ON PREPONDERANT MAXIMA

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Let \( \lambda_n \) (respectively, \( \lambda^*_n \)) stand for Lebesgue (respectively, outer Lebesgue) measure on the Euclidean space \( E_n \). We denote by \( B(x, r) \) the closed ball of centre \( x \) and radius \( r \). Let \( A \subseteq E_n \) be an arbitrary set. The number

\[
\overline{D}_x(A) = \lim_{h \to 0^+} \frac{\lambda^*_n(A \cap B(x, h))}{\lambda_n(B(x, h))}
\]

is called the outer upper symmetric density of \( A \) at \( x \).

Let \( f \) be an arbitrary (possibly infinite-valued) function on \( E_n \) and let \( 0 < a \leq 1 \). Following Foran [2] put

\[
\overline{M}_a(f) = \{ x : \overline{D}_x(\{ t : f(t) \geq f(x) \}) < a \}.
\]

Foran [2] proved that \( \overline{M}_a(f) \) is a set of measure zero for \( a = 2^{-n} \) and any \( f \). He raised the problem (P 1019) whether \( a = 2^{-n} \) can be improved.

It is natural to say that \( f \) has a preponderant maximum at \( x \) if \( x \in \overline{M}_{1/2}(f) \) and we will use this terminology. Since for any linear non-constant function \( f \) on \( E_n \) and \( 1/2 < a \leq 1 \) we have obviously \( \overline{M}_a(f) = E_n \), the following theorem solves completely the Foran problem.

**Theorem.** Let \( f \) be an arbitrary function on \( E_n \). Then the set \( \overline{M}_{1/2}(f) \) of all points at which \( f \) has a preponderant maximum is of Lebesgue measure zero.

**Proof.** Let \( u(x) \) be the upper measurable boundary of \( f \) defined by Blumberg in [1]. The upper boundary

\[
u(x) = \inf \{ t : \overline{D}_x(\{ y : f(y) > t \}) = 0 \}
\]

is a measurable (possibly infinite-valued) function. Modifying slightly the proof in [3], p. 504, it is easy to show that \( \lambda_n(\overline{M}_{1/2}(u)) = 0 \) implies \( \lambda_n(\overline{M}_{1/2}(f)) = 0 \). Thus it is sufficient to prove the Theorem only for an arbitrary measurable (possibly infinite-valued) function \( f \).
For integers \( m > 0 \) and \( k > 0 \) define
\[
A_{m,k} = \{ x : \lambda_n(\{ y : f(y) \geq f(x) \} \cap B(x, r)) < (2^{-1}m^{-1})\lambda_n(B(x, r)) \text{ for any } 0 < r < k^{-1} \}.
\]

Since obviously
\[
\mathcal{M}_{1/2}(f) = \bigcup_{m,k=1}^{\infty} A_{m,k},
\]
it is sufficient to show that all sets \( A_{m,k} \) are of measure zero. Suppose on the contrary that for some \( m, k \) we have \( \lambda_n^*(A_{m,k}) > 0 \). Then we can choose \( a \in A_{m,k} \) which is a point of the outer density for \( A_{m,k} \). Choose \( \varepsilon > 0 \) such that
\[
(1) \quad (1 - \varepsilon)(1/2 + 1/m) > 1/2.
\]

Further choose \( \delta > 0 \) such that
\[
(2) \quad \frac{\lambda_n^{*}(A_{m,k} \cap B(a, \delta))}{\lambda_n(B(a, \delta))} > 1 - \varepsilon \quad \text{for } 0 < \Delta < \delta.
\]

Finally, it follows from (1) that we can choose \( r > 0 \) such that
\[
(3) \quad r < \min(\delta, 1/k) \quad \text{and} \quad (1 - \varepsilon)(1/2 + 1/m)(\delta^n + (\delta - r)^n) > \delta^n.
\]

Define the subset \( C \) of \( E_{2n} = E_n \times E_n \) as
\[
C = \{ (x, y) : x \in B(a, \delta), y \in B(x, r) \}
\]
and put \( S = C \cap \{ (x, y) : f(x) > f(y) \} \). The sets \( C \) and \( S \) are obviously measurable and by the Fubini theorem we have
\[
\lambda_{2n}(S) = \int_{B(a, \delta)} g(x) d\lambda_n(x),
\]
where \( g(x) = \lambda_n(\{ y \in B(x, r) : f(x) > f(y) \}) \). Let \( V_n \) denote the volume of the unit ball in \( E_n \). By the definition of \( A_{m,k} \), for \( x \in A_{m,k} \cap B(a, \delta) \) we have \( g(x) \geq (1/2 + 1/m)V_n^{1/n}\delta^n \). Therefore, using (2) we obtain
\[
(4) \quad \lambda_{2n}(S) \geq (1/2 + 1/m)(1 - \varepsilon)V_n^{1/n}\delta^n.
\]

Further, put
\[
T = \{ (x, y) : y \in B(a, \delta - r), x \in B(y, r), f(x) < f(y) \}.
\]

Obviously, \( T \subset C \), \( T \cap S = \emptyset \), and \( T \) is measurable. By the Fubini theorem we have
\[
\lambda_{2n}(T) = \int_{B(a, \delta - r)} h(y) d\lambda_n(y),
\]
where \( h(y) = \lambda_n(\{ x \in B(y, r) : f(x) < f(y) \}) \).
Similarly as above we obtain

\begin{equation}
\lambda_{2n}(T) \geq (1/2 + 1/m)(1 - \varepsilon) V_n^2 (\delta - r)^n.
\end{equation}

Using (4) and (5) we get

\begin{align*}
V_n^2 \delta^n r^n &= \lambda_{2n}(C) \geq \lambda_{2n}(S) + \lambda_{2n}(T) \\
&\geq V_n^2 r^n (1/2 + 1/m)(1 - \varepsilon)(\delta^n + (\delta - r)^n),
\end{align*}

which contradicts (3).

REFERENCES


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