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On the monotonicity of nonnegative solutions and the uniqueness of eigenvalues*

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Dedicated to the memory of Jacek Szarski

Abstract. We consider a second order nonlinear differential equation of the form $u''+T(t)U(u) = 0$. In Part I, we prove some monotonicity properties of nonnegative solutions. In Part II, using the results of Part I, we deal with the question of uniqueness of eigenvalues of some associated boundary value problems. Motivations and statements of results are given in the first sections (Sections 1 and 6) of each part.

Part I

1. Statement of results. Some results in recent mathematical physics literature can be formulated in terms of singular boundary value problems on $(-\infty \leqslant)\ a < t < \omega\ (\leqslant \infty)$ involving a differential equation of the form

(1.1) $$u''+T(t)U(u) = 0,$$

and interest lies in solutions $u(t)$ on (a, ω) such that

(1.2) $$u(a) = \lim_{t\to a} u(t) \quad \text{exists and} \quad u(t) \geqslant u(a) \quad \text{on } (a, \omega).$$

For example, suppose that one boundary condition is

(1.3) $$u(t) \to 0 \quad \text{as } t \to a$$

and that

(1.4) $$0 \not\equiv u(t) \geqslant 0 \quad \text{on } (a, \omega).$$

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We shall assume that the functions $T(t)$, $U(u)$ satisfy

$$0 < T \in C^0(a, \omega) \quad \text{and} \quad U \in C^0[0, \infty), \tag{1.5}$$

$$T(t) \text{ is strictly monotone on } (a, \omega). \tag{1.6}$$

For examples of such problems, see [1], [3], [4], [6], [8], [9], [10] and earlier references in these papers. A related type of problem was recently discussed by J. Serrin in a lecture at Johns Hopkins University. In [6], the problem is formulated as above; in the others, the results concern problems for the partial differential equation

$$\Delta v + U(v) = 0, \tag{1.7}$$

which yield radially symmetric solutions $v(r) \not\equiv 0$ on $0 < r < R$ $(\leqslant \infty)$, where $r = |x|$, $x \in \boldsymbol{R}^n$, and $n \geqslant 2$,

$$0 \leqslant v \to 0 \quad \text{as } r = |x| \to R \ (\leqslant \infty). \tag{1.8}$$

In $n > 2$ or $n = 2$ and $0 < R < \infty$, the change variables

$$u(t) = v(r), \quad t = \int_r^R s^{1-n} ds$$

transforms (1.7)–(1.8) into (1.1), (1.3)–(1.4), where $0 < r < R$ corresponds to $\infty > t > 0$ (and $T(t) = \text{const}/t^{2(n-1)/(n-2)}$ if $n > 2$). If $n = 2$ and $R = \infty$, the change of variables

$$u(t) = v(r), \quad t = \int_r^1 ds/s = -\log r$$

transforms (1.7)–(1.8) into (1.1), (1.3)–(1.4), where $0 < r < R = \infty$ corresponds to $\infty > t > -\infty$ (and $T = e^{-2(n-1)t}$).

Papers [1], [3], [6], [8], [9], [10] give existence theorems for the problems treated (with $(a, \omega) = (0, \infty)$ or $R = \infty$), and in [1], [3] and [10] it is also verified that the solutions constructed satisfy

$$u'(t) \geqslant 0 \quad \text{on } a < t < \omega. \tag{1.9}$$

Existence in these three papers is proved by considering an appropriate problem in the calculus of variations, while the radial symmetry and the analogue of (1.9) is obtained from symmetrization principles. In [4], it is shown, by the use of maximum principles, that if $U \in C^1$ and $v \in C^2\{|x| \leqslant R < \infty\}$ is a solution of (1.7)–(1.8) with $v(x) > 0$ for $|x| < R$, then v is radially symmetric and satisfies the analogue of (1.9). It turns out that the solutions in [6] also satisfy (1.9). In fact, the main result to be proved in Part I is the following

THEOREM 1.1. *Let* $T(t)$, $U(u)$ *satisfy* (1.5)–(1.6). *Let* $u = u(t)$ *be a solution of* (1.1) *satisfying* (1.3) *and* (1.4).

(i) *Then either* (1.9) *holds or there exists a t-value* $t = t_0$, $a < t_0 < \omega$, *such that*

$$(1.10) \qquad u'(t) \geqslant 0 \quad on\ (a, t_0) \quad and \quad u'(t) < 0 \quad on\ (t_0, \omega);$$

in which case, there are t-values β *and* t^0, $a \leqslant \beta < t^0 \leqslant t_0$, *such that*

$$(1.11) \quad u' > 0 \quad on\ (\beta, t^0), \quad u' \equiv 0 \quad on\ [t^0, t_0], \quad and \quad u(\beta) = u(\omega).$$

(ii) *If* $\omega = \infty$, *or* $\int^{\omega} T^{1/2} dt = \infty$ *or, more generally, if*

$$(1.12) \qquad \liminf u'^2/T = 0 \quad as\ t \to \omega,$$

then (1.9) *holds.*

(iii) *Also, if solutions of* (1.1) *are uniquely determined by initial conditions* (*e.g., if* $U(u)$ *is locally uniformly Lipschitz continuous*), *then* $u' \geqslant 0$ *can be replaced by* $u' > 0$ *in* (1.9) *and* (1.10) *in assertions* (i) *and* (ii).

Note that, in (i) and (ii), there is no assumption on $U(u)$ except continuity, and the only assumption on u is that it *attains* its minimum at $t = a$. The example in [4], where (1.1) is

$$(1.13) \quad u'' + (u-1) = 0 \quad \text{and} \quad u = 1 - \cos t, \quad a = 0,\ \omega = \infty$$

(i.e., $T(t) \equiv 1$ and $U(u) = u - 1$) shows that *strictly* cannot be omitted in hypothesis (1.6).

COROLLARY 1.1. *Let* $n \geqslant 2$ *and* $U(v) \in C^0[0, \infty)$. *Let* $a \geqslant 0$ *and* $0 < \varrho(r) \in C^0(a, \infty)$ *be a nondecreasing function or, more generally, let*

$$(1.14) \qquad r^{2(n-1)} \varrho(r) \quad \textit{be strictly monotone}.$$

Let $v = v(r)$, $r = |x|$ *and* $x \in \mathbf{R}^n$, *be a radially symmetric* C^2-*solution of*

$$(1.15) \qquad \Delta v + \varrho(|x|) U(v) = 0$$

on $(0 \leqslant)\ a < |x| < \infty$ *satisfying*

$$(1.16) \qquad 0 \not\equiv v \geqslant 0 \quad and \quad v \to 0, \quad r \to \infty.$$

Then $dv/dr \leqslant 0$ *on* $a < r < \infty$.

This is not contained in [4] since U does not satisfy a Lipschitz condition and it is not assumed that $v > 0$; cf. [4], Section 2.3.

Remark. Theorem 1.1 implies that the solutions y_k, where $k = 0, 1, \ldots$, in Theorem 3.1 of [6], have nonvanishing derivatives to the right of their largest zero.

The same is true for Theorem 3.2 if the condition $r' \leqslant 0$ is strengthened to $r' < 0$ and if g is locally uniformly Lipschitz.

For comparison of Theorem 1.1 and a result of [4] on ordinary differential equations, we state the following consequence of Theorem 3 in [4]. According as $T(t)$ is decreasing or increasing, it may supply additional information on (a, t^0) or on (t_0, ω) in (1.10) or (1.11).

THEOREM 1.2. *Let $T(t)$, $U(u)$ satisfy* (1.5) *and suppose that $T(t)$ is nonincreasing and that $U(u)$ is locally uniformly Lipschitz continuous. Let $u(t)$ be a solution of* (1.1) *on (a, ω) satisfying*

(1.17) $$u > 0 \quad on \ (a, \omega) \quad and \quad u(a+0) = 0.$$

If $u'(t) = 0$ for some $t \in (a, \omega)$, then there is a t-value t_0 such that

(1.18) $$u' > 0 \quad on \ (a, t_0), \quad u'(t_0) = 0,$$

(1.19) $$\int_{t_0}^{\omega} T^{1/2} dt \leqslant \int_{a}^{t_0} T^{1/2} dt \leqslant \infty;$$

also, if the second integral is finite and

(1.20) $$\int_{t_0}^{\omega} T^{1/2} dt = \int_{a}^{t_0} T^{1/2} dt \ (< \infty),$$

then $T(t)$ is a constant, $-\infty < a < \omega < \infty$, and u is symmetric with respect o $t = t_0$.

Notice, in contrast to Theorem 1.1, this theorem assumes a Lipschitz condition on U, replaces (1.6) by T *is nonincreasing*, and (1.3)–(1.4) by (1.18). We indicate the proof of Theorem 1.2 in Section 5 below. When (1.1) is a linear equation $U(u) = u$ and $T(t) \in C^0[a, \omega]$, where $-\infty < a < \omega < \infty$, Theorem 1.2 is an immediate consequence of the Sturm comparison theorems; cf., e.g., [7], p. 531.

2. Proof of Theorem 1.1(i). Suppose, if possible, that (1.9) does not hold, so that there is a t-value $t = t_1$ such that

(2.1) $$u'(t_1) < 0 \quad \text{and} \quad a < t_1 < \omega$$

(hence $u(t_1) > 0$). Then there exists a t-value $t = t_0$,

(2.2) $$0 < t_0 < t_1 \quad \text{and} \quad u(t_0) = \max u(t) \quad \text{on } (a, t_1],$$

so that $u(t_0) > 0$ and

(2.3) $$u'(t_0) = 0.$$

We have the following two possibilities:

Case (1). There exists a smallest t-value $t = t_2$,

(2.4) $$t_1 < t_2 \quad \text{and} \quad u'(t_2) = 0.$$

Case (2). Or t_2 does not exist, so that

(2.5) $$u'(t) < 0 \quad \text{on } t_1 < t < \omega.$$

Consider the *conjugate energy*, i.e., the Lyapunov function

(2.6) $H(t) = u'^2/2T + V(u)$, where $u = u(t)$ and $a < t < \omega$,

$$V(u) = \int_0^u U(s)\,ds. \tag{2.7}$$

Then $H(t) \in C^0(a, \omega)$ is locally of bounded variation. In the sense of Riemann-Stieltjes integration, (1.1) implies that

$$dH(t) = (u'^2/2)\,d(1/T), \tag{2.8}$$

so that H is monotone.

For sake of definiteness, assume that T is strictly decreasing, so that H is nondecreasing. (The case that T is strictly increasing is similar.) Let $u_0 = u(t_0)$. Then $u'(t_0) = 0$ and (2.6), (2.8) show that

$$H(t) \equiv u'^2/2T + V(u) \leqslant V(u_0) \quad \text{for } a < t \leqslant t_0. \tag{2.9}$$

In particular,

$$V\big(u(t)\big) \leqslant V(u_0) \quad \text{for } a < t \leqslant t_0, \tag{2.10}$$

so that, by (2.2),

$$V(u) \leqslant V(u_0) \quad \text{for } 0 \leqslant u \leqslant u_0. \tag{2.11}$$

On Case (1). Assume the existence of $t = t_2$ as in Case (1). Then, a similar argument gives $V\big(u(t)\big) \leqslant V(u_2)$ for $a < t \leqslant t_2$ if $u_2 = u(t_2)$. In particular $V(u_0) \leqslant V(u_2)$. But $0 \leqslant u_2 < u_0$ and (2.11) imply that $V(u_2) \leqslant V(u_0)$. Hence $V(u_0) = V(u_2)$. It follows from (2.3), (2.4) and (2.6) that $H(t_0) = H(t_2)$, hence $H(t) \equiv H(t_0)$ on $t_0 \leqslant t \leqslant t_2$. Consequently, (1.6) and (2.8) show that $u'(t) \equiv 0$ on $t_0 \leqslant t \leqslant t_2$. In particular, $u_2 = u_0$. But the definitions of t_0 and t_2 give $u_0 > u(t_1) > u_2$. This contradicts he assumption that t_2 exists, so that Case (1) cannot occur.

On Case (2). The arguments just completed show that (1.10) holds in Case (2). In order to verify the conclusion (1.11), let $u(\omega) = \lim u(t)$ as $t \to \omega$, so that $u(t_0) > u(\omega) \geqslant 0$. Let β be the largest t-value on $a \leqslant t < t_0$ such that $u(\beta) = u(\omega)$ (where it is understood that $u(a) = 0$). Let $t = t^0$ be the smallest t-value, $\beta < t \leqslant t_0$, such that $u(\beta) < u(t^0) = u(t_0)$. The arguments above, in which we replace t by $-t$ and a by ω (and ω by β), imply (1.11). This completes the proof of (i).

3. Proof of Theorem 1.1(ii). First assume (1.12). We shall show that (1.9) holds. If not, (1.10) holds. Let $t_2 = \omega$ and $u_2 = \lim u(t)$ as $t \to \omega$, and $H(\omega) = V(u_2)$. Arguing as in Case (1) of the last section, it follows that $V(u_2) = V(u_0)$ and that $H(t) \equiv H(u_0)$ for $t_0 \leqslant t \leqslant \infty$. We now obtain the same contradiction as in Case (1) above.

Note that if

$$(3.1) \qquad u \geqslant 0 \quad \text{and} \quad u' \leqslant 0 \quad \text{for } t \text{ near } \omega,$$

then $\lim u(t)$, $t \to \omega$, exists. Thus, from (2.6) and (2.8), it follows that H, hence u'^2/T, has a limit, possibly ∞. If

$$(3.2) \qquad u'^2/T \to 0 \quad \text{as } t \to \omega$$

does not hold, there is a constant $c > 0$ such that $-u' \geqslant cT^{1/2}$ for t near ω. By (3.1),

$$(3.3) \qquad -\int^{\omega} u'dt < \infty, \quad \text{hence} \quad \int^{\omega} T^{1/2}(t)dt < \infty.$$

Thus the proof of (ii) will be complete if we verify

PROPOSITION 3.1. *Assume* (1.5) *with* $\omega = \infty$ *and* T *is monotone. Let* $u(t)$ *be a solution of* (1.1) *satisfying* (3.1). *Then* (3.2) *holds.*

Proof. From the argument leading to (3.3), we can suppose that $T > 0$ is nonincreasing to 0 and that

$$(3.4) \qquad \int^{\infty} T^{1/2}dt < \infty, \quad \text{hence} \quad \int^{\infty} Tdt < \infty.$$

Equation (1.1) and the boundedness of u shows that $|u''| \leqslant CT$ for large t and some constant C. The last part of (3.4) implies that $\int^{\infty} |u''|dt < \infty$, so that u' has a limit as $t \to \infty$, which is 0 by (3.3). Hence

$$0 \leqslant -u'/T^{1/2} \leqslant C \int_t^{\infty} T(s)T^{-1/2}(t)dt \leqslant C \int_t^{\infty} T^{1/2}(s)ds,$$

since T is nonincreasing. This proves (3.2).

4. Proof of Theorem 1.1(iii). Let $u(t)$ be a solution of (1.1) satisfying $u' \geqslant 0$ on some open interval (a, t_0) and that $u'(t^*) = 0$ for some $t^* \in (a, t_0)$. Then $T > 0$ implies that $U(u^*) = 0$, where $u^* = u(t^*)$. For otherwise, u has a strict maximum or minimum at $t = t^*$, contradicting $u' \geqslant 0$. Consequently, $u(t) \equiv u^*$ by the uniqueness of solutions of (1.1). This is impossible if $u(a) < u(t_0)$, so that $u' > 0$ on (a, t_0).

5. Proof of Theorem 1.2. We merely indicate the proof. Since solutions of (1.1) are uniquely determined by initial conditions, we can suppose that $T \in C^1$ and that $\int_a T^{1/2}dt < \infty$. For otherwise, we approximate (a, ω) by compact intervals and T by smooth functions. The change of variables

$$u(t) = v(x) \quad \text{and} \quad x = \int_a^t T^{1/2}(s)ds$$

changes (1.1) into

$$d^2v/dx^2 + b(x)dv/dx + U(v) = 0, \quad \text{where } b(x) \equiv T'/2T^{3/2} \leqslant 0.$$

This makes it clear that Theorem 1.2 is a consequence of Theorem 3 in [4].

Part II

6. Statement of results. Consider the differential equation

$$(6.1) \qquad u'' + T(t)U(u) = 0,$$

where $-\infty \leqslant a < t < \omega \leqslant \infty$. A number of singular or nonsingular boundary value problems for (6.1) can be formulated as follows: Let $u = u(t, \lambda)$ be a solution of (6.1), say, determined by a condition at $t = a$. For example, if (a, ω) is replaced by $[a, \omega)$, $-\infty < a < \omega \leqslant \infty$, we can let $u(t, \lambda)$ be determined by

$$(6.2) \qquad u = 0 \quad \text{and} \quad u' = \lambda \ (>0) \quad \text{at } t = a.$$

The boundary value problem involves the question of existence and uniqueness of λ-values such that $u(t, \lambda)$ satisfies a *condition at* $t = \omega$, e.g., a *smallness* or L^2-*condition* at $t = \omega$, or if (a, ω) is replaced by $(a, \omega]$, $-\infty \leqslant a < \omega < \infty$, the boundary condition can be

$$(6.3) \qquad u = 0 \quad \text{or} \quad u' = 0 \quad \text{at } t = \omega.$$

Often, an auxiliary condition is also imposed, namely,

$$(6.4) \qquad u \text{ has exactly } k \text{ zeros on } (a, \omega),$$

where $k = 0, 1, \ldots,$ is fixed. For analogous formulations of some singular boundary value problems, see, e.g., [6] and earlier references there.

In [2], Coffman discusses a uniqueness question for a more general equation, but when (a, ω) is replaced by $[a, \omega]$ and the corresponding boundary conditions are $u(a) = u(\omega) = 0$ or $u(a) = u'(\omega) = 0$. His results are most complete for an equation of the form (6.1). The following is an improvement of his result (with a somewhat neater and more natural arrangement of his proof, but with the additional assumption that $T \in C^2$, instead of $T \in C^1$ and $T^{-1/2}$ is concave).

THEOREM 6.1. *Let* $T(t) \in C^2(a, \omega)$, $-\infty \leqslant a < \omega \leqslant \infty$, *satisfy*

$$(6.5) \qquad T > 0, \quad T' < 0, \quad (T^{-1/2})'' \leqslant 0,$$

and let $U(u) \in C^1(-\infty, \infty)$ *satisfy*

$$(6.6) \qquad U(0) = 0 \quad \textit{and} \quad U_u > U/u \quad \textit{for } u \neq 0,$$

where $U_u = \partial U/\partial u$. *For a fixed* λ, $0 < \lambda < \Lambda$ $(\leqslant \infty)$, *let* $u = u(t, \lambda)$ *be a solution of* (6.1) *on* (a, ω) *such that* $u, u' \in C^1[(a, \omega)\times(0, \Lambda)]$,

$$(6.7) \qquad u > 0 \quad \textit{for } t\ (> a) \quad \textit{near } a \quad (\textit{for fixed } \lambda),$$

$$(6.8) \qquad \limsup_{t\to a}(u_\lambda u' - u'_\lambda u) \geqslant 0,$$

$$(6.9) \qquad \limsup_{t\to a} T^{-1/2}[u'u'_\lambda + (TU + T'u'/2T)u_\lambda] \geqslant 0.$$

Then the zeros of $u_\lambda = \partial u/\partial\lambda$, *if any, cannot cluster at* $t = a$. *Assume that*

$$(6.10) \qquad u_\lambda > 0 \quad \textit{for } t\ (> a) \quad \textit{near } a \quad (\textit{for fixed } \lambda).$$

If the ordered zeros of $u(\cdot, \lambda)$, *if any, are*

$$(6.11) \qquad (a <)\ t_1(\lambda) < t_2(\lambda) < \ldots$$

then $t_k(\lambda)$ *satisfies*

$$(6.12) \qquad dt_k(\lambda)/d\lambda < 0$$

on its interval of existence. Furthermore, the zeros of $u'(\cdot, \lambda)$, *if any, are simple (hence isolated) and do not cluster at* $t = a$. *If* $(a <)\ s_1(\lambda) < s_2(\lambda) < \ldots$ *are the ordered zeros of* $u'(\cdot, \lambda)$, *then*

$$(6.13) \qquad (a <)\ s_1(\lambda) < t_1(\lambda) < s_2(\lambda) < \ldots,$$

and $s_k(\lambda)$ *satisfies*

$$(6.14) \qquad ds_k(\lambda)/d\lambda < 0$$

on its interval of existence. In particular, for any $b \in (a, \omega)$, *there is at mos one* λ-*value such that* $u = 0$ [*or* $u' = 0$] *at* $t = b$ *and* u *has exactly* k *zeros on* (a, b), $k = 0, 1, \ldots$

In Coffman's results (in his Theorem 2.4 and the remarks following it), a above is replaced by a nonsingular point at which (6.2) holds, so that conditions (6.7)–(6.10) are trivial. Also, he assumes that $U(u)$ is an odd function and that $U/u > 0$ for $u \neq 0$. (He makes a redundant assumption $(t^2T(t))' \geqslant 0$, which is implied by the other conditions of (6.5), in which $(T^{-1/2})'' \leqslant 0$ is replaced by an assumption equivalent to $T^{-1/2}$ is concave.)

COROLLARY 6.1. *In addition to the hypothesis of Theorem* 6.1, *assume that* $U(u)/u > 0$ *for* $u \neq 0$ *and that* γ, δ *are nonnegative constants*, $\gamma + \delta > 0$. *When it exists, let* $t = s_1^*(\lambda)$ *be the unique* t-*value on* $(a, s_1(\lambda)]$ *and* $t = s_k^*(\lambda)$, $k > 1$, *the unique* t-*value on* $[t_{k-1}(\lambda), s_k(\lambda)]$, *where*

$$(6.15) \qquad \gamma u'(t, \lambda) - \delta u(t, \lambda) = 0.$$

Then, on its interval of existence, $s_k^*(\lambda)$ *satisfies*

$$(6.16) \qquad ds_k^*(\lambda)/d\lambda < 0.$$

In particular, if $a < b < \omega$, then there is at most one λ-value such that (6.15) *holds at $t = b$ and u has exactly k zeros on (a, b), $k = 0, 1, \ldots$*

The proof of this corollary is similar to the proof of (6.14) and will be omitted.

We shall not consider the more interesting situation where $(T^{-1/2})'' \geqslant 0$.

Remark. On (6.8)–(6.9). As remarked above, (6.8)–(6.9) is trivial if a is a (finite) nonsingular point and (6.2) holds. It may also be possible to verify (6.8)–(6.9) in some singular cases. For example, assume that

$$\int_a T^{1/2}dt = \infty \quad \text{and} \quad \int_a |5T'^2/16T^3 - T''/4T^2|T^{1/2}dt < \infty,$$

that $U_u(0) = -c^2$ with $c > 0$, and that $U(u)/u \in C^1$. Then (6.1) has a family of solutions $u(t, \lambda)$ satisfying, as $t \to a$,

$$u \sim \lambda T^{-1/4}\exp \quad \text{and} \quad u' \sim c\lambda T^{1/4}(1 - T'/4T^{3/2})\exp,$$

where

$$\exp = \exp\Big(-c\int_t^\alpha T^{1/2}(s)ds\Big), \qquad \alpha \in (a, \omega) \text{ fixed}.$$

These asymptotic formulas can be differentiated with respect to λ. Thus, for example, (6.8)–(6.9) hold with $\limsup \geqslant 0$ replaced by $\lim = 0$ if

$$(T'^2/T^3 - T'/T^{3/2})(\exp)^2 \to 0 \quad \text{as } t \to a.$$

(For example, let $a = 0$ and $T \sim 1/t^2$, $T' \sim -2/t^3$, $T'' \sim 6/t^4$ as $t \to 0$.)

7. Proof of Theorem 6.1. We can suppose that $u(\cdot, \lambda)$ has some zeros, enumerated as in (6.11). It follows from Theorem 1.1 (iii) in Part I above that $u'(\cdot, \lambda)$, hence

$$w = g(t)u'(t, \lambda), \quad \text{where } g = T^{-1/2} > 0, \tag{7.1}$$

has at most one zero on $(a, t_1(\lambda))$ and on each of the intervals $(t_{j-1}(\lambda), t_j(\lambda))$. Let the zeros of w (or u') be $(a <)\ s_1(\lambda) < s_2(\lambda) < \ldots$, so that (6.13) holds. It is easy to verify, from (6.1) and (7.1), that w satisfies the differential equation

$$w'' + [T(t)U_u(u) - g''(t)/g(t)]w = 0, \quad \text{where } u = u(t, \lambda). \tag{7.2}$$

We write (6.1) as a linear differential equation

$$u'' + T(t)[U(u)/u]u = 0, \quad \text{where } u = u(t, \lambda). \tag{7.3}$$

Differentiation of (6.1) with respect to λ shows that

$$v = u_\lambda(t, \lambda) \tag{7.4}$$

satisfies

(7.5) $$v''+T(t)U_u(u)v = 0, \quad \text{where } u = u(t, \lambda).$$

Since $g'' \leqslant 0$ in (7.2), the Sturm comparison theorem (cf. [5], p. 333–337) applied to (7.2), (7.5) shows that w has a zero between any pair of zeros of v. Hence the zeros of $v = u_\lambda$ cannot cluster at $t = a$ and, if any, can be ordered as $(a<)\ \tau_1(\lambda) < \tau_2(\lambda) < \ldots$ Differentiation of (7.1) gives

$$w' = -gTU+g'u'.$$

From this, it follows that (6.9) is equivalent to

(7.6) $$\limsup_{t\to a}(wv'-w'v) \geqslant 0.$$

Hence $g'' \leqslant 0$, (7.2), (7.5) and the Sturm comparison theorem show that the zeros $\tau_j(\lambda)$ of v satisfy

(7.7) $$s_j(\lambda) \leqslant \tau_j(\lambda),$$

and every interval $[\tau_{j-1}(\lambda), \tau_j(\lambda))$ contains a zero of w.

Similarly, it follows from (6.8), (7.3), (7.5), $U_u > U/u$ and the Sturm comparison theorems that if u has at least k zeros, then v has at least k zeros and

(7.8) $$\tau_j(\lambda) < t_j(\lambda),$$

and that v has at least one zero on every open interval $(t_{j-1}(\lambda), t_j(\lambda))$. Furthermore, if u and v have the same number of zeros on (a, t^*) for some t^*, then

(7.9) $$u'/u > v'/v \quad \text{at } t = t^*.$$

Inequalities (7.7), (7.8) and the remarks following them imply that

(7.10) $$(a<)\ s_1(\lambda) \leqslant \tau_1(\lambda) < t_1(\lambda) < s_2(\lambda) \leqslant \tau_2(\lambda) < t_3(\lambda) < \ldots$$

Consequently,

(7.11) $$(-1)^k u' > 0 \quad \text{and} \quad (-1)^k v > 0 \quad \text{at } t = t_k(\lambda).$$

By differentiating $u(t_k(\lambda), \lambda) \equiv 0$ with respect to λ, it is seen that $dt_k/d\lambda = -v(t_k, \lambda)/u'(t_k, \lambda) < 0$. This proves (6.12).

In order to verify (6.14), let $\sigma_1(\lambda)$ be the largest zero of v' on $(a, \tau_1(\lambda))$ and $\sigma_k(\lambda)$ the largest zero of v' on $(\tau_{k-1}(\lambda), \tau_k(\lambda))$, so that $(-1)^{k+1}v(\sigma_k, \lambda) > 0$. From (7.9) and (7.10), it follows that

(7.12) $$u'/u > 0 \quad \text{at } t = \sigma_k(\lambda).$$

Hence $u'(\sigma_1, \lambda) > 0$, i.e., $\sigma_1(\lambda) < s_1(\lambda)$. Also $u'/u < 0$ on $(s_1, t_1) \supset (\tau_1, t_1)$, so that $t_1 < \sigma_2 < \tau_2$, where $u < 0$. Thus (7.12) gives $u' < 0$ at $t = \sigma_2(\lambda)$, so that $\sigma_2(\lambda) < s_2(\lambda)$. Continuing this argument, it follows that

$$\sigma_k(\lambda) < s_k(\lambda), \quad \text{hence } (-1)^k v'(s_k, \lambda) > 0. \tag{7.13}$$

The condition $U_u > U/u$ in (6.6) means that U/u is increasing for $u > 0$ and decreasing for $u > 0$. Hence either $U > 0$ on $(0, \infty)$, or $U < 0$ on $(0, \infty)$, or there is a u-value u_0 such that $U < 0$ on $(0, u_0)$ and $U > 0$ on (u_0, ∞). An analogous assertion holds for $-\infty > u > 0$. Since u' has exactly one zero $s_1(\lambda)$ on (a, t_1) and exactly one zero $s_k(\lambda)$ on (t_{k-1}, t_k), a simple argument involving convexity shows that $U > 0$ or $U < 0$ at $u = u(s_k, \lambda)$ according as $u(s_k, \lambda) > 0$ or < 0. Since $(-1)^{k+1}u(s_k, \lambda) > 0$, we have $(-1)^{k+1} U > 0$ for $u = u(s_k, \lambda)$. Hence

$$(-1)^k u'' = (-1)^{k+1} TU > 0 \quad \text{at } t = s_k(\lambda).$$

Differentiating $u'(s_k(\lambda), \lambda) \equiv 0$, we get $ds_k(\lambda)/d\lambda = -v'(s_k, \lambda)/u''(s_k, \lambda) < 0$. This is (6.14), and completes the proof.

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