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RANDOM LINEAR FUNCTIONALS AND RANDOM INTEGRALS

BY

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I have attempted to survey what is called the Riesz representation of random linear functionals. Let X be a separable, complete locally convex linear metric space over the real field R and let μ be a Borel probability measure on X. We say that f is a *random linear functional* or, more precisely, f is a *μ-measurable linear functional* on X whenever f is defined and μ-measurable on a μ-measurable linear manifold D_f with $\mu(D_f) = 1$ and, for any pair $\alpha, \beta \in R$ and $x, y \in D_f$, the equality

$$f(\alpha x+\beta y) = \alpha f(x)+\beta f(y)$$

holds. It is evident that each continuous linear functional on X is μ-measurable for every Borel probability measure μ. In other words, denoting by X^* the dual space, and by X^*_μ the space of all μ-measurable linear functionals on X, we have the inclusion $X^* \subset X^*_\mu$. The space X^*_μ is equipped with the convergence in the measure μ and, moreover, functionals equal μ-almost everywhere are treated here as identical. It is evident that the space X^*_μ is separable and complete. Moreover, a non-homogeneous norm $\|\cdot\|$ can be defined on X^*_μ by means of the formula

$$\|f\| = \int_{-\infty}^{\infty} \frac{1-\Phi(tf)}{1+t^2}\, dt,$$

where $f \in X^*_\mu$, Φ being the characteristic functional of μ (cf. [6], p. 614) such that

$$\Phi(f) = \int_X e^{if(x)}\mu(dx).$$

Cameron and Graves (see [3] and [7]) seem to have been the first authors to discuss measurable linear functionals and, more generally, measurable additive functionals in the case of the Wiener measure on the space of continuous functions. A few additional sidelights on this theory are mentioned in [21] and [5]. The case of product measures μ

has also been treated by Smolyanov in [20]. The theory of random linear functionals on separable Hilbert spaces has been presented in [6], Chapter 8.

It is well known that each Borel probability measure μ on X is tight (see [2], Theorem 1.4). Applying Lusin Theorem (see [11], p. 159) to a μ-measurable linear functional f we infer that for every $\varepsilon > 0$ there exists a compact subset A of D_f with $\mu(A) \geqslant 1-\varepsilon$ such that the restriction $f|A$ is continuous. Słowikowski considered in [19] a slightly strengthened property. Namely, he called an element f of X_μ^* a *Lusin functional* provided for every $\varepsilon > 0$ there exists a convex compact subset B of D_f with $\mu(B) \geqslant 1-\varepsilon$ such that the restriction $f|B$ is continuous.

Let X_μ^L denote the subset of X_μ^* consisting of all Lusin functionals. It is clear that $X^* \subset X_\mu^L$. Moreover, denoting by $\mu\text{-}\mathrm{cl}X^*$ the closure of X^* in the topology of the convergence in the measure μ, we have the following statement:

THEOREM 1. $X_\mu^L = \mu\text{-}\mathrm{cl}X^*$.

Proof. The inclusion $X_\mu^L \subset \mu\text{-}\mathrm{cl}X^*$ is a simple consequence of Corollary I.1.5 in [1] which states that every linear functional continuous on a convex compact subset of X is a uniform limit on this subset of a sequence of linear functionals continuous on the whole space X, i.e. belonging to X^*.

To prove the converse let us suppose that $f \in \mu\text{-}\mathrm{cl}X^*$. Without loss of generality we may assume that

$$f = \lim_{n\to\infty} f_n \quad \mu\text{-almost everywhere}, \qquad f_n \in X^* \ (n = 1, 2, \ldots),$$

and that D_f coincides with the set of all points x in X for which the sequence $\{f_n(x)\}$ has a limit. Given $\varepsilon > 0$, by Egorov Theorem (see [11], p. 157) there exists a compact subset A of D_f with $\mu(A) \geqslant 1-\varepsilon$ such that the sequence $\{f_n\}$ is on A uniformly convergent to f. Let B be the closed convex hull of A. Obviously, each element y of B is the limit of a sequence of elements

$$y_k = \sum_{j=1}^{n_k} a_{j,k} x_{j,k} \qquad (k = 1, 2, \ldots),$$

where $x_{j,k} \in A$, $a_{j,k} \geqslant 0$ and $\sum_{j=1}^{n_k} a_{j,k} = 1$ $(j = 1, 2, \ldots, n_k;\ k = 1, 2, \ldots)$. From the inequality

$$|f_m(y) - f_n(y)| = \lim_{k\to\infty} |f_m(y_k) - f_n(y_k)| \leqslant \overline{\lim_{k\to\infty}} \sum_{j=1}^{n_k} a_{j,k} |f_m(x_{j,k}) - f_n(x_{j,k})| \leqslant \max_{x\in A} |f_m(x) - f_n(x)|$$

it follows that $\{f_n\}$ is a Cauchy sequence on B with respect to the uniform convergence. Thus f being a uniform limit of $\{f_n\}$ on B is continuous on B which implies that f is a Lusin functional. The theorem is thus proved.

The following statement will be frequently used in the sequel:

PROPOSITION 1. *For every μ-measurable linear functional f there exists a Borel linear submanifold $\tilde{D}_f$ of D_f such that $\mu(\tilde{D}_f) = 1$ and the restriction $f|\tilde{D}_f$ is Borel measurable.*

Proof. We know, by virtue of Lusin Theorem, that for every integer n there exists a compact subset K_n of D_f such that $\mu(K_n) \geqslant 1-1/n$ and the restriction $f|K_n$ is continuous. The set

$$L_{kn} = \Big\{\sum_{j=1}^{k} c_j x_j:\ x_j \in K_n,\ c_j \in R,\ |c_j| \leqslant k\ (j = 1, 2, \ldots, k)\Big\},$$

being the continuous image

$$\langle c_1, c_2, \ldots, c_k, x_1, x_2, \ldots, x_k\rangle \to \sum_{j=1}^{k} c_j x_j$$

of the compact set

$$[-k, k]\times[-k, k]\times\ldots\times[-k, k]\times K_n\times K_n\times\ldots\times K_n,$$

is compact and the restriction $f|L_{kn}$ is continuous. Put

$$\tilde{D}_f = \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} L_{kn}.$$

It is clear that $\tilde{D}_f$ is a linear submanifold of D_f being F_σ-set, the restriction $f|\tilde{D}_f$ is Borel measurable and $\tilde{D}_f \supset K_n$ $(n = 1, 2, \ldots)$. Hence it follows that $\mu(\tilde{D}_f) = 1$ which completes the proof.

We say that the probability measure μ has the *Riesz property* whenever $X^*_\mu = \mu\text{-}\mathrm{cl}\, X^*$. In other words, μ has the Riesz property if and only if μ-measurable linear functionals are Lusin. It is obvious that each probability measure on a finite-dimensional linear space X has the Riesz property. The first non-trivial result is due to Cameron and Graves [3] who proved that the Wiener measure considered on the space of continuous functions has the Riesz property. This theorem was strengthened by Kanter ([9], p. 448) who proved that each Gaussian measure with zero mean has the Riesz property. He established also this property for symmetric stable measures of index p satisfying the inequality $1 < p \leqslant 2$. The case of product measures $\mu = \nu_1\times\nu_2\times\ldots$ on R^∞, where ν_j are equivalent to the Lebesgue measure, was considered by Smolyanov in [18]. At a first glance it might seem that each probability measure on

X has the Riesz property. In particular, it was claimed in [6] (Chapter 8, Theorem 1) that each non-degenerate measure μ on a separable Hilbert space, i.e. each measure vanishing on every proper subspace, has the Riesz property. Unfortunately, this is not true. A simple counterexample is due to Kanter ([9], p. 447).

The aim of the present note is to investigate the Riesz property for probability measures on the L^2-space over the unit interval induced by symmetric, homogeneous, separable and continuous in probability stochastic process with independent increments. It should be mentioned that the probability measure induced by such a process is, in fact, concentrated on the subset of L^2 consisting of bounded functions having no discontinuities of the second kind ([4], Theorem 7.2).

We review here a few facts about the random integral. Suppose we have a symmetric, homogeneous and separable stochastic process $\{x(t)\colon 0 \leqslant t \leqslant 1\}$ with independent increments satisfying the initial condition $X(0) = 0$. This process defines a random measure M with independent values by means of the formula $\mu\big((a, b]\big) = x(b) - x(a)$. For the definition of an integral with respect to the random measure M we refer to papers [13]-[17] and [22]. In the sequel we shall use the notation

$$\int_0^1 \varphi(t)\,dx(t) = \int_0^1 \varphi(t) M(dt).$$

It is clear that for every M-integrable function φ the formula

$$\tilde{\varphi}(x) = \int_0^1 \varphi(t)\,dx(t) \tag{1}$$

defines a random functional on the space L^2. Moreover, it follows from the definition of the random integral that the functional $\tilde{\varphi}$ is the limit in probability of a sequence of continuous functionals. Consequently, $\tilde{\varphi}$ is a Lusin functional. Since each continuous linear functional can be treated as a random one, we infer that formula (1) establishes a one-to-one correspondence between M-integrable functions and Lusin functionals. This formula can be also regarded as an analogue of the Riesz integral representation of continuous linear functionals on classical function spaces. The algebraic and topological structure of the space of all M-integrable functions was determined in [22]. Namely, this space is linearly isomorphic and homeomorphic to the Orlicz space $L(\Psi)$ of all Borel functions φ satisfying the condition

$$\int_0^1 \Psi\big(|\varphi(t)|\big)\,dt < \infty, \qquad \text{where } \Psi(t) = \int_{1/t}^{\infty} \frac{G(u)}{u^3}\,du,$$

and G is the Lévy-Khinchine function of the process in question determined by means of the formula for the characteristic function γ of the increment $x(b)-x(a)$,

$$(2) \qquad \gamma(t) = \exp(b-a)\int_0^\infty (\cos tu - 1)\frac{1+u^2}{u^2}\,dG(u),$$

and the condition $G(0) = 0$. Consequently, we have the following statement:

THEOREM 2. *If μ is a probability measure on the L^2-space over the unit interval induced by a symmetric, homogeneous, separable and continuous in probability stochastic process with independent increments, then each Lusin functional f on L^2 is of the form*

$$f(x) = \int_0^1 \varphi(t)\,dx(t),$$

where φ belongs to the Orlicz space $L(\Psi)$ with

$$\Psi(t) = \int_{1/t}^\infty \frac{G(u)}{u^3}\,du,$$

G being the Lévy-Khinchine function corresponding to the process in question. Consequently, $(L^2)^L_\mu = \mu\text{-}\mathrm{cl}\,L^2$.

Another proof of this theorem was given by Nguyen Chi Bao in [12].

Now consider the case of *compound Poisson processes*, i.e. processes whose sample functions are step functions with a finite number of jumps. A necessary and sufficient condition for a process to be compound Poisson can be expressed in terms of its Lévy-Khinchine function G as follows (see [18], p. 90):

$$\int_0^\infty \frac{1+u^2}{u^2}\,dG(u) < \infty.$$

Following Ito ([8], Section 17) we associate with every compound Poisson process $\{x(t)\colon 0 \leqslant t \leqslant 1\}$ a random measure N_x defined on Borel subsets S of the product $[0,1]\times(R\setminus\{0\})$ as follows: $N_x(S)$ is the cardinality of the set

$$S \cap \{\langle t, x(t+0)-x(t-0)\rangle\colon 0 \leqslant t \leqslant 1\}.$$

The random variable $N_x(S)$ has a Poisson probability distribution with the parameter $\lambda(S)$ given by the formula

$$\lambda(S) = \int_S \frac{1+u^2}{u^2}\,dG(u)\,dt.$$

Moreover, for any collection of disjoint Borel sets $S_1, S_2, \ldots, S_k$, the random variables $N_x(S_1), N_x(S_2), \ldots, N_x(S_k)$ are mutually independent (see [8], Section 17). Further, it is very easy to verify that each Borel function is integrable with respect to any compound Poisson process.

THEOREM 3. *Probability measures on L^2 induced by compound Poisson processes have the Riesz property.*

Proof. Let μ be such a measure and let f be an arbitrary μ-measurable linear functional on L^2. By Proposition 1, we may assume, without loss of generality, that the functional f is Borel measurable and the domain D_f is a Borel set. For every $s \in [0,1]$ we put $y_s(t) = 0$ if $t \leqslant s$ and $y_s(t) = 1$ otherwise. It is clear that the mapping $h\colon s \to y_s$ is a homeomorphism from the unit interval $[0,1]$ into L^2. Consequently, E is a Borel subset of $[0,1]$ if and only if $h(E)$ is that of L^2. Put

$$E_0 = \{s\colon y_s \in D_f\}.$$

Obviously,

$$h(E_0) = h([0,1]) \cap D_f,$$

which shows that E_0 is a Borel set. Further, since D_f is a linear manifold, we infer that the sets D_f and

$$A = \{x\colon N_x\big(([0,1]\setminus E_0)\times(R\setminus\{0\})\big) = 1,\ N_x\big(E_0\times(R\setminus\{0\})\big) = 0\}$$

are disjoint. Thus $0 = \mu(A) = c(1-|E_0|)e^{-c}$, where c is a non-negative constant. Of course, we may assume that $c > 0$, since in the opposite case the measure μ is concentrated at the origin and our statement is trivial. Hence we infer that $|E_0| = 1$. Consequently, the linear manifold D_f^0 spanned by E_0 consisting of all step functions with jumps at the set E_0 has the property $\mu(D_f^0) = 1$. Thus we may consider f on the domain D_f^0. Setting $\varphi(s) = f(y_s)$ for $s \in E_0$ and $\varphi(s) = 0$ otherwise, we get a Borel function which, of course, is integrable with respect to the process in question. Since each element x from D_f^0 can be represented in the form

$$x = \sum_{j=1}^{n} c_j y_{s_j}, \qquad \text{where } c_j \in R,\ s_j \in E_0\ (j = 1, 2, \ldots, n),$$

we have the integral representation

$$f(x) = \sum_{j=1}^{n} c_j \varphi(s_j) = \int_0^1 \varphi(t)\,dx(t).$$

Thus, by Theorem 2, the measure μ has the Riesz property which completes the proof.

We say that the process in question has a *Gaussian component* whenever its Lévy-Khinchine function G appearing in formula (2) satisfies the inequality $G(0+) > 0$. We note that this inequality, together with the condition that the function G is constant on the open half-line $(0, \infty)$, characterizes the Brownian motion process. It is well known that each symmetric, homogeneous, separable and continuous in probability stochastic process with independent increments is the sum of two independent processes of the same type: one the Brownian motion process and the other without a Gaussian component (see [8], Section 17, and [18], p. 85). We already mentioned the result due to Cameron and Graves [3] that the measure induced by the Brownian motion, i.e. the Wiener measure, has the Riesz property. Using an idea due to Kanter ([9], p. 447) we prove the following theorem:

THEOREM 4. *Suppose that μ is induced on L^2 by a symmetric, homogeneous, separable and continuous in probability process with independent increments and having a Gaussian component. Further, suppose that μ is not the Wiener measure. Then μ has not the Riesz property.*

Proof. Contrary to this, let us assume that each μ-measurable linear functional is the limit in the measure μ of a sequence of continuous linear functionals on L^2. Further, consider the decomposition of the process in question into independent components

$$x(t) = x_1(t) + x_2(t), \tag{3}$$

where the equality is taken is the sense of the L^2-space, x_1 is the Brownian motion process, and x_2 has no Gaussian component. Since

$$x_1(t) = \sum_{s<t} \big(x(s+0) - x(s-0)\big),$$

the correspondence $x \to x_1$ is μ-measurable and linear (see [18], p. 85). Consequently, the linear functional f defined by the formula $f(x) = x_2(1-0)$ is μ-measurable. Consequently, we can find a sequence $\{f_n\}$ of continuous linear functionals on L^2 which converges to f in the measure μ. Taking into account the classical Riesz representation

$$f_n(x) = \int_0^1 \varphi_n(t) x(t)\, dt \qquad (n = 1, 2, \ldots), \tag{4}$$

we infer, by (3), that the sequence $\{f_n(x_1) + f_n(x_2)\}$ is convergent in the measure μ. Since the random variables $f_n(x_1)$ and $f_n(x_2)$ are stochastically independent and, in view of linearity of f_n, symmetrically distributed, we infer that both sequences $\{f_n(x_1)\}$ and $\{f_n(x_2)\}$ are convergent in the measure μ to μ-measurable linear functionals, say $g_1(x_1)$ and $g_2(x_2)$, respectively. Put

$$\psi_n(t) = \int_t^1 \varphi_n(u)\, du \qquad (n = 1, 2, \ldots).$$

Then (4) can be rewritten in the form of the random integral

$$f_n(x) = \int_0^1 \psi_n(t)\,dx(t) \qquad (n = 1, 2, \ldots).$$

Hence and from Theorem 2 it follows that the sequence $\{\psi_n\}$ of functions on $[0, 1]$ converges in the sense of both Orlicz spaces $L(\Psi_1)$ and $L(\Psi_2)$ corresponding to processes x_1 and x_2, respectively. Thus there exists a function ψ belonging to $L(\Psi_1)\cap L(\Psi_2)$ and such that

$$g_1(x_1) = \int_0^1 \psi(t)\,dx_1(t) \tag{5}$$

and

$$g_2(x_2) = \int_0^1 \psi(t)\,dx_2(t) \tag{6}$$

μ-almost everywhere.

Hence, in particular, it follows that the random variable $g_1(x_1)$ has a Gaussian or degenerate probability distribution and the random variable $g_2(x_2)$ has an infinitely divisible probability distribution without a Gaussian component. Moreover, both random variables $g_1(x_1)$ and $g_2(x_2)$ are independently and symmetrically distributed. Since the functional $x_2(1-0)$ has no Gaussian component, we infer, in view of the equality

$$x_2(1-0) = g_1(x_1)+g_2(x_2), \tag{7}$$

that $g_1(x_1)$ vanishes μ-everywhere. Thus, by (5), ψ vanishes almost everywhere in the sense of the Lebesgue measure on $[0, 1]$. Consequently, by (6), $g_2(x_2) = 0$ μ-almost everywhere which, by (7), implies the equality $x_2(1-0) = 0$ μ-almost everywhere. Thus, by (3), $x(1-0)$ has a Gaussian distribution. Applying well-known Cramer's Theorem ([10], Section 19), we infer that for every $t \in [0, 1)$ the random variable $x(t)$ has a Gaussian probability distribution. Hence it follows immediately that $\{x(t): 0 \leqslant t \leqslant 1\}$ is the Brownian motion process which contradicts the assumption. The theorem is thus proved.

It is not known whether each measure induced on L^2 by a symmetric, homogeneous, separable and continuous in probability stochastic process with independent increments having no Gaussian component has the Riesz property (**P 949**). Theorem 3 gives an affirmative answer to this problem only in the case of compound Poisson processes.

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