary function (Theorem 1), and we prove a theorem on the existence of a one-parameter family of regular solutions of equation (2) in this case (Theorem 2). Theorem 3 shows that a small change of the assumptions of Theorem 2 implies that equation (2) has a regular solution depending on an arbitrary function or only the trivial solution

$\varphi(x) \equiv 0$. In Theorem 4 we investigate the non-homogeneous equation (1) and we obtain results like those of Theorems 1 and 2. These theorems complete the results of our previous paper [1], where we have found some conditions for every continuous solution of equation (1) or (2) in I to be regular.

Equation (1) or (2) may have a continuous solution depending on a parameter only in the indeterminate case where $g(0) = 1$; similarly, a regular solution depending on a parameter may occur only in the indeterminate case $g(0) = f'(0)$. In the present paper we shall assume that

$$g(0) = f'(0) = 1;$$

thus we have the indeterminate case for continuous as well as for regular solutions. We shall make use of the results of paper [2] by Kuczma and the present author, which gives a theory of continuous solutions of equations (1) and (2) in the indeterminate case.

§ 2. Preliminaries. We shall make the following assumptions:

(i) *The function $f(x)$ is continuous and strictly increasing in an interval $I = \langle 0, a)$, $0 < f(x) < x$ in $(0, a)$, $f(0) = 0$.*

(ii) *The function $g(x)$ is continuous and positive in I, $g(0) = 1$.*

(iii) *The function $h(x)$ is continuous in I, $h(0) = 0$* [1].

(iv) *The function $f(x)$ has at the point $x = 0$ the right-sided derivative $f'(0)$ and $f'(0) = 1$.*

(v) *The function $h(x)$ has at the point $x = 0$ the right-sided derivative $h'(0)$.*

Let $f^n(x)$ be the sequence of iterates of the function $f(x)$:

$$f^0(x) = x\,, \quad f^{n+1}(x) = f[f^n(x)]\,, \quad n = 0, 1, 2, \ldots, \ x \in I\,.$$

It is not difficult to prove the following

LEMMA 1. *If the function $f(x)$ fulfils hypothesis* (i), *then for every $x \in I$, $x \neq 0$ the sequence $f^n(x)$ is strictly decreasing and $\lim\limits_{n\to\infty} f^n(x) = 0$.*

Now we write

$$G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)]\,. \tag{3}$$

The family of continuous solutions of equation (2) in I depends on the behaviour of sequence (3). Three cases are possible (cf. [2]):

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[1] Setting $x = 0$ in (1) we obtain in view of (i) and (ii): $\varphi(0) = \varphi(0) + h(0)$; it follows that in the case where $h(0) \neq 0$ equation (1) cannot have any solution in the whole interval I.

(A) The limit $G(x) = \lim_{n\to\infty} G_n(x)$ exists and $G(x)$ is continuous in I, $G(x) \neq 0$ in I.

(B) There exists an interval $J \subset I$ such that $\lim_{n\to\infty} G_n(x) = 0$ uniformly in J.

(C) Neither (A) nor (B) occurs.

LEMMA 2 (cf. [2], [3]). *Let hypotheses* (i) *and* (ii) *be fulfilled. In case* (A) *the functions* $\varphi_c(x) = c/G(x)$, *where* c *is an arbitrary real number, are the only continuous solutions of equation* (2) *in* I. *In case* (B) *equation* (2) *has in* I *a continuous solution depending on an arbitrary function, and all continuous solutions fulfil the condition* $\varphi(0) = 0$. *In case* (C) *the function* $\varphi(x) \equiv 0$ *is the only continuous solution of equation* (2) *in* I.

We also introduce the auxiliary equation

$$\psi[f(x)] = \frac{xg(x)}{f(x)}\psi(x) + \frac{h(x)}{f(x)}, \tag{4}$$

where for $x = 0$ the function $x \cdot g(x)/f(x)$ is defined as 1, and the function $h(x)/f(x)$ is defined as $h'(0)$. In virtue of hypotheses (i)-(v) these functions are continuous in the whole of I. The following evident lemma allows us to replace the investigation of the regular solutions of equation (1) by the investigation of continuous solutions of equation (4).

LEMMA 3 (cf. [1]). *Let hypotheses* (i)-(v) *be fulfilled. If* $\psi(x)$ *is a continuous solution of equation* (4) *in* I, *then the function* $\varphi(x) = x\psi(x)$ *is a regular solution of equation* (1) *taking on the value* 0 *for* $x = 0$. *Conversely, if the function* $\varphi(x)$ *is a regular solution of equation* (1) *and* $\varphi(0) = 0$, *then the function*

$$\psi(x) \overset{\text{def}}{=} \begin{cases} \varphi(x)/x & \text{for} \quad x \in I,\ x \neq 0, \\ \varphi'(0) & \text{for} \quad x = 0 \end{cases}$$

is a continuous solution of equation (4) *in* I.

Since equation (4) is of form (1), we can also calculate sequence (3) for it. We denote the sequence obtained by $\Gamma_n(x)$, i.e. we put

$$\Gamma_n(x) = \prod_{i=0}^{n-1} \frac{f^i(x)g[f^i(x)]}{f^{i+1}(x)} \quad (\text{for } x \neq 0), \quad \Gamma_n(0) = 1. \tag{5}$$

Sequences (3) and (5) are linked by the formula

$$\Gamma_n(x) = xG_n(x)/f^n(x), \quad \text{for} \quad x \in I \tag{6}$$

(for $x = 0$ the right-hand side of (6) is defined as the limit if $x \to 0+0$ and is equal to 1).

We shall also make use of estimates due to Thron [4] and concerning the behaviour of the sequence $f^n(x)$.

LEMMA 4 (cf. [4], Theorem 3.1). *Let the function $f(x)$ fulfil hypothesis* (i) *and suppose that there exist a number $k>0$ (not necessarily an integer!) and a function $p(x)$ continuous and positive in the interval I and such that*

$$f(x) = x - p(x)x^{k+1} \quad \text{for} \quad x \in I . \tag{7}$$

Then for every $x_0 \in I$, $x_0 \neq 0$ there exist positive constants K and M such that for an arbitrary $x \in (0, x_0)$ one can find an index $N(x)$ such that

$$(Kn)^{-1/k} < f^n(x) < (Mn)^{-1/k} \quad \text{for} \quad n > N(x) . \tag{8}$$

§ 3. Results. The results of the present paper are contained in the following four theorems:

THEOREM 1. *Let hypotheses* (i) *and* (ii) *be fulfilled and suppose that there exist functions $R_1(x)$ and $R_2(x)$ and positive constants a_1, a_2, k, μ, ν, such that for $x \in I$ we have*

$$f(x) = x - a_1 x^{k+1} + R_1(x) , \tag{9}$$

$$g(x) = 1 - a_2 x^k + R_2(x) , \tag{10}$$

and, moreover,

$$R_1(x) = O(x^{k+1+\mu}) , \quad R_2(x) = O(x^{k+\nu}) , \quad x \to 0+0 . \tag{11}$$

Then equation (2) *has in I a continuous solution depending on an arbitrary function and every continuous solution $f(x)$ of equation* (2) *fulfils the condition*

$$\varphi(0) = 0 . \tag{12}$$

THEOREM 2. *Suppose that the functions $f(x)$ and $g(x)$ fulfil the hypotheses of Theorem* 1 *and, besides, that hypothesis* (iv) *is fulfilled and that we have*

$$a_2 = a_1 . \tag{13}$$

Then 1° *sequence* (5) *uniformly converges in I to a positive limit, and* 2° *equation* (2) *has in I a one-parameter family of regular solutions given by the formula*

$$\varphi_c(x) = c \cdot \lim_{n\to\infty} f^n(x)/G_n(x) , \tag{14}$$

where c is an arbitrary real number.

THEOREM 3. *Suppose that the functions $f(x)$ and $g(x)$ fulfil the hypotheses of Theorem* 1 *and that hypothesis* (iv) *is fulfilled and that we have*

$$a_2 \neq a_1 . \tag{15}$$

If $a_2 > a_1$, then equation (2) *has in I a regular solution depending on an arbitrary function; if $a_2 < a_1$, then the function $\varphi(x) \equiv 0$ is the only regular solution of equation* (2) *in I.*

THEOREM 4. *Suppose that the functions $f(x)$ and $g(x)$ fulfil the hypotheses of Theorem 2 and the function $h(x)$ fulfils hypotheses* (iii) *and* (v). *Suppose further that there exists a positive number λ such that*

$$h(x) = O(x^{k+1+\lambda}), \quad x \to 0+0. \tag{16}$$

Then equation (1) *has in I a one-parameter family of regular solutions given by the formula*

$$\varphi(x) = \varphi^*(x) + \varphi_c(x), \tag{17}$$

where $\varphi_c(x)$ are given by formula (14) *and*

$$\varphi^*(x) \stackrel{\text{def}}{=} -\sum_{n=0}^{\infty} h[f^n(x)]/G_{n+1}(x). \tag{18}$$

On the other hand, the continuous solution of equation (1) *depends on an arbitrary function.*

Remark. The number k postulated above need not be an integer.

§ 4. Proofs. Now we are going to supply proofs of Theorems 1-4. We shall use the following notation

$$I_0 = \langle 0, x_0 \rangle, \quad \text{where} \quad x_0 \in I, \ x_0 \neq 0.$$

Proof of Theorem 1. According to Lemma 2 it is enough to show that for the sequence $G_n(x)$ (cf. (3)) case (B) occurs.

It follows from (i), (9) and (11) that we can choose an $x_0 \in I$, $x_0 \neq 0$, and an $r > 0$ such that for $x \in I_0$, $x \neq 0$, we have

$$(a_1 - r)x^{k+1} < R_1(x) < a_1 x^{k+1}. \tag{19}$$

Assume that the x_0 has been chosen in such a manner that for $x \in I_0$, $x \neq 0$, we have

$$g(x) < 1 - M_0 x^k, \tag{20}$$

where M_0 is a positive constant (cf. (10) and (11)). We consider sequence (3) in the interval $J \stackrel{\text{def}}{=} \langle f(x_0), x_0 \rangle$. In virtue of (20) we have for every $x \in J$

$$g[f^n(x)] < 1 - M_0[f^n(x)]^k < 1 - M_0[f^{n+1}(x_0)]^k, \tag{21}$$

since $f^n(x) > f^{n+1}(x_0)$ for $x \in J$ (cf. Lemma 1). The function $f(x)$ fulfils the hypotheses of Lemma 4 (we put $p(x) = a_1 - R_1(x)x^{-k-1}$, for $x \neq 0$; and $p(0) = a_1$); consequently we have for $n > N = N(x_0)$

$$f^n(x_0) > (Kn)^{-1/k}. \tag{22}$$

We get by (21) and (22) for $n > N$ and $x \in J$

$$0 < G_n(x) < G_{N+1}(x) \prod_{i=N+1}^{n-1} \big(1 - M_0/K(i+1)\big) . \tag{23}$$

The function $G_{N+1}(x)$ is continuous, and thus bounded in J, and the other factor in (23) tends to zero, since the series $\sum M_0/K(n+1)$ diverges. Consequently $\lim_{n\to\infty} G_n(x) = 0$ uniformly in J, i.e. for sequence (3) case (B) occurs, which was to be proved.

Proof of Theorem 2. In virtue of Theorem 1 (relation (12)) and of Lemma 3 it is enough to determine all the continuous solutions of the equation

$$\psi[f(x)] = \frac{xg(x)}{f(x)} \psi(x) \tag{24}$$

in I. For this purpose we are going to investigate the behaviour of sequence (5). Let us write

$$c_n(x) \overset{\text{def}}{=} \frac{f^n(x) g[f^n(x)] - f^{n+1}(x)}{f^{n+1}(x)} \quad \text{for} \quad x \neq 0; \ c_n(0) = 0 \tag{25}$$

so that

$$\Gamma_n(x) = \prod_{i=0}^{n-1} \big(1 + c_i(x)\big) . \tag{26}$$

We shall prove that the series

$$\sum_{n=0}^{\infty} |c_n(x)| \tag{27}$$

is uniformly convergent in every interval $I_0 \subset I$. In the sequel x_0 is regarded as fixed.

We have by (9), (10), (11) and (13)

$$xg(x) - f(x) = xR_2(x) - R_1(x) = O(x^{k+1+\varkappa}) , \quad x \to 0+0 ,$$

where $\varkappa = \min(\mu, \nu) > 0$. Thus we have for $x \in I_0$, $x \neq 0$

$$|c_n(x)| \leqslant \frac{M_1[f^n(x)]^{k+1+\varkappa}}{f^{n+1}(x)} \tag{28}$$

with a certain constant $M_1 > 0$. It follows from Lemma 1 that for every positive $x \in I_0$ the value $f^n(x)$ can be made arbitrarily small, provided n is sufficiently large. Thus inequality (19) is fulfilled at the point $f^n(x)$ and, by (9), we infer that the inequalities

$$f^{n+1}(x) = f[f^n(x)] > f^n(x)\{1 - r[f^n(x)]^k\} > \tfrac{1}{2} f^n(x)$$

are valid for n sufficiently large. We make use of it in (28) and, owing to Lemma 1, we get the estimate

$$|c_n(x)| \leqslant 2M_1[f^n(x)]^{k+\varkappa} \leqslant 2M_1[f^n(x_0)]^{k+\varkappa} \ (^2)$$

for $x \in I_0$, provided n is sufficiently large. In virtue of Lemma 4 we have

$$f^n(x_0) < (Mn)^{-1/k} \tag{29}$$

for large n. Consequently series (27) has in I_0 the convergent majorant

$$\sum_{n=0}^{\infty} 2M_1(Mn)^{-1-\varkappa/k}, \quad k > 0, \ \varkappa > 0,$$

and thus uniformly converges in I_0. Hence it follows that also the sequence $\Gamma_n(x)$ (cf. (26)) uniformly converges in I_0 and its limit

$$\Gamma(x) = \lim_{n\to\infty} \Gamma_n(x) \tag{30}$$

is different from zero (and thus positive) in I_0. Since x_0 has been arbitrary, function (30) is continuous and positive in the whole interval I. Assertion 1° of the theorem is proved. Consequently for sequence (5) case (A) occurs and, according to Lemma 2, the functions $\psi_c(x) = c/\Gamma(x)$ are the only continuous solutions of equation (24) in I. By Lemma 3, in view of relation (12), the functions

$$\varphi_c(x) = x\psi_c(x) = cx/\Gamma(x) \tag{31}$$

are the only regular solutions of equation (2) in I. Formula (14) results from (31), (30) and (6).

Proof of Theorem 3. Theorem 1 remains valid also in the present case. Consequently, all the continuous solutions of equation (2) fulfil relation (12) and we may confine ourselves to the study of the continuous solutions of equation (24). Those are determined by the behaviour of the sequence $\Gamma_n(x)$, which, in turn, depends on the behaviour of the series $\sum c_n(x)$, where $c_n(x)$ are given by (25).

We have assumed (15); thus, to begin with, let $a_2 < a_1$. Then

$$xg(x) - f(x) = (a_1 - a_2)x^{k+1} + O(x^{k+1+\varkappa}), \quad x \to 0+0, \quad \varkappa = \min(\mu, \nu).$$

Hence it follows that there exist a constant $P > 0$ and an $x_0 \in I$, $x_0 \neq 0$, such that for $x \in I_0$ we have $xg(x) - f(x) \geqslant Px^{k+1}$, whence

$$\frac{xg(x) - f(x)}{f(x)} \geqslant \frac{Px^{k+1}}{x - rx^{k+1}} \geqslant Px^k \quad \text{for} \quad x \in I_0, \ x \neq 0$$

(cf. (19)), provided x_0 has been chosen small enough.

(2) This inequality is also valid for $x = 0$, though this is not implied by formula (28) since the right-hand side of (28) is defined only for $x \neq 0$.

Now let us fix an $x \in I$, $x \neq 0$ and let $N(x)$ be such that $f^n(x) \leqslant x_0$ for $n > N(x)$ (Lemma 1). Then, for $n > N(x)$,

$$c_n(x) > P[f^n(x)]^k .$$

We may assume that $N(x)$ has been chosen so large that we may apply Lemma 4. Consequently, for $n > N(x)$,

$$c_n(x) > P_1/n$$

with a certain constant $P_1 > 0$. This shows that the series $\sum c_n(x)$ diverges for every $x \in I$, $x \neq 0$, and the same is true for the sequence $\Gamma_n(x)$. The corresponding assertion of our Theorem now follows immediately from Lemma 2.

Let $a_2 > a_1$. Then, for n sufficiently large, we have $c_n(x) < 0$, and by an argument similar to the preceding one we get the estimation

$$-c_n(x) \geqslant Q_0[f^n(x)]^k \quad \text{for } x \in I ,$$

with a certain constant $Q_0 > 0$. Let us fix an $x_0 \in I$, $x_0 \neq 0$. Then for $x \in \langle f(x_0), x_0 \rangle$ and n sufficiently large, say $n > N = N(x_0)$, we have

$$-c_n(x) \geqslant Q_0[f^{n+1}(x_0)]^k > Q/(n+1) ,$$

whence

$$\prod_{i=N+1}^{n-1} \big(1 + c_i(x)\big) < \prod_{i=N+1}^{n-1} \big(1 - Q/(i+1)\big) ,$$

where $Q > 0$ is a certain constant and we may assume that $N > Q$. The last inequality proves that the sequence $\Gamma_n(x)$ tends to zero uniformly in the interval $\langle f(x_0), x_0 \rangle$. The corresponding assertion of our Theorem now follows immediately from Lemma 2.

Proof of Theorem 4. Let us consider equation (4). On account of 1° of Theorem 2, case (A) occurs for the sequence $\Gamma_n(x)$. As has been proved in [2], Theorem 5, equation (4) has a continuous solution in I if and only if the series

$$(32) \qquad -\sum_{n=0}^{\infty} h[f^n(x)]/\big(f^{n+1}(x)\Gamma_{n+1}(x)\big) \; (^3)$$

converges in I to a continuous function $\psi^*(x)$. The function $\psi^*(x)$ is then itself a continuous solution of equation (4) in I.

Let us fix an $x_0 \in I$, $x_0 \neq 0$. In virtue of (16) and (iii) we have for $x \in I_0$

$$|h(x)| \leqslant K_0 x^{k+1+\lambda}$$

(3) As the value of the n-th term of series (32) for $x = 0$ we take the limit of this term when x tends to zero from the right.

with a certain constant $K_0 > 0$. By a similar argument as those in the proof of Theorem 2 (inequality (27)) we obtain the estimate

$$\left|\frac{h[f^n(x)]}{f^{n+1}(x)}\right| \leqslant 2K_0[f^n(x_0)]^{k+\lambda} \quad \text{for } x \in I, \tag{33}$$

valid for large n. Estimate (29) is also valid for large n. From 1° of Theorem 2 we infer that there exists a positive constant K_1 such that for large n and for $x \in I_0$ we have

$$\Gamma_{n+1}(x) > K_1 . \tag{34}$$

Relations (33), (29) and (34) yield in I_0 the majorant $\sum K^* n^{-\beta}$ for series (32), where $K^* = 2K_0K_1^{-1}M^{-\beta}$ and $\beta = 1+\lambda/k > 1$. This proves that series (32) uniformly converges in every interval $I_0 \subset I$. Thus its sum $\psi^*(x)$ is a continuous solution of equation (4) in I, and the function $\varphi^*(x) = x\psi^*(x)$ is a regular solution of equation (1) in I (Lemma 3). Inserting in (32) expression (6) in place of $\Gamma_n(x)$ we obtain relation (18).

It follows from Theorem 2 that the general regular solution of equation (1) in I is given by formula (17).

Equation (1) has in I a continuous solution depending on an arbitrary function, since it has a continuous solution in I (viz. any of the regular solutions, e.g. $\varphi^*(x)$), and the corresponding homogeneous equation (2) has in I a continuous solution depending on an arbitrary function (Theorem 1). This completes the proof.

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Reçu par la Rédaction le 20. 8. 1967