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A REMARK ON THE EXISTENCE OF THE ERGODIC HILBERT TRANSFORM

BY

J. WOŚ (WROCŁAW)

Let $T\colon X \to X$ be an invertible ergodic measure-preserving transformation of a probability space and let $h\colon X \to \boldsymbol{R}$ be an integrable function. The almost everywhere (a.e.) existence of the ergodic Hilbert transform

$$\tilde{h} = \lim_{n\to\infty} \sum_{\substack{k=-n \\ k\neq 0}}^{n} \frac{h\circ T^k}{k}$$

was proved by M. Cotlar [2]. Other proofs were given by A. P. Calderón [1], R. L. Jones [4], and K. Petersen [5]. All these proofs depend on the corresponding weak type (1, 1) inequality.

In the present note we give a direct proof of the existence of $\tilde{h}$ which, however, presupposes the existence of $\tilde{h}$ for $h \in L_\infty \subseteq L_2$ (as in Jones' paper [4]). Moreover, we show that the assumption of the integrability of h can be somewhat weakened. Namely, $\tilde{h}$ exists a.e. if $h^* = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} h\circ T^k$ exists and is finite a.e.

The proof is based on the uniform approximation lemma (see below) which can be also of independent interest. We note that the corollary to that lemma, which states in particular that $L_1 \subseteq L_\infty + \mathscr{T}\circ(I-T)$, can be thought as a global variant of the Calderón–Zygmund decomposition in the ergodic theory setting (cf. [4], p. 287).

Throughout the paper we fix a probability space $(X, \mathscr{A}, \mu)$ and an ergodic measure-preserving transformation $T\colon X \to X$. We consider only measurable functions.

Let $\mathscr{T}$ denote the vector lattice consisting of all functions $v\colon X \to \boldsymbol{R}$ such that

$$\lim_{n\to\infty} \frac{v\circ T^n}{n} = 0 \quad \text{a.e.}$$

It follows by the Proposition in [6] that a function v belongs to $\mathscr{T}$ iff there exists a function $w \geqslant |v|$ such that $w - w \circ T \in L_1$. Observe that for an invertible transformation T the space $\mathscr{T}$ for the inverse T^{-1} is the same as for T.

It is a classical result that an integrable function h with $\int h = 0$ can be approximated in the L_1-norm by the functions of the form $v - v \circ T$, where $v \in L_\infty$. We shall prove that this approximation can be made uniform (i.e. in the L_∞-norm), if we agree that v will now belong to $\mathscr{T}$. The following lemma is an improvement of Corollary 3 in [6].

UNIFORM APPROXIMATION LEMMA. *For a function h: $X \to \mathbf{R}$ the following statements are equivalent*:

(i) $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=0}^{n-1} h \circ T^k = 0$ a.e.;

(ii) *there exists a sequence $(v_n)_{n\geqslant 0}$ of elements of $\mathscr{T}$ such that*

$$\lim_{n\to\infty} (\operatorname{ess\,sup}_X |h - (v_n - v_n \circ T)|) = 0.$$

Proof. (i) $\Rightarrow$ (ii). Take an arbitrary $\varepsilon > 0$ and assume that (i) is valid. By Corollary 3 in [6] we can approximate h by the class $\mathscr{T} \circ (I - T)$ in the L_1-norm, i.e. there exists a function $w \in \mathscr{T}$ such that $\int |z| < \varepsilon$, where $z = h - (w - w \circ T)$.

Denote $z_1 = z^+ - \varepsilon$, $z_2 = z^- - \varepsilon$. By Birkhoff ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} z_i \circ T^k = \int z_i < 0 \quad \text{a.e.}$$

for $i = 1, 2$. Therefore, the function $H_i = \sup\limits_{n\geqslant 0} \sum\limits_{k=0}^{n} z_i \circ T^k$ is finite a.e. and

$$z_i = -H_i^- + H_i^+ - H_i^+ \circ T \quad \text{a.e.}$$

for $i = 1, 2$ (cf. [6], Theorem 1). Clearly, $0 \leqslant H_i^- \leqslant z_i^- \leqslant \varepsilon$, and $H_i^+ \in \mathscr{T}$ (this follows from the integrability of $H_i^+ - H_i^+ \circ T = z_i + H_i^-$). Consequently

$$\begin{aligned} h &= z + w - w \circ T \\ &= z_1 - z_2 + w - w \circ T \\ &= -H_1^- + H_2^- + (w + H_1^+ - H_2^+) - (w + H_1^+ - H_2^+) \circ T \\ &= u + v - v \circ T \quad \text{a.e.,} \end{aligned}$$

where $|u| \leqslant \varepsilon$ and $v \in \mathscr{T}$.

(ii) $\Rightarrow$ (i). Let $\varepsilon > 0$ be arbitrary. By (ii) there exist a function u with

$|u| \leqslant \varepsilon$ a.e. and a function $v \in \mathscr{T}$ such that $h = u + v - v \circ T$. Consequently

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} h \circ T^k\right| \leqslant \varepsilon + \frac{1}{n}(|v| + |v| \circ T^n) \quad \text{a.e.}$$

Since $v \in \mathscr{T}$, we have $\limsup_n \left|\frac{1}{n}\sum_{k=0}^{n-1} h \circ T^k\right| \leqslant \varepsilon$ a.e., which completes the proof.

Let $\mathscr{H}$ denote the vector space consisting of all functions $h\colon X \to \boldsymbol{R}$ such that

$$h^* = \lim_{n \to \infty} \frac{1}{n}\sum_{k=0}^{n-1} h \circ T^k$$

exists and is finite a.e.

It follows by Theorem 4 in [6] that $\mathscr{H} = L_1 + \mathscr{T} \circ (I - T)$, i.e. a function h belongs to $\mathscr{H}$ iff there exist a function $u \in L_1$ and a function $v \in \mathscr{T}$ such that $h = u + v - v \circ T$. (Here I denotes the identity transformation on X).

The following corollary follows from the uniform approximation lemma and from the fact that the limit function h^* is a constant a.e.

COROLLARY. $\mathscr{H} = L_\infty + \mathscr{T} \circ (I - T)$.

The proof of the next lemma uses a standard technique from probability theory.

LEMMA. *If $v \in \mathscr{T}$ and $f \in L_1$, then the series*

$$\sum_{n=1}^{\infty} \frac{(vf) \circ T^n}{n^2}$$

converges a.e. to a finite limit.

Proof. We may assume that v and f are non-negative. Since $v \in \mathscr{T}$, $v \circ T^n/n$ is a.e. not less than 1 only finitely many times. Therefore it suffices to check the a.e. convergence of the series

$$\sum_{n=1}^{\infty} \frac{(vf) \circ T^n}{n^2} 1_{\{v \circ T^n < n\}}.$$

Actually, we shall prove that it is integrable.

We have

$$(vf) \circ T^n 1_{\{v \circ T^n < n\}} \leqslant \sum_{k=1}^{n} kf \circ T^n 1_{\{k-1 \leqslant v \circ T^n < k\}}.$$

Consequently,

$$\int (vf) \circ T^n 1_{\{v \circ T^n < n\}} \leqslant \sum_{k=1}^{n} k \int f 1_{\{k-1 \leqslant v < k\}}.$$

Now, denoting $a_k = \int f\, 1_{\{k-1 \leqslant v < k\}}$, we obtain

$$\int \left(\sum_{n=1}^{\infty} \frac{(vf)\circ T^n}{n^2} 1_{\{v\circ T^n < n\}} \right) \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} k a_k \leqslant 2 \sum_{k=1}^{\infty} a_k = 2 \int f,$$

which completes the proof.

Finally, we have

THEOREM. *If T is invertible and $h \in \mathscr{K}$, then the ergodic Hilbert transform*

$$\tilde{h} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{h\circ T^k - h\circ T^{-k}}{k}$$

exists and is finite a.e.

Proof. By the Corollary, $h = u + v - v\circ T$ where $u \in L_\infty \subseteq L_2$ and $v \in \mathscr{T}$.

Now, the existence of $\tilde{u}$ follows by the results of Cotlar [2] (one can also transfer a well-known strong type (2, 2) inequality for the Hilbert transform on the integers [3], p. 212, and then use the Banach convergence principle).

Next, we have

$$\sum_{k=1}^{n} \frac{(v - v\circ T)\circ T^k}{k} = v\circ T - \sum_{k=1}^{n-1} \frac{v\circ T^{k+1}}{k(k+1)} - \frac{v\circ T^{n+1}}{n},$$

and

$$\sum_{k=1}^{n} \frac{(v - v\circ T)\circ T^{-k}}{k} = -v + \sum_{k=1}^{n-1} \frac{v\circ T^{-k}}{k(k+1)} + \frac{v\circ T^{-n}}{n}.$$

Thus, the existence of $(v - v\circ T)^{\sim}$ follows from the Lemma (with $f \equiv 1$), the fact that $\frac{v\circ T^n}{n} \to 0$ a.e. as $n\to\infty$, and from the remark that the space $\mathscr{T}$ for T^{-1} is the same as for T.

Remark. The implication converse to that of the Theorem is an open problem (**P 1321**).

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INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY
WROCŁAW, POLAND

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