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SEMI-IDEALS IN SEMI-LATTICES

BY

P. V. VENKATANARASIMHAN (TRIVANDRUM, KERALA)

Introduction. In this paper* certain results about semi-ideals of a semi-lattice are obtained; some of these are analogues of known results about ideals of a distributive lattice. Also, some known results about distributive lattices are extended to semi-lattices. The notion of disjunction is extended to semi-lattices and various sets of necessary and sufficient conditions are obtained for a semi-lattice to be a disjunction semi-lattice. It is proved that a pseudo-complemented disjunction semi-lattice is a Boolean algebra. The results about semi-ideals are applied to study some special features of a topology on the set of all prime semi-ideals of a semi-lattice. A natural topology is introduced on the set of all proper dual ideals of a semi-lattice. It is proved that this topology is T_0 and that it is compact and non-regular if the semi-lattice has the greatest and the least elements. The subspace of maximal dual ideals is proved to be T_3.

1. Preliminaries. This section is devoted to a summary of known concepts and results which will be used in subsequent sections.

First we shall recall some concepts introduced in [8] and [9]. For lattice-theoretic and topological concepts which have now become commonplace the reader is referred to [4], [6] and [7]. A non-null subset A of a poset (partially ordered set) P is called a *semi-ideal* if $a \in A$, $b \leqslant a$ $(b \in P)$ $\Rightarrow b \in A$. A semi-ideal A of P is called an *ideal* if the lattice-sum of any finite number of elements of A, whenever it exists, belongs to A. An element a of a poset P with 0 is said to have a *pseudo-complement* a^* if there exists an element a^* in P such that $(a]\cap(a^*] = (0]$ and for $b \in P$, $(a]\cap(b] = (0]$ $\Rightarrow (b] \subseteq (a^*]$. A semi-ideal of a poset with 0 is said to be *normal* (*dense*) if it is a normal (dense) element of S_μ (the set of all semi-ideals of a poset with 0 forms a lattice under set-inclusion; this lattice is denoted by S_μ).

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A proper ideal or a semi-ideal A of a poset is said to be *prime* if $(a]\cap(b] \subseteq A \Rightarrow (a]\subseteq A$ or $(b]\subseteq A$. A prime semi-ideal is called a *minimal prime semi-ideal* if it does not contain any other prime semi-ideal. The dual concepts are defined in an obvious way.

A point of a topological space is called an *anti-T_1 point* if it does not belong to the closure of any other point.

Set-inclusion is denoted by $\subseteq$. The lattice-sum and lattice-product in S_μ coincide with set-union and set-intersection and these are denoted by $\cup$ and $\cap$, respectively. In the poset of ideals of a poset, the lattice--sums, if they exist, are denoted by $\vee$. The same symbol is also used to denote lattice-sums in the poset of dual ideals, whenever the sums exist. $(a]$ denotes the principal ideal generated by a. The principal dual ideal generated by a is denoted by $[a)$. The set of all prime semi-ideals of a poset is denoted by $\mathscr{P}$ and $F(A)$ denotes the set of prime semi-ideals containing a semi-ideal A. $F'(A)$ stands for $\mathscr{P}-F(A)$.

We collect below some known results used in the sequel.

LEMMA I. *Any proper ideal (dual ideal) of a poset with* 1 (0) *is contained in a maximal ideal (dual ideal).*

LEMMA II. *Any semi-ideal of a poset is the product of all the prime semi-ideals containing it.*

LEMMA III. *If the product of a finite number of semi-ideals of a poset with* 0 *is* $(0]$, *then any prime semi-ideal contains at least one of them.*

LEMMA IV. S_μ *is a complete Σ, π-distributive lattice; consequently, it is closed for pseudo-complements.*

LEMMA V. *The normal elements of a semi-lattice closed for pseudo--complements form a Boolean algebra.*

LEMMA VI. *If P is a poset with* 0, *then*

(i) $F\big(\bigcup_{i\in I} A_i\big) = \bigcap_{i\in I} F(A_i)$,

(ii) $F(A_1\cap A_2\cap \ldots \cap A_n) = F(A_1)\cup F(A_2)\cup \ldots \cup F(A_n)$,

(iii) $F(P) = \emptyset$,

(iv) $F\big((0]\big) = \mathscr{P}$.

(*Here the A_i are semi-ideals of P.*) *Consequently, F defines a closure operation in $\mathscr{P}$, thereby giving rise to a topology on $\mathscr{P}$.*

LEMMA VII. *In the topological space $\mathscr{P}$,* $\operatorname{Int} F(A) = F'(A^*)$, *where A^* is the pseudo-complement of A in S_μ.*

LEMMA VIII. *$\mathscr{P}$ is semi-regular if and only if every semi-ideal of P is a union of normal semi-ideals.*

LEMMA IX. *An open subset $F'(A)$ of P is compact if and only if A is a union of a finite number of semi-ideals.*

Lemmas I-IV are proved in [9]; Lemma V is proved in [5] and the rest in [8].

2. Prime semi-ideals in semi-lattices. In this section we obtain some results about prime semi-ideals of a semi-lattice. These are analogues of results obtained by Balachandran [2] for prime ideals of a distributive lattice. Throughout this section S denotes a semi-lattice with 0.

It is easily seen that a subset of S is a prime semi-ideal if and only if its set-complement is a dual ideal and we have the following

THEOREM 1. *A subset A of S is a minimal prime semi-ideal if and only if its set-complement $\complement A$ is a maximal dual ideal.*

Proof. Suppose $\complement A$ is a maximal dual ideal. Then A is a prime semi--ideal. Let $B \subseteq A$, B being a prime semi-ideal. Then $\complement B \supseteq \complement A$. Now $\complement B$ is a dual ideal and $\complement A$ is a maximal dual ideal. Hence $\complement B = \complement A$ and so $B = A$. Thus A is minimal prime.

Conversely, suppose A is a minimal prime semi-ideal and $\complement A \subseteq B$, B being a proper dual ideal of S. Then $\complement B$ is a prime semi-ideal and $A \supseteq \complement B$. Hence, as A is minimal prime, it follows that $A = \complement B$. Consequently, $\complement A = B$. Thus $\complement A$ is a maximal dual ideal.

Remark. By Lemma I, maximal dual ideals exist in S and so Theorem 1 establishes the existence of minimal prime semi-ideals of S.

As a consequence of Lemma I and Theorem 1 we have the following

THEOREM 2. *Any prime semi-ideal of S contains a minimal prime semi-ideal.*

COROLLARY. *The product of all the minimal prime semi-ideals of S is* $(0]$.

The corollary follows from Lemma II and Theorem 2.

THEOREM 3. *If a prime semi-ideal of S meets the lattice-sum of a family of dual ideals, then it meets at least one of them.*

The proof of this theorem is similar to that of the corresponding known result about ideals in a lattice (vide [1], Theorem 3).

The following theorem gives a necessary and sufficient condition for a prime semi-ideal to be minimal prime:

THEOREM 4. *A prime semi-ideal A of S is minimal prime if and only if A contains precisely one of $(x]$, $(x]^*$ for every x in S.*

Proof. Suppose A is minimal prime. Then, by Lemma III, A contains at least one of $(x]$, $(x]^*$. Suppose $A \supseteq (x]$. Then $x \notin \complement A$ and so $\complement A \vee [x) = S$. Hence $xy = 0$ for some $y \in \complement A$. Clearly, $y \in (x]^*$ and $y \notin A$. Therefore $(x]^* \nsubseteq A$.

Conversely, suppose A satisfies the given condition and $x \in A$. Then $(x]^* \nsubseteq A$. Hence there exists $y \in (x]^*$ such that $y \notin A$. Clearly, $y \in \complement A$ and $xy = 0$, so that $0 \in \complement A \vee [x)$. Hence $\complement A \vee [x) = S$. Thus $\complement A$ is maximal. By Theorem 1, it follows that A is a minimal prime semi-ideal.

COROLLARY. *If M is a minimal prime semi-ideal of S, then* $x \in M \Rightarrow (x]^{**} \subseteq M$.

This corollary follows from Theorem 4 and Lemma III.

We obtain below a sufficient condition for a semi-ideal to be contained in a minimal prime semi-ideal. The condition is, in general, not necessary, as can be easily seen by considering a non-principal prime ideal of a Boolean algebra. (In a Boolean algebra every prime semi-ideal is a prime ideal.) However, in the case of a principal ideal, the condition turns out to be necessary.

THEOREM 5. *Any non-dense semi-ideal of S is contained in a minimal prime semi-ideal. Any principal ideal contained in a minimal prime semi--ideal is non-dense.*

Proof. Let A be a non-dense semi-ideal of S. Then $A^* \neq (0]$, and so there exists $x \in A^*$, $x \neq 0$. By Lemma I, there exists a maximal dual ideal M containing $[x)$. Clearly, $x \notin \complement M$ and so $A^* \nsubseteq \complement M$. By Theorem 1, $\complement M$ is a minimal prime semi-ideal and, by Lemma III, $A \subseteq \complement M$.

Now suppose $(a] \subseteq A$, A being a minimal prime semi-ideal of S. Then, by Theorem 4, $(a]^* \nsubseteq A$. Hence $(a]^* \neq (0]$.

The following theorem gives a characterization of the pseudo-complement of a semi-ideal:

THEOREM 6. *The pseudo-complement of a semi-ideal A of S is the product of all the minimal prime semi-ideals not containing A.*

Proof. Let B be the product of all the minimal prime semi-ideals not containing A. By Lemma III it follows that $A^* \subseteq B$. Suppose $A^* \neq B$. Then there exists $x \in B - A^*$. Clearly, $xy \neq 0$ for some $y \in A$. By Lemma I, there exists a maximal dual ideal M containing $[xy)$. Clearly, $x, y \in M$ and so $x, y \notin \complement M$. Consequently, $A, B \nsubseteq \complement M$. This is a contradiction to the choice of B since, by Theorem 1, $\complement M$ is a minimal prime semi-ideal. Hence $A^* = B$.

Since any normal semi-ideal of S is the pseudo-complement of some semi-ideal, we have the following

COROLLARY 1. *Any normal semi-ideal of S is the product of all the minimal prime semi-ideals containing it.*

COROLLARY 2. *Any normal prime semi-ideal of S is minimal prime.*

Corollary 2 is an immediate consequence of Corollary 1.

THEOREM 7. *A necessary and sufficient condition for a principal ideal of S to be a normal semi-ideal is that it is the product of all the minimal prime semi-ideals containing it.*

Proof. In view of Corollary 1, under Theorem 6, we need prove only the sufficiency of the condition. Let

$$(a] = \bigcap_{i \in I} M_i,$$

M_i being minimal prime semi-ideals. By the corollary under Theorem 4 it follows that

$$(a]^{**} \subseteq \bigcap_{i\in I} M_i.$$

Consequently, $(a] = (a]^{**}$, thus proving the result.

COROLLARY. *Any principal ideal, which is a minimal prime semi-ideal, is a normal semi-ideal.*

3. Disjunction semi-lattices. In this section we generalize the work of Balachandran [3] on disjunction lattices.

A poset P with 0 is called a *disjunction poset* if $a, b \in P$ *and* $a \neq b$ imply that there exists $c \in P$ such that exactly one of the ideals $(a] \cap (c]$, $(b] \cap (c]$ is $(0]$.

In the case of a semi-lattice the above definition can obviously be reformulated as follows:

A semi-lattice S with 0 is called *a disjunction semi-lattice* if $a, b \in S$ and $a \neq b$ imply that there exists $c \in S$ such that exactly one of the products ac, bc is 0.

THEOREM 8. *If a disjunction poset P has 1, then 1 is the only dense element of P.*

Proof. Let $a \in P$ and $a \neq 1$. Then, as P is a disjunction poset, there exists $c \in P$ such that exactly one of the ideals $(a] \cap (c]$, $(1] \cap (c]$ is $(0]$. Now $(1] \cap (c] = (c] \neq (0]$. Hence $(a] \cap (c] = (0]$. It follows that a is not dense.

We obtain below a necessary and sufficient condition for a poset to be a disjunction poset.

THEOREM 9. *A poset P with 0 is a disjunction poset if and only if distinct principal ideals of P have distinct pseudo-complements in S_μ.*

Proof. Suppose P is a disjunction poset and $(a]$, $(b]$ two distinct principal ideals of P. Then $a \neq b$ and so there exists $c \in P$ such that exactly one of $(a] \cap (c]$, $(b] \cap (c]$ is $(0]$. Hence c belongs exactly to one of $(a]^*$, $(b]^*$. It follows that $(a]^* \neq (b]^*$.

Conversely, suppose distinct principal ideals of P have distinct pseudo-complements in S_μ. Let $a, b, \in P$ and $a \neq b$. Then $(a] \neq (b]$ and so $(a]^* \neq (b]^*$. Hence there exists $c \in P$ such that c belongs exactly to one of $(a]^*$, $(b]^*$. It follows that exactly one of the ideals $(a] \cap (c]$, $(b] \cap (c]$ is $(0]$. Thus P is a disjunction poset.

For a semi-lattice the notion of disjunction can be sharpened as in the following

LEMMA 1. *A semi-lattice S with 0 is a disjunction semi-lattice if and only if $a, b \in S$ and $a \lneq b$ imply that there exists $c \in S$ such that $ac = 0$ and $bc \neq 0$.*

Proof. Suppose S satisfies the given condition and a, b are any two distinct elements of S. Then $ab \lneq a$ or $ab \lneq b$. Let us take $ab \lneq a$. Then, by hypothesis, there exists $d \in S$ such that $abd = 0$, $ad \neq 0$. Taking $ad = c$, we have $bc = bad = abd = 0$; $ac = aad = ad \neq 0$. Hence S is a disjunction semi-lattice.

The converse is obvious.

We shall now obtain various sets of necessary and sufficient conditions for a semi-lattice to be a disjunction semi-lattice.

THEOREM 10. *A semi-lattice S with 0 is a disjunction semi-lattice if and only if every principal ideal of S is a normal semi-ideal.*

Proof. Let S be a disjunction semi-lattice. Suppose $(a] \subsetneq (a]^{**}$ for some principal ideal $(a]$. Then there exists $b \in (a]^{**} - (a]$. Now $(b] \subseteq (a]^{**}$ and so

$$(1) \qquad (b]^* \supseteq (a]^*.$$

Also $b \nleq a$ and so $ab \lneq b$. Hence, by Lemma 1, there exists $d \in S$ such that $abd = 0$, $bd \neq 0$. Taking $bd = c$, we have $ac = 0$, $bc = bbd = bd \neq 0$. It follows that $c \in (a]^*$, $c \notin (b]^*$. Consequently, $(a]^* \nsubseteq (b]^*$. This contradicts (1). Hence $(a] = (a]^{**}$. Thus $(a]$ is a normal semi-ideal.

Conversely, suppose every principal ideal of S is a normal semi-ideal and $(a] \neq (b]$. Then $(a]^* \neq (b]^*$. Hence, by Theorem 9, S is a disjunction semi-lattice.

COROLLARY. *A semi-lattice S with 0 is a disjunction semi-lattice if and only if every principal ideal of S is the product of all the minimal prime semi-ideals containing it.*

This corollary follows from Theorems 7 and 10.

THEOREM 11. *A semi-lattice S with 0 is a disjunction semi-lattice if and only if every principal dual ideal of S is the product of all the maximal dual ideals containing it.*

Proof. In view of the corollary under Theorem 10, it is sufficient to establish the equivalence of the following two conditions:

(i) Every principal ideal of S is the product of all the minimal prime semi-ideals containing it.

(ii) Every principal dual ideal of S is the product of all the maximal dual ideals containing it.

We shall now show that (i) is equivalent to

(iii) $a, b \in S$ and $a \lneq b$ imply that there exists a minimal prime semi-ideal containing a but not b. That (i) $\Rightarrow$ (iii) is clear. Suppose (iii) holds and A is the product of all the minimal prime semi-ideals containing $(a]$. Suppose $(a] \subsetneq A$. Then there exists $b \in A$ such that $b \nleq a$. Clearly, $ab \lneq b$ and so, by (iii), there exists a minimal prime semi-ideal M such that

$ab \in M$, $b \notin M$. As M is prime, $a \in M$. It follows that $A \nsubseteq M$, $(a] \subseteq M$; this is a contradiction to the choice of A and so $(a] = A$. Thus (iii) $\Rightarrow$ (i). In a similar manner we can show that (ii) is equivalent to:

(iv) $a, b \in S$ and $a \gneq b$ imply that there exists a maximal dual ideal containing a but not b.

By Theorem 1, it follows that (iii) $\Leftrightarrow$ (iv). Consequently, (i) $\Leftrightarrow$ (iii), which completes the proof.

THEOREM 12. *A disjunction semi-lattice closed for pseudo-complements is a Boolean algebra and conversely.*

Proof. Suppose S is a disjunction semi-lattice closed for pseudo-complements and $a \in S$. Then, by Theorem 10, $(a] = (a]^{**} = (a^{**}]$. Hence $a = a^{**}$ and so, by Lemma V, S is a Boolean algebra.

The converse is obvious.

4. The topological space $\mathscr{P}$. This section is a sequel to Section 6 of [8]. Throughout this section S denotes a semi-lattice with 0 and $\mathscr{P}$ the space of prime semi-ideals of S. $\mathscr{N}$ and $\mathscr{N}_1$ denote the set of minimal prime semi-ideals and the set of normal prime semi-ideals of S, respectively. By Corollary 2 under Theorem 6, $\mathscr{N}_1 \subseteq \mathscr{N}$.

THEOREM 13. *If S is a disjunction semi-lattice, $\mathscr{P}$ is semi-regular.*

Proof. Suppose S is a disjunction semi-lattice. Then, by Theorem 10, every principal ideal of S is a normal semi-ideal. Clearly, every semi-ideal is a union of principal ideals. It follows that every semi-ideal of S is a union of normal semi-ideals. Hence, by Lemma VIII, $\mathscr{P}$ is semi-regular.

THEOREM 14. *If S is closed for pseudo-complements, then the exterior of every compact open subset of $\mathscr{P}$ is compact.*

Proof. Suppose S is closed for pseudo-complements and g a compact open subset of $\mathscr{P}$. Then, in view of Lemma IX, we can write

$$g = F'\big((a_1]\cup(a_2]\cup \ldots \cup(a_n]\big),$$

where $a_1, a_2, \ldots, a_n \in S$. By Lemma VII, it follows that $\operatorname{Ext} F'(A) = F'(A^*)$. Hence

$$\begin{aligned}\operatorname{Ext} g &= F'\big(((a_1]\cup(a_2]\cup \ldots \cup(a_n])^*\big) = F'\big((a_1]^*\cap(a_2]^*\cap \ldots \cap(a_n]^*\big)\\ &= F\big((a_1^* a_2^* \ldots a_n^*]\big).\end{aligned}$$

Now the result follows by Lemma IX.

Theorems 15 and 16 proved below generalize the corresponding results of Balachandran [2] on distributive lattices.

THEOREM 15. *The subspace $\mathscr{N}_1$ is discrete.*

Proof. Let $X = \{N_i \mid i \in I\}$ be any subset of $\mathscr{N}_1$ and $N \in \operatorname{Cl} X$. Then

$$N \supseteq \bigcap_{i \in I} N_i.$$

Suppose $N \nsupseteq N_i$ for any $i \in I$. Then, for every $i \in I$, $N \supseteq N_i^*$ and so $N \supseteq \bigcup N_i^*$. It follows that

$$N \supseteq \Big(\bigcap_{i \in I} N_i\Big) \cup \Big(\bigcup_{i \in I} N_i^*\Big).$$

Consequently

$$N^* \subseteq \Big(\bigcap_{i \in I} N_i\Big)^* \cap \Big(\bigcup_{i \in I} N_i^*\Big)^* = \Big(\bigcap_{i \in I} N_i\Big)^* \cap \Big(\bigcap_{i \in I} N_i\Big) = (0].$$

This is contrary to the fact that N is normal. Hence $N \supseteq N_j$ for some $j \in I$. By Corollary 2 under Theorem 6 it follows that $N = N_j$. Thus $N \in X$ and so $\mathrm{Cl}\, X = X$. Hence $\mathcal{N}_1$ is discrete.

THEOREM 16. *The subspace $\mathcal{N}$ is T_3.*

Proof. Since no minimal prime semi-ideal contains any other minimal prime semi-ideal, $\mathcal{N}$ is T_1.

Let X be any non-void closed subset of $\mathcal{N}$ and $A \notin X$ $(A \in \mathcal{N})$. Then $X = \mathcal{N} \cap F(B)$ for some $B \in S_\mu$ and $A \nsupseteq B$. Hence there exists $b \in B$ such that $b \notin A$. By Theorem 4, $A \supseteq (b]^*$. Thus $A \in F\big((b]^*\big)$. Set

$$X_1 = \mathcal{N} \cap F\big((b]\big) \quad \text{and} \quad X_2 = \mathcal{N} \cap F\big((b]^*\big).$$

Then, clearly, $X_1 \supseteq X$, $A \notin X_1$ and $A \in X_2$. Now

$$X \cap X_2 = \mathcal{N} \cap \big(F(B) \cap F\big((b]^*\big)\big) \subseteq \mathcal{N} \cap \big(F\big((b]\big) \cap F\big((b]^*\big)\big)$$
$$= \mathcal{N} \cap F\big((b] \cup (b]^*\big) = \emptyset$$

by Theorem 4. Also

$$X_1 \cup X_2 = \mathcal{N} \cap \big(F\big((b]\big) \cup F\big((b]^*\big)\big) = \mathcal{N} \cap F\big((b] \cap (b]^*\big) = \mathcal{N} \cap F\big((0]\big) = \mathcal{N}.$$

Hence it follows that $\mathcal{N}$ is regular. This completes the proof.

5. A topology for the set of proper dual ideals. Throughout this section S denotes a semi-lattice with 0 and $\mathcal{D}$ the set of all proper dual ideals of S. The set of maximal dual ideals and the set of dual ideals disjoint with a semi-ideal A are denoted by $\mathcal{M}$ and $G(A)$, respectively. $G'(A)$ stands for $\mathcal{D} - G(A)$.

LEMMA 2. $A \subseteq B \Leftrightarrow G(A) \supseteq G(B) \Leftrightarrow G'(A) \subseteq G'(B)$.

Proof. Clearly, $A \subseteq B \Rightarrow G(A) \supseteq G(B)$. Suppose $G(A) \supseteq G(B)$. If M is any prime semi-ideal containing B, $\complement M$ is a dual ideal and $B \cap \complement M = \emptyset$. Hence, by hypothesis, $A \cap \complement M = \emptyset$ and so $M \supseteq A$. By Lemma II, it follows that $A \subseteq B$. Thus

$$A \subseteq B \Leftrightarrow G(A) \supseteq G(B).$$

Since $G'(A) = \mathcal{D} - G(A)$,

$$G(A) \supseteq G(B) \Leftrightarrow G'(A) \subseteq G'(B).$$

Since the set-complement of a dual ideal is a prime semi-ideal, as a consequence of Lemma VI, we have the following

THEOREM 17. (i) $G\left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} G(A_i)$.

(ii) $G(A_1 \cap A_2 \cap \ldots \cap A_n) = G(A_1) \cup G(A_2) \cup \ldots \cup G(A_n)$.

(iii) $G(S) = \emptyset$.

(iv) $G((0]) = \mathscr{D}$.

This theorem shows that G defines a closure operation in $\mathscr{D}$ thereby giving rise to a topology on $\mathscr{D}$.

THEOREM 18. (i) $G'\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} G'(A_i)$.

(ii) $G'(A_1 \cap A_2 \cap \ldots \cap A_n) = G'(A_1) \cap G'(A_2) \cap \ldots \cap G'(A_n)$.

(iii) $G'(S) = \mathscr{D}$.

(iv) $G'((0]) = \emptyset$.

Since $G'(A) = \mathscr{D} - G(A)$, Theorem 18 follows from Theorem 17.

As a consequence of Lemma 2, Theorem 17 and Theorem 18 we have the following

THEOREM 19. *The lattice of all open (closed) sets of $\mathscr{D}$ is isomorphic (dually isomorphic) to S_μ and the mapping $A \to G'(A)$ ($A \to G(A)$) takes arbitrary lattice-sums into corresponding set-unions (set-intersections).*

THEOREM 20. *If X is any subset of $\mathscr{D}$, $\mathrm{Cl}\, X = G(X_0)$, where X_0 is the product of the set-complements of all the members of X.*

Proof. Clearly, $G(X_0)$ is a closed subset of $\mathscr{D}$ containing X. If $G(Y_0)$ is any closed subset of $\mathscr{D}$ containing X, each member of X is disjoint with Y_0 and so the set-complement in S of each member of X contains Y_0. Hence $X_0 \supseteq Y_0$ and so, by Lemma 2, $G(X_0) \subseteq G(Y_0)$. Hence the theorem.

From Theorem 20 it follows that the closure of a singleton set consisting of an element A of $\mathscr{D}$ is the set of all dual ideals disjoint with the set-complement of A; this set is the same as the set of all dual ideals contained in A. Since, of any two distinct dual ideals, one is not contained in the other, it follows that distinct points of $\mathscr{D}$ have distinct closures. Hence we have the following

THEOREM 21. *$\mathscr{D}$ is T_0.*

Remark. Theorems 25 to 32 of [8], which we have proved for $\mathscr{P}$, hold good for $\mathscr{D}$ also. The proofs are similar to those of the corresponding results of [8] but for obvious modifications.

THEOREM 22. (i) $\mathrm{Cl}\, G'(A) = G(A^*)$.

(ii) $\mathrm{Int}\, G(A) = G'(A^*)$.

(iii) $\mathrm{Ext}\, G'(A) = G'(A^*)$.

Proof. (i) $\mathrm{Cl}\, G'(A) = G(B)$, where B is the product of the set-complements of dual ideals meeting A. Clearly, B is the product of all the prime semi-ideals not containing A. Hence $\mathrm{Cl}\, G'(A) = G(A^*)$.

(ii) $\mathrm{Int}\, G(A) = \complement\, \mathrm{Cl}\, G'(A) = G'(A^*)$.

(iii) $\mathrm{Ext}\, G'(A) = \mathrm{Int}\, G(A) = G'(A^*)$.

THEOREM 23. *The subspace $\mathscr{M}$ is T_3.*

Proof. Since no maximal dual ideal of S contains any other maximal dual ideal, the closure of no point of M contains any other point of $\mathscr{M}$. Hence $\mathscr{M}$ is T_1.

Let X be a non-void closed subset of $\mathscr{M}$ and $M \notin X$ ($M \in \mathscr{M}$). Then $X = \mathscr{M} \cap G(A)$ for some $A \in S_\mu$ and $M \cap A \neq \emptyset$. Let $a \in M \cap A$. Since $a \in M$, from Theorems 1 and 4 it follows that $M \in G((a]^*)$. Let $X_1 = \mathscr{M} \cap \cap G((a])$, $X_2 = \mathscr{M} \cap G((a]^*)$. Then, clearly, $X_1 \supseteq X$, $M \notin X_1$ and $M \in X_2$. Now

$$X_1 \cap X_2 = \mathscr{M} \cap \big(G(A) \cap G((a]^*)\big) \subseteq \mathscr{M} \cap \big(G((a]) \cap G((a]^*)\big)$$
$$= \mathscr{M} \cap G\big((a] \cup (a]^*\big) = \emptyset$$

since from Theorems 1 and 4 it follows that no maximal dual ideal can be disjoint with $(a] \cup (a]^*$. Also

$$X_1 \cup X_2 = \mathscr{M} \cap \big(G((a]) \cup G((a]^*)\big)$$
$$= \mathscr{M} \cap G\big((a] \cap (a]^*\big) = \mathscr{M} \cap G((0]) = \mathscr{M}.$$

From this it follows that $\mathscr{M}$ is regular. Hence the result.

We conclude with the following result which is easily proved:

THEOREM 24. *The maximal dual ideals of S are precisely the anti-T_1 points of D.*

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KERALA

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