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ON A THEOREM OF W. HUREWICZ ON MAPPINGS WHICH LOWER DIMENSION

BY

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1. Introduction. The generalized form of a well-known theorem of Hurewicz (see [4], p. 91) states that

If $f\colon X \to Y$ is a continuous closed mapping of a metric separable space X onto a metric separable space Y, then

$$\dim X \leqslant \dim Y + \dim f, \tag{1}$$

where $\dim f = \sup\{\dim f^{-1}(y); y \in Y\}$ and $\dim A$ is the dimension of A.

This result, which has a simple geometrical character, became a source of similar theorems in which the spaces X and Y are more general (see, e.g., [8], [9], and [10], p. 129). Some results related to (1) were obtained in [7] and [11].

"Purely geometrical" directions in which (1) can be generalized were given, e.g., in [5] and in [15]. To see one of these directions let us consider the following

Example 1. Let $X = \{(x, y, z); x^2+y^2+z^2 \leqslant 1\}$ be the unit ball and $Y = \{(x, y, 0); x^2+y^2 \leqslant 1\}$ be the circular region in the 3-dimensional Euclidean space E^3. Let $f(x, y, z) = (x, y, 0)$ be the projection of X onto Y. Then, for the 1-dimensional closed subset $A = \{(x, 0, 0); -1 \leqslant x \leqslant 1\}$ of Y, the set $f^{-1}(A)$ is 2-dimensional and we have

$$\dim X \leqslant \dim Y - 1 + \dim f^{-1}(A).$$

Thus replacing in (1) the point inverses $f^{-1}(y)$ by inverses $f^{-1}(A)$, where A is 1-dimensional, one can replace $\dim Y$ in (1) by $\dim Y - 1$.

In Section 3 of this paper ([1]) we note first that, for closed mappings $f\colon X \to Y$, we have $\dim_0 f = \dim f$, where $\dim_0 f = \sup\{\dim f^{-1}(A); \dim A \leqslant 0, A \subset Y \text{ and } A \text{ closed}\}$.

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Thus the right-hand side in formula (1) is equal to the right-hand side of the following formula:

$$\dim X \leqslant \dim Y + \dim_0 f. \tag{2}$$

We then define a class P *larger than the class of closed and continuous mappings* (for example, each local homeomorphism with compact point inverses belongs to P) for which the following result holds (see Theorem 1):

If $f\colon X \to Y$ is a continuous mapping of X onto Y, where X and Y are metric separable spaces, and if $f \in P$, then, for every k satisfying $-1 \leqslant k \leqslant \dim Y$, one has

$$\dim X \leqslant \dim Y - k + \dim_k f, \tag{3}$$

where $\dim_k f = \sup\{\dim f^{-1}(A);\ \dim A \leqslant k,\ A \subset Y \text{ and } A \text{ closed}\}$ for $k \geqslant 0$ and $\dim_{-1} f = \dim X$.

Let us note that we *do not assume* that $\dim Y \leqslant \dim X$. Thus, for example, if $f \in P$, $\dim X = 3$ and $\dim Y = 4$, then not all closed subsets A of Y with $\dim A \leqslant 2$ can be images of 0-dimensional subsets of X. Let us also note that, for *closed* mappings $f\colon X \to Y$ of metric spaces X and Y, inequality (3) was stated in [5], p. 240 [2].

In Section 4 generalizations of some results of [7] are given. At the end of this paper two problems are posed.

In the sequel the spaces under consideration are metric separable (unless otherwise stated) and all mappings are assumed to be continuous.

2. Notation. We denote by (X^*, h) a *metric compactification of* X (i.e., $h\colon X \to X^*$ is a homeomorphism of X onto a dense subset of a compact *metric* space X^* — thus X is necessarily a metric separable space) and by $\operatorname{def} X$ the *deficiency of* X (i.e., $\operatorname{def} X = \min \dim\big(X^* \setminus h(X)\big)$, where (X^*, h) varies over all compactifications of X, see [3], p. 50). For a given mapping $f\colon X \to Y$ of X onto Y and for a compactification (X^*, h) of X we put $G = \{(x, y);\ x \in h(X) \text{ and } y = f\big(h^{-1}(x)\big)\}$, $\bar{G}$ is the closure of G in $X^* \times Y$, $\bar{f} = \bar{f}(f, X^*, h)$ — the projection of $\bar{G}$ onto Y, and $\overline{\dim_k f} = \min(\dim_k \bar{f})$, where the minimum is taken over all compactifications (X^*, h) of X.

3. We begin this section by showing that $\dim_0 f = \dim f$.

We then prove inequality (3).

Let $f\colon X \to Y$ be a closed mapping of X onto Y. Since $\dim f \leqslant \mathit{dim}_0 f$, it suffices to show that, for every *closed* 0-*dimensional* subset A of Y, one has $\dim f^{-1}(A) \leqslant \dim f$. This last inequality, however, follows imme-

[2] For this reference and for the necessity of a particular definition of $\dim_{-1} f$, I am indebted to A. Lelek.

diately by applying inequality (1) to the mapping $f|X_1: X_1 \to A$, where $X_1 = f^{-1}(A)$.

We introduce now the following

Definition 1. Let $f: X \to Y$ be a mapping of a topological space X onto a topological space Y. We say that f *has property* $P(f \in P)$ iff

(*) For every open neighbourhood U of $f^{-1}(y)$ and every sufficiently small open neighbourhood V of y there exists an open neighbourhood W of $f^{-1}(y)$ such that $W \subset U$ and $f(\mathrm{Bd} W) \subset \mathrm{Bd} V$, where Bd denotes the boundary.

We note first that *if A is a closed subset of X and $f: X \to Y$ has property P, then the partial mapping $f|A: A \to f(A)$ has property P.*

Indeed, let $U = U_1 \cap A$ be an arbitrary neighbourhood of $f^{-1}(y) \cap A$ and $V = V_1 \cap f(A)$ be a neighbourhood of $y \in f(A)$, where U_1 is open in X and V_1 is a sufficiently small open (in Y) neighbourhood of y. Since A *is closed*, the set $U_1 \cup (X \setminus A)$ is an open neighbourhood of $f^{-1}(y)$. By $f \in P$ there exists an open neighbourhood $W_1 \subset U_1 \cup (X \setminus A)$ with $f(\mathrm{Bd} W_1) \subset \mathrm{Bd} V_1$. Put $W = W_1 \cap A$. Then $f^{-1}(y) \cap A \subset W$, $W \subset U$ and $f(\mathrm{Bd} W) \subset \mathrm{Bd} V$.

We prove now that

(4) *If $f: X \to Y$ is a closed mapping of X onto Y, where X and Y are topological spaces, then $f \in P$.*

Indeed, let $y \in Y$ and let U be an arbitrary open neighbourhood of $f^{-1}(y)$. Since $X \setminus U$ is closed and f is a closed mapping, the set $f(X \setminus U)$ is closed. Since $y \notin f(X \setminus U)$, the set $Y \setminus f(X \setminus U)$ is an open neighbourhood of y. Let V be an arbitrary neighbourhood of y such that $V \subset Y \setminus f(X \setminus U)$. Then, by the continuity of f, one infers for $W = f^{-1}(V)$ that $f(\mathrm{Bd} W) \subset \mathrm{Bd} V$ and evidently $W \subset U$.

We show also that

(5) *If $f: X \to Y$ is a local homeomorphism of X onto Y, where X and Y are topological spaces, and if for every $y \in Y$ the set $f^{-1}(y)$ is compact, then $f \in P$.*

Indeed, let U be an arbitrary open neighbourhood of $f^{-1}(y)$. For every point $x \in f^{-1}(y)$ let U_x be an open neighbourhood of x such that $U_x \subset U$ and such that $f|U_x$ is a homeomorphism of U_x onto some open neighbourhood $V_y(x)$ of y. Since $f^{-1}(y)$ is compact, the covering $\{U_x; x \in f^{-1}(y)\}$ contains a finite subcovering $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$. The set

$$V = \bigcap_{i=1}^{n} V_y(x_i),$$

where $V_y(x_i) = f(U_{x_i})$ is an open neighbourhood of y and $W = f^{-1}(V) \cap U$ satisfies (*).

Remark 1. Let $X = \{(x, \frac{1}{2}); 0 < x < \frac{1}{2}\} \cup \{(x, 1); 0 < x < 1\}$ and $Y = \{(x, 0); 0 < x < 1\}$ be subspaces of the Euclidean plane and let $f\colon X \to Y$ be the projection of X onto Y. Then f is a local homeomorphism and $f^{-1}(y)$ is compact for every $y \in Y$. Hence $f \in P$. However f is not a closed mapping.

THEOREM 1. *If $f\colon X \to Y$ is a mapping of X onto Y, where X and Y are metric separable spaces, and if $f \in P$, then, for every k satisfying $-1 \leqslant k \leqslant \dim Y$, one has* $\dim X \leqslant \dim Y - k + \dim_k f$ *(i.e., inequality* (3) *holds).*

Proof. The proof is by induction on the dimension of Y and is similar to the proof of inequality (1) given by Hurewicz. For $\dim Y = -1$ the theorem is obvious. Suppose that the theorem is proved for all Y with $\dim Y \leqslant n$ and all $k \leqslant n$ and let $\dim Y = n+1$. For $k = n+1$ the theorem is again obvious, since one can take $A = Y$ in the definition of $\dim_k f$. So, we can assume that $k \leqslant n$. Now take $y \in Y$ and let U be an arbitrary open neighbourhood of $f^{-1}(y)$. Since $f \in P$ and $\dim Y \leqslant n+1$, one can find arbitrarily small neighbourhoods V of y with $\dim \mathrm{Bd}(V) \leqslant n$ and neighbourhoods W of $f^{-1}(y)$, $W \subset U$, such that $f(\mathrm{Bd} W) \subset \mathrm{Bd} V$. By the inductive assumption applied to $Y = f(\mathrm{Bd} W)$ (note that $\mathrm{Bd} W$ is a closed subset of X), one has

$$\dim \mathrm{Bd} W \leqslant \dim \mathrm{Bd} V - k + \dim_k f \leqslant n - k + \dim_k f.$$

Thus, for every open set U containing $f^{-1}(y)$, there exists an open set $W \subset U$ with $\dim \mathrm{Bd} W \leqslant n - k + \dim_k f$ and the theorem follows by Proposition G of [4], p. 90.

Remark 2. Note that for $k = 0$ Theorem 1 follows by (1) (for closed mappings) from $\dim f \leqslant \dim_0 f$. Note also that the separability of X was used only in application of Proposition G of [4], p. 90 (which is proved there for metric separable spaces).

4. In this section two theorems generalizing some results obtained in [7] are proved. First, we introduce — similarly to the notion of an "inductive invariant" in [7], p. 223 (also in [1] and [12]) — the following

Definition 2. Let S and T be two classes of sets and let X be a topological space. We put $I_{S,T}(X) = I(X) = -1$ iff $X \in T$ and define $I(X) \leqslant n+1$ iff, for every subset A of X such that $A \in S$, there exist arbitrarily small neighbourhoods U of A with $I(\mathrm{Bd}\, U) \leqslant n$.

Let us note that if T is the empty set and S is the set of all one-point subsets of X, then $I(X)$ is the weak inductive dimension of X. If S is as above and T is the class of all topologically complete spaces or $T = P$ is a topologically closed family of spaces, then $I(X) = I\mathrm{cd} X$ (see [1]) or $I(X) = \mathrm{in} PX$ (see [12]), respectively. If S is the set of all closed subsets of X and $T = \emptyset$, then $I(X)$ is the *large inductive dimension* of X.

Let us consider the class T of all metric separable spaces Z such that Z contains a compact subset C with $\dim Z = \dim C$ and let S be the set of all one-point subsets of a metric separable space X. Then, as in [7], we write $\operatorname{sub\,com} X$ for $I(X)$. The proof of the following theorem is similar to that of 3.1 in [7]:

THEOREM 2. *For every mapping $f\colon X \to Y$ of X onto Y and for every k satisfying $-1 \leqslant k \leqslant \dim Y$ one has*

$$(6) \quad \dim X \leqslant \dim Y - k + \sup\{\dim_k f|C;\ C \subset X \text{ and } C \text{ compact}\} + \\ + \operatorname{sub\,com} X + 1.$$

Proof. Let C be an arbitrary compact subset of X. By Theorem 1, we have $\dim C \leqslant \dim f(C) - k + \dim_k f|C$. Since, as easily seen (see [7], p. 224), $\dim X \leqslant \sup\{\dim C;\ C \subset X \text{ and } C \text{ compact}\} + \operatorname{sub\,com} X + 1$, we infer (6).

Remark 3. Let us note that, for every compact subset C of X, we have (see the proof of the formula $\dim_0 f \leqslant \dim f$) $\dim_0 f|C \leqslant \dim f$. Thus, putting $k = 0$ in (6), we infer that

$$\dim X \leqslant \dim Y + \dim f + \operatorname{sub\,com} X + 1.$$

This is the inequality proved in 3.1 of [7].

Let us also note that, for every subset C of X, one has $\dim_k f|C \leqslant \dim_k f$. Hence, by (6),

$$\dim X \leqslant \dim Y - k + \dim_k f + \operatorname{sub\,com} X + 1.$$

THEOREM 3. *For every mapping $f\colon X \to Y$ and every k satisfying $-1 \leqslant k \leqslant \dim Y$, one has*

$$(7) \qquad \dim X \leqslant \dim Y - k + \overline{\dim}_k f.$$

Proof. Let (X^*, h) be an arbitrary metric compactification of X. Since, as easily seen (see [6], p. 4), the projection $\bar{f}$ of $\bar{G}$ onto Y (see notation) is a closed mapping, it follows by Theorem 1 that $\dim X = \dim G \leqslant \dim \bar{G} \leqslant \dim Y - k + \overline{\dim}_k f$.

Remark 4. A mapping $f\colon X \to Y$ is called *locally compact* iff, for every $y \in Y$, the set $f^{-1}(y)$ is locally compact. It was proved in [7], p. 225, that for a locally compact mapping $f\colon X \to Y$ one has

$$(8) \qquad \dim X \leqslant \dim Y + \max\{\dim f, \operatorname{def} X\}.$$

We give now an example showing that inequality (7) is much stronger than inequality (8).

Example 2. Consider the subset $X = (a_0) \cup \big[\bigcup_{j=1}^{\infty} (a_j) \times J\big]$ of the plane, where $a_0 = 0$ and $a_j = 2^{-j+1}, j = 1, 2, \ldots,$ are real numbers on the real axis and $J = [0, 1]$ and put $f(x) = x$. Evidently, f is a locally compact mapping. Taking

$$X^* = \bigcup_{j=0}^{\infty} [(a_j) \times J] \quad \text{and} \quad h(x) = x,$$

one infers that (X^*, h) is a compactification of X. As easily seen, one has (for $k = 0$) $\overline{\dim}_0 f = 0$ and inequality (7) gives $\dim X \leqslant 1$. On the other hand, one has $\operatorname{def} X = 1$ (see [14], p. 71, Example 1) and inequality (8) gives $\dim X \leqslant 2$.

A similar example can be obtained by using the set constructed in [13], p. 806. Using Theorem 4 of [14], p. 72, one could also construct for every $n \geqslant 1$ examples of spaces X and locally compact mappings f (even identity mappings) such that $\dim X = n$ and such that (7) gives $\dim X \leqslant n$ whereas inequality (8) gives $\dim X \leqslant 2n$.

The set X in Example 2 is not semicompact. Using a result of Freudenthal (see [2]) which (for semicompact spaces X for which the space of quasicomponents is compact) implies $\operatorname{def} X \leqslant 0$, one can show that in some cases (8) gives the same result as (7) used for $k = 0$.

We conclude this paper with the two problems:

PROBLEM 1. Find classes S and T for which inequality (3) holds with "dim" replaced by "$I_{S,T} = I$" (see Definition 2). Characterize these classes. (**P 809**)

PROBLEM 2. Let $I(X)$ and $I_1(X)$ be given by Definition 2. Prove theorems similar to Theorem 2 with "dim" replaced by "I" and "sub com" replaced by "I_1". (**P 810**)

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