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RECENT RESULTS IN SYMPLECTIC GEOMETRY

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In the mid-sixties, Vladimir Arnol'd stated remarkable conjectures in symplectic geometry, related to celestial mechanics and the Poincaré–Birkhoff fixed point theorem. The first of these problems ([A1], Appendix 9) was solved at the end of 1982 by Charles Conley and Eduard Zehnder [CZ82], using rather elementary ideas. Since then, much progress has been accomplished in the field, with the help of various techniques, some rather hard, some much softer. We shall concentrate on the latter, applied first to the Arnol'd conjectures, and then to a related conjecture of Alan Weinstein – first proved by Claude Viterbo by a quite different method (V86).

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1. Work on the Arnol'd conjectures

1.1. Phase functions on compact manifolds

A *phase function* on a manifold M is a smooth function $M \times E \to \mathbf{R}$, where E denotes a finite-dimensional real vector space. Such an F is called *quadratic* when there exists a *non-degenerate* quadratic form Q on E such that the mapping

$$M \times E \ni (a, v) \to \frac{\partial F}{\partial v}(a, v) - dQ(v) \in E^*$$

is bounded. The following result is classical (see, for instance, [ChZ83], pp. 82–95):

THEOREM. *The number of critical points of a quadratic phase function on a compact manifold M is greater than the cup-length* $\mathrm{cl}(M)$ *of M in any case, and at least equal to the sum* $\mathrm{SB}(M)$ *of its Betti numbers when none of these critical points is degenerate.*

EXAMPLE. $\mathrm{cl}(T^n) = n$ and $\mathrm{SB}(T^n) = 2^n$.

1.2. First statement and proof of the Conley–Zehnder theorem

Let $T^* \boldsymbol{R}^n = \boldsymbol{R}^n \times (\boldsymbol{R}^n)^*$ be endowed with its canonical vector space structure. By a *lattice Z of* $T^* \boldsymbol{R}^n$ (resp. $\boldsymbol{R}^n$), we mean a discrete subgroup such that the quotient $T^* \boldsymbol{R}^n / Z$ (resp. $\boldsymbol{R}^n / Z$) is compact–hence diffeomorphic to a torus.

THEOREM (CZ82). *Given a lattice Z of* $T^* \boldsymbol{R}^n$, *let* H: $T^* \boldsymbol{R}^n \times [0, 1] \to \boldsymbol{R}$ *be smooth and Z-periodic with respect to the variable* $(q, p) \in \boldsymbol{R}^n \times (\boldsymbol{R}^n)^* = T^* \boldsymbol{R}^n$. *Consider the hamiltonian system*

$$\dot{q} = -\frac{\partial H}{\partial p}(q, p, t) \quad \textit{and} \quad \dot{p} = \frac{\partial H}{\partial q}(q, p, t) \tag{1}$$

and the boundary condition

$$q(0) = q(1) \quad \textit{and} \quad p(0) = p(1). \tag{2}$$

Then, one can find at least $2n+1$ *solutions of* (1)–(2) *which cannot be deduced from one another by Z-translation, and at least* 2^{2n} *if* (1)–(2) *has no degenerate solution.*

Proof. We shall construct a quadratic phase function F^H on $T^* \boldsymbol{R}^n / Z$, the critical set of which is in 1-1 correspondence with the mod Z solutions of (1)–(2).

Given a positive integer N and arbitrary points $v_i = (q_i, p_i) \in T^* \boldsymbol{R}^n$, $0 \leqslant i \leqslant N$, let $v_{i+1}^H(v_i) = \left(q_{i+1}^H(v_i), p_{i+1}^H(v_i)\right) \in T^* \boldsymbol{R}^n$, $0 \leqslant i \leqslant N$, be given by

$$v_{i+1}^H(v_i) = \left(q\left(\frac{i+1}{N+1}\right), p\left(\frac{i+1}{N+1}\right)\right),$$

where (q, p) satisfies (1) and $\left(q\left(\frac{i}{N+1}\right), p\left(\frac{i}{N+1}\right)\right) = v_i$. *Choose N large enough for each* $v_i \to \left(q_{i+1}^H(v_i), p_i\right)$ *to be a diffeomorphism* (this is possible, because H is Z-periodic).

Let V denote the set of all points $v = (v_0, \ldots, v_N) \in (T^* \boldsymbol{R}^n)^{N+1}$, and let f^H be the (smooth) function on V given by

$$f^H(v) = \oint_{\hat{c}_v} p\,dq + \int_0^1 H(c_v(t), t)\,dt,$$

where the path c_v: $[0, 1) \to T^* \mathbf{R}^n$ and the loop $\hat{c}_v$ are defined as follows:

– on $\left[\frac{j}{N+1}, \frac{j+1}{N+1}\right)$, c_v is the solution of (1) such that $\left(q\left(\frac{j}{N+1}\right), p\left(\frac{j}{N+1}\right)\right) = v_j$;

– the loop $\hat{c}_v$ is obtained by adding "corners" to c_v as in Fig. 1, so as to close it.

In other words,

$$f^H(v) = \sum_{j=0}^{N} \int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}} \left(c_v^*(pdq) + H(c_v(t), t)dt\right) + \sum_{j=1}^{N+1} p_j\left(q_j - q_j^H(v_{j-1})\right),$$

where $q_{N+1} = q_0$ and $p_{N+1} = p_0$.

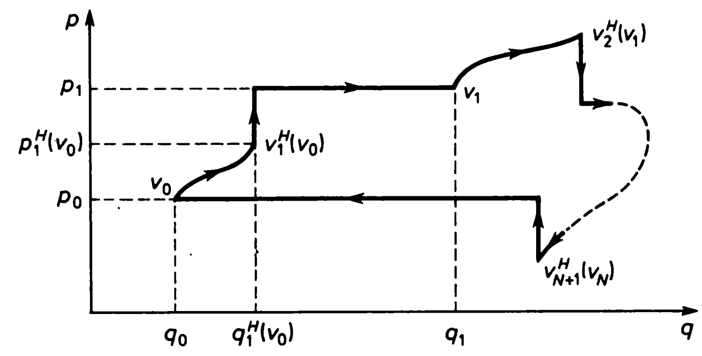

Fig. 1

From this and the definition of c_v, we deduce that

$$\text{(3)} \qquad df^H(v) = \sum_{0}^{N} \left(\left(p_{j+1}^H(v_j) - p_{j+1}\right) dq_{j+1}^H(v_j) + \left(q_{j+1} - q_{j+1}^H(v_j)\right) dp_{j+1}\right).$$

Therefore, by our choice of N, v is a critical point of f^H if and only if $v_{j+1} = v_{j+1}^H(v_j)$ for $0 \leq j < N$ and $v_{N+1}^H(v_N) = v_0$, i.e. if and only if c_v satisfies (1)–(2).

Let W denote the set of all points

$$(x, y) = \left((x_0, y_0), \ldots, (x_N, y_N)\right) \in (T^* \mathbf{R}^n)^{N+1},$$

and let $\tilde{F}^H = f^H \circ h^H$: $W \to \mathbf{R}$, where h^H: $W \to V$ is defined as follows: for each $(x, y) \in W$, $h^H(x, y) = v$ is given by

$$\text{(4)} \qquad (x_0, y_0) = \left(q_1^H(v_0), p_0\right)$$

$$\text{and} \quad x_j = q_{j+1}^H(v_j) - q_j^H(v_{j-1}), \; y_j = p_j - p_0, \; 1 \leq j \leq N.$$

By our choice of N; h^H is a well-defined smooth diffeomorphism. Since $\tilde{F}^H$ is Z-periodic with respect to (x_0, y_0), it induces a smooth function $(T^* \boldsymbol{R}^n/Z) \times E \to \boldsymbol{R}$, where $E = (T^* \boldsymbol{R}^n)^N$, and we just have to show that this phase function F^H is quadratic. Let $\tilde{F}^0$ and F^0 be defined in the same fashion (with the same N) for $H = 0$. Clearly,

$$\tilde{F}^0(x, y) = \sum_1^N y_j x_j.$$

Therefore, F^0 is of the form $(a, v) \to Q(v)$, where Q is a non-degenerate quadratic form on E, and our theorem will be proven if we can show that $d(\tilde{F}^H - \tilde{F}^0)$: $W \to W^*$ is bounded. Now, if we set $h^H(x, y) = v$, (3)–(4) yield

$$d(\tilde{F}^H - \tilde{F}^0) = \sum_{j=0}^N (p_{j+1}^H(v_j) - p_j) d \sum_0^j x_k + \sum_{j=1}^N (q_j - q_{j+1}^H(v_j)) d(y_0 + y_j) + (q_0 - q_1^H(v_0)) dy_0,$$

which is bounded, because H is Z-periodic.

The fact that non-degenerate critical points of F^H correspond to non-degenerate solutions of (1)–(2) is an easy exercise (this is one of the advantages of our method, introduced in [Ch 84]). ∎

1.3. Symplectic formulation of the same result

Recall that a *symplectic form* on a manifold X is a closed 2-form σ such that the bilinear form $\sigma(x)$ is non-degenerate for every $x \in X$ (thus, X must be even-dimensional).

EXAMPLES. On the cotangent bundle $T^* M$ of a manifold M, define the *Liouville form pdq* to be the 1-form such that $\alpha^*(pdq) = \alpha$ for every 1-form α on M (viewed as a mapping $M \to T^* M$ on the left-hand side). The 2-form $d(pdq)$ is symplectic; it is called the *canonical symplectic form on* $T^* M$.

A *standard symplectic form on* $\boldsymbol{T}^{2n}$ is a 2-form ω such that, for some lattice Z of $T^* \boldsymbol{R}^n$ and some smooth diffeomorphism g: $T^* \boldsymbol{R}^n/Z \to \boldsymbol{T}^{2n}$, $(g \circ \pi)^* \omega = d(pdq)$, where π: $T^* \boldsymbol{R}^n \to T^* \boldsymbol{R}^n/Z$ denotes the canonical projection.

An *isotopy of* a manifold X is a family $(g_t)_{0 \leqslant t \leqslant 1}$ of smooth diffeomorphisms of X such that $g_0 = \mathrm{Id}$ and that $(t, x) \to g_t(x)$ is smooth. Such an isotopy (g_t) is determined by its *infinitesimal generator*, which is the family $(\dot{g}_t)_{0 \leqslant t \leqslant 1}$ of vector fields on X given by $\frac{d}{dt} g_t = \dot{g}_t \circ g_t$. Given a symplectic form σ on X, an *isotopy of* (X, σ) (or, simply, of σ) is an isotopy (g_t) of X such that $g_t^* \sigma = \sigma$ for every t, or, equivalently, such that the Lie derivative

$L_{\dot{g}_t}\sigma$ equals zero for every t; in other words, by the homotopy formula, the interior product $\dot{g}_t \lrcorner \sigma$ is a closed 1-form for every t. If $\dot{g}_t \lrcorner \sigma$ is exact for every t, isotopy (g_t) is called *hamiltonian*; one can then choose a *smooth* family $(H_t)_{0 \leqslant t \leqslant 1}$ of smooth functions on X, called a *hamiltonian of* (g_t), such that

(5) $$\dot{g}_t \lrcorner \sigma = dH_t \quad \text{for every } t.$$

Examples. An isotopy (g_t) of $T^* M$ is hamiltonian (for the canonical symplectic form $d(pdq)$) if and only if $g_t^*(pdq) - pdq$ is exact for every t.

If $X = T^* \boldsymbol{R}^n$ and $\sigma = d(pdq)$, (5) is equivalent to $g_t(x) = (q(t), p(t))$, where (q, p) satisfies $(q(0), p(0)) = x$ and (1), setting $H(q, p, t) = H_t(q, p)$; hence, the following formulation of Theorem (1.2) (the link with the Poincaré–Birkhoff theorem is described in (Al), Appendix 9, and – painfully – detailed in [ChZ83]):

Theorem. *For each hamiltonian isotopy (g_t) of a standard symplectic form on the $2n$-torus, g_1 has at least $2n+1$ fixed points x such that the loop $t \to g_t(x)$ is homotopic to a point, and at least 2^{2n} if none of these fixed points is degenerate.*

One of the Arnol'd conjectures is that this is still true for an arbitrary symplectic form on a compact manifold M – replacing $2n+1$ by $\mathrm{cl}(M)+1$ and 2^{2n}; by $\mathrm{SB}(M)$. This has been proven by Barry Fortune and Alan Weinstein (FW 84) for $M = CP^n$ (equipped with its standard Kähler symplectic form), and by Jean-Claude Sikorav (S 83) and Andreas Floer (F 84) for large classes of symplectic manifolds admitting compatible riemannian structures with non-negative sectional curvatures (these classes include all surfaces of positive genus). In (1.5) below, we shall sketch a very elegant proof of such a result, again due to Sikorav (S 85). In the general case, Mikhael Gromov (G 85) and Andreas Floer (F 86) use Gromov's theory of holomorphic curves, but this is too hard a work for us...

1.4. Lagrangian intersections

Theorem (Ch83). *Let M denote the n-torus $\boldsymbol{T}^n = \boldsymbol{R}^n/\boldsymbol{Z}^n$, and let O_M be the zero section $M \times \{0\}$ of $T^* M = M \times (\boldsymbol{R}^n)^*$. For every hamiltonian isotopy (g_t) of $(T^* M, d(pdq))$, there are at least $n+1$ points in $O_M \cap g_1(O_M)$, and at least 2^n if all of them are transversal intersection points.*

Proof. As $\bigcup_t g_t(O_M)$ is compact, multiplying a hamiltonian of (g_t) by a bump function, *we may assume that (g_t) has a compactly supported hamiltonian* $(\underline{H}_t)$. Define $H_t(q, p) = H(q, p, t)$ by $H_t = \underline{H}_t \circ P$, where P: $T^* \boldsymbol{R}^n \to T^* \boldsymbol{T}^n$ denotes the canonical covering projection. Then, our problem is to count the $\mathrm{mod}(\boldsymbol{Z}^n \times \{0\})$ solutions of (1) which satisfy

(2′) $$p(0) = p(1) = 0.$$

Now, our hypothesis on (H_t) allows us to do this exactly as in (1.2), except that here v_0 lies in $\boldsymbol{R}^n \times \{0\}$, hence $p_0 = y_0 = p_{N+1} = 0$ and

$$(3') \quad df^H(v) = \sum_1^N \big((p_j^H(v_{j-1}) - p_j)\, dq_j^H(v_{j-1}) + (q_j - q_j^H(v_{j-1})\, dp_j\big) + p_{N+1}^H(v_N)\, dq_{N+1}^H(v_N).$$

From this, it is clear that $O_M \cap g_1(O_M)$ is in 1-1 correspondence with the critical set of the quadratic phase function F^H on M, hence our result by Theorem (1.1). ∎

Helmut Hofer (H 84) proved that this result can be extended from the torus to an arbitrary compact manifold, but his argument required the strength of this cheerful giant. A much simpler proof of a somewhat better result was given by François Laudenbach and Jean-Claude Sikorav (LS 85) a few months later, *via* the method described above – with an additional idea. In (1.5) below, we shall show how Sikorav made this proof about as simple for a general compact manifold as for the torus, but let us first explain why Theorem (1.4) is more general – and, in a way, more natural – then Theorem (1.2): under the hypotheses of Hofer's theorem (i.e. taking for M an arbitrary compact manifold in the hypotheses of Theorem (1.4)), if g_1 is C^1-close enough to the identity, there exists a (unique) 1-form α: $M \to T^* M$ on M such that $g_1(O_M) = \alpha(M)$, and the fact that (g_t) is hamiltonian implies that α is *exact*, since so is $g_1^*(pdq) - pdq$ (and $\alpha^*(pdq) = \alpha$); thus, in this case, Hofer's theorem comes at once from ordinary Morse–Lyusternik–Schnirelmann theory for smooth functions on M, since α vanishes precisely at the critical points of its primitives. To deduce the Conley–Zehnder theorem from Theorem (1.4), we shall use the following easy

PROPOSITION. *For every standard symplectic form ω on $M = \boldsymbol{T}^{2n}$, let $\overset{2}{\omega}$ be the symplectic form on $M \times M$ given by $\overset{2}{\omega} = \mathrm{pr}_2^* \omega - \mathrm{pr}_1^* \omega$, where pr_1, pr_2: $M \times M \to M$ denote the projections. There exists a covering projection P: $T^* M \to M \times M$ such that $P|_{O_M}$ is a diffeomorphism onto the diagonal ΔM and that $P^* \overset{2}{\omega} = d(pdq)$.*

Proof. (Ch 83). Let A: $(T^* \boldsymbol{R}^n)^2 \to T^*(T^* \boldsymbol{R}^n)$ be the linear isomorphism given by

$$(6) \qquad A((q, p), (q', p')) = \left(\left(\frac{q+q'}{2}, \frac{p+p'}{2}\right), (p'-p, q-q')\right).$$

Clearly, the inverse image of the canonical symplectic form by A is $\mathrm{pr}_2^* d(pdq) - \mathrm{pr}_1^* d(pdq)$ and A maps the diagonal onto the zero section. Thus, if Z is a lattice of $T^* \boldsymbol{R}^n$ and $T_Z = T^* \boldsymbol{R}^n / Z$, we get from A^{-1} a covering projection

P_Z: $T^* T_Z \to Z_Z \times T_Z$ such that, denoting by ω_Z the symplectic form on T_Z obtained by projecting the canonical form of $T^* \mathbf{R}^n$, we have $P_Z^*(\overset{2}{\omega}_Z) = d(pdq)$. Moreover, P_Z maps the zero section diffeomorphically onto the diagonal. Now, if h: $M \to T_Z$ is a diffeomorphism such that $h^* \omega_Z = \omega$, then the diffeomorphism $\tilde{h}$: $T^* M \to T^* T_Z$ given by $\tilde{h}(a, p) = (h(a), p \overset{2}{\circ} (T_a h)^{-1})$ sends one Liouville form onto the other, and the diffeomorphism $\overset{2}{h}$: $M \times M \to M_Z^2$ defined by $\overset{2}{h}(x, x') = (h(x), h(x'))$ sends $\overset{2}{\omega}$ onto $\overset{2}{\omega}_Z$, hence our result with $P = (\overset{2}{h})^{-1} \circ P_Z \circ \tilde{h}$. ∎

COROLLARY. *For every standard symplectic form* ω *on* $M = \mathbf{T}^{2n}$ *and every hamiltonian isotopy* (h_t) *of* $\overset{2}{\omega}$, *there are at least* $2n+1$ *points in* $\Delta M \cap h_1(\Delta M)$, *and at least* 2^{2n} *if all of them are transversal intersection points.*

Proof. If (K_t) is a hamiltonian of (h_t), the unique isotopy $(\tilde{h}_t)$ of $T^* M$ such that $h_t \circ P = P \circ \tilde{h}_t$ for every t is hamiltonian, with hamiltonian $(K_t \circ P)$. As P maps $O_M \cap \tilde{h}_t(O_M)$ injectively into $h_t(\Delta M) \cap \Delta M$, Theorem (1.4) yields our result. ∎

The Conley–Zehnder theorem is a particular case of this situation: given a hamiltonian isotopy (g_t) of ω, with hamiltonian (H_t), we can define a hamiltonian isotopy (h_t) of $\overset{2}{\omega}$, with hamiltonian $(K_t\colon (x, y) \mapsto H_t(y))$, by $h_t(x, y) = (x, g_t(y))$. As the fixed point set of g_1 is the 1-1 image of $\Delta M \cap h_1(\Delta M)$ by the (first or second) projection, we just have to remark that the loops $t \to g_t(x)$ obtained by the proof of our Corollary are homotopic to constant loops.

1.5. A short account of recent work by Sikorav

Given a symplectic form σ on a manifold X, an immersion j of some manifold L into X is called *lagrangian* if it is smooth, satisfies $j^* \sigma = 0$, and if the dimension of L equals half the dimension of X — since $\sigma(x)$ is non-degenerate for each x, this is the maximal dimension allowing $j^* \sigma = 0$.

Following Lars Hörmander (H 71), call a phase function F: $M \times E \to \mathbf{R}$ on an n-dimensional manifold M *non-degenerate* if the "vertical derivative" $d_V F$: $M \times E \to E^*$, defined by $d_V F(a, v) = \frac{\partial F}{\partial v}(a, v)$, admits $0 \in E^*$ as a regular value. Given such a phase function, $(d_V F)^{-1}(0) = \Sigma_F$ is an n-dimensional submanifold of $M \times E$, and the mapping j_F: $\Sigma_F \to T^* M$ obtained by restricting the partial differential $\frac{\partial F}{\partial a}$: $M \times E \to T^* M$ is an immersion such that $j_F^*(pdq)$ is exact (in particular, j_F is lagrangian).

A lagrangian immersion j: $L \to T^* M$ is *generated by a phase function* if there exist a non-degenerate phase function F on M and a diffeomorphism g: $L \to \Sigma_F$ such that $j = j_F \circ g$.

Remark. Then, $j^{-1}(O_M)$ clearly is the image by g^{-1} of the critical set of F.

EXAMPLE. Sikorav's starting point was the following observation, due to Alan Weinstein: going back to the proof of Theorem (1.4), (3′) implies that the solutions of (1) satisfying $p(0) = 0$ are precisely those c_v such that (taking $p_1, \ldots, p_N, q_1^H, \ldots, q_{N+1}^H$ as coordinates on V) the partial derivatives of f^H with respect to p_j and q_j^H vanish at v for $1 \leqslant j \leqslant N$; moreover, the endpoint of such a c_v admits $q_{N+1}^H(v_N)$ as its q-component, and the partial derivative of f^H with respect to q_{N+1}^H as its p-component. In other words, if we compose $\tilde{F}^H$ with the diffeomorphism $(a, (x_j, y_j)_{1 \leqslant j \leqslant N}) \to (a - \Sigma x_j, (x_j, y_j)_{1 \leqslant j \leqslant N})$ of $\boldsymbol{R}^n \times (T^* \boldsymbol{R}^n)^N$, we get a function $\tilde{F}$ such that the induced function F on $M \times (T^* \boldsymbol{R}^n)^N$ generates the lagrangian immersion $g_1|_{O_M}$.

Noticing that the canonical embedding of M as O_M is generated by a quadratic phase function – any constant function on M will do ! – the above remark shows that Hofer's theorem is a corollary of the following

THEOREM (S 85). *Let L and M be two compact manifolds, and let j: $L \to T^* M$ denote a lagrangian immersion. If j is generated by a quadratic phase function, then, for every hamiltonian isotopy (g_t) of $T^* M$, so is $g_1 \circ j$.* ∎

The proof of this result is quite simple – see (Ch86) for an account when $M = T^n$. Roughly speaking, the idea is to consider each "jump" in the proof of Theorem (1.4) separately.

(S 85) contains a wonderful contribution to the following problem: can one extend Proposition (1.4) to other symplectic forms on compact manifolds (hence a generalisation of the Conley–Zehnder theorem to such symplectic forms)? He gets a large class of such symplectic forms – including all volume forms on compact surfaces of positive genus – to which *both Proposition* (1.4) *and its proof* extend naturally.

Remarks. Proposition (1.4) is "locally true" for every symplectic form ω on a compact manifold M: by Weinstein's symplectic tubular neighbourhood theorem ((W 71) – see also (ChZ 83)), there exists a smooth embedding I of an open neighbourhood U of O_M in $T^* M$ into $M \times M$ such that $I(O_M) = \Delta M$ and $I^* \overset{2}{\omega} = d(pdq)$. Therefore, by Hofer's theorem, the Conley–Zehnder theorem extends to those hamiltonian isotopies (g_t) of ω such that the graph of every g_t lies in $I(U)$: this statement in a C^0-neighbourhood of the identity is originally due to Alan Weinstein (W 83).

Even though Proposition (1.4) is not true in general, there is some hope to deduce the generalised Conley–Zehnder theorem from Hofer's theorem *via* some weakened form of Proposition (1.4). In the author's opinion, this problem in "symplectic topology" is important.

2. A proof of Viterbo's theorem

This is a short account of joint work with Biancamaria D'Onofrio (ChDO 87). Viterbo's theorem is one of the most general known results on periodic orbits of hamiltonian systems, whereas its original proof was rather simple (V 86), Hofer's proof (HZ 86), much simpler, and the proof outlined below, probably even simpler.

2.1. Topological prerequisites

LEMMA. *Let $F: E \to \boldsymbol{R}$ be a "quadratic phase function on $\{0\}$" (!), and let (g^t) denote the flow of its gradient with respect to some euclidean metric on E (g^t is defined for every t because F is quadratic). For each non-empty compact subset K of E, if there exists a number b satisfying*

(7) $\min F(g^t(K)) \leqslant b$ *for every non-negative t,*

then the closure of $\bigcup_{t\geqslant 0} g^t(K)$ contains a critical point of a F with $F(a) \leqslant b$.

Proof. By (7), as K is compact and non-empty, so is $K_0 \leqslant \bigcap_{t\geqslant 0} g^{-t}(F^{-1}((-\infty, b]) \cap K$. For each $x \in K_0$, the set $\{F(g^t(x)) \colon t \geqslant 0\}$ is bounded above by b. Therefore, $\frac{d}{dt} F(g^t(x)) = |dF(g^t(x))|^2$ tends to 0 when $t \to +\infty$. Now, since F is quadratic, this implies that $\{g^t(x) \colon t \geqslant 0\}$ is bounded and that the compact subset $\bigcap_{t\geqslant 0} \overline{\{g^s(x) \colon s \geqslant t\}}$ is non-empty and consists of critical points of F. ∎

We shall use this lemma under the following form:

COROLLARY. *Given F as in the above lemma, assume that there exist a topologically embedded sphere S in E and a topological embedding j of some euclidean ball B into E with the following properties:*

(i) *The two spheres $j(\partial B)$ and S are linked in E (hence in particular $\dim S + \dim B = \dim E$ and $S \cap j(B) \neq \emptyset$).*

(ii) $\min F \circ j(\partial B) > \max F(S)$.

Then, F admits a critical point a with $F(a) \leqslant \max F(S)$.

Proof. Since F is non-decreasing along the flow-lines of its gradient, we have $\min F(g^t(j(\partial B))) > \max F(S)$ for every non-negative t. Therefore, the two spheres S and $g^t(j(\partial B))$ never intersect for such t's, which proves that they are linked, hence in particular $g^t(j(B)) \cap S \neq \emptyset$, for every non-negative t. Applying the lemma with $K = j(B)$ and $b = \max F(S)$, we obtain our corollary. ∎

2.2. Statement of the theorem – The Viterbo–Hofer construction

Let $M \overset{j}{\hookrightarrow} T^* \mathbf{R}^n$ be a smooth hypersurface. For each $x \in M$, the kernel K_x of the bilinear from $j^* d(pdq)(x)$ is a line. Hypersurface M is *of contact type* if there is a primitive α of $j^* d(pdq)$ such that $\alpha(x)(K_x) = \{0\}$ has no solution $x \in M$ (such an α is a contact form on M, hence the name given to these M's).

EXAMPLE. If M is an embedded sphere, (strictly) star-like, it is of contact type.

The *characteristics of M* are those (smooth) curves $C \subset M$ such that $T_x C = K_x$ for every $x \in C$. They define a smooth foliation, called the *characteristic foliation of M*. Equivalently, if $M = H^{-1}(b)$ for some regular value b of a smooth real function H, defined in a neighbourhood of M, then, every solution of the hamiltonian system

$$\dot{q} = -\frac{\partial H}{\partial p}(q, p) \quad \text{and} \quad \dot{p} = \frac{\partial H}{\partial q}(q, p) \tag{1'}$$

which starts at some point of M remains in M, and the characteristics of M are precisely the images of such solutions. A characteristic of M is *closed* if it is an embedded circle – hence a periodic orbit of such hamiltonian systems.

The following result had been conjectured by Weinstein (W 79):

THEOREM (V 86). *If a compact hypersurface M of $T^* \mathbf{R}^n$ is of contact type, it admits a closed characteristic.*

The proof in (2.3) below applies equally well to the more general theorem proven in (HZ 86), from which we borrow the following key construction:

LEMMA. *Denote by $v \to |v|$ the standard euclidean norm on $T^* \mathbf{R}^n$, and by Q the quadratic form $Q(v) = \frac{3\pi}{2}|v|^2$. Under the hypotheses of the theorem, there exists a smooth real function H on $T^* \mathbf{R}^n$ with the following properties:*

(i) *It does not take negative values, and its only critical values are* 0 *and* $c > 0$.

(ii) *M is a level surface of H, it admits $H^{-1}((0, c))$ as a tubular neighbourhood and, for each $b \in (0, c)$, there exists a diffeomorphism of M onto $H^{-1}(b)$ sending one characteristic foliation onto the other.*

(iii) *$H^{-1}(0)$ is the bounded component of $T^* \mathbf{R}^n \setminus H^{-1}((0, c))$. Via a translation of $T^* \mathbf{R}^n$, we may – and SHALL – assume that* 0 *lies in its interior.*[1]

[1] Of course, we also assume that M is connected, which is not a restriction either.

(iv) *$H^{-1}(c)$ is the intersection of a (large) closed ball $\{|v| \leqslant R\}$ with the unbounded component of $T^* \boldsymbol{R}^n \setminus H^{-1}((0, c))$.*

(v) *For $|v| > R$, $H(v) = f(|v|^2)$, where $f\colon \boldsymbol{R} \hookleftarrow$ is a smooth function satisfying $f(x) = c$ for $x \leqslant R^2$, $f(x) = \frac{3\pi}{2} x$ for every large enough $x > R^2$ (hence $H(v) = Q(v)$ for every large enough v), and, for each $x > R^2$, $f(x) \geqslant \frac{3\pi}{2} x$ and $0 < f'(x) \leqslant \frac{3\pi}{2}$.*

Idea of the proof. By our hypothesis on M, there exists a tubular neighbourhood of M, (trivially) foliated by copies of M such that the characteristic foliations on any two of them are diffeomorphic. From this, the construction of H is quite easy. ∎

COROLLARY. *Under the hypotheses of the above lemma,*

(i) *if (1′) admits a 1-periodic solution contained in $H^{-1}(b)$ for some $b \in (0, c)$, then M has a closed characteristic;*

(ii) *If a 1-periodic solution γ of (1′) has its image in $H^{-1}(b)$ for some $b \notin (0, c)$, then $I(\gamma) = \int_0^1 (\gamma^* pdq + H(\gamma(t))dt)$ is non-negative.*

Proof. (i) comes from part (ii) of the lemma. Under the hypotheses of (ii), if b equals 0 or c, then γ is constant, hence $I(\gamma) = b \geqslant 0$. For $b > c$, we have that $J(\gamma) := \int_0^1 (\gamma^* pdq) = \int_0^1 \left(\gamma^* \left(\frac{pdq - qdp}{2}\right)\right)$ because γ is a loop. Therefore, (1′) yields $J(\gamma) = -\int_0^1 dH(\gamma(t)) \cdot \gamma(t)\, dt/2$, hence, by part (v) of the lemma,

$$J(\gamma) = -\int_0^1 f'(|\gamma(t)|^2)|\gamma(t)|^2\, dt$$

$$\geqslant -\int_0^1 \frac{3\pi}{2} |\gamma(t)|^2\, dt \geqslant -\int_0^1 f(|\gamma(t)|^2)\, dt = -\int_0^1 H(\gamma(t))\, dt,$$

hence $I(\gamma) \geqslant 0$. ∎

This corollary shows that *Viterbo's theorem would follow from the existence of a 1-periodic solution γ of (1′) with $I(\gamma) < 0$. This is precisely what we are going to establish now.*

2.3. An elementary proof of Viterbo's theorem

For every smooth real function H on $T^* \boldsymbol{R}^n$, let (g_H^t) denote the flow of the associated hamiltonian vector field X_H, defined by $X_H \lrcorner\, d(pdq) = dH$ (in other words, its integral curves are precisely the solutions of (1′)).

IN THE SEQUEL, H AND Q ARE AS IN LEMMA (2.2).

Since (g^t_Q) is a 1-parameter group of linear automorphisms and $H-Q$ has compact support, the following two properties hold:

(a) (g^t_H) is a one-parameter group of diffeomorphisms.

(b) Using the same notation as in (1.2) – with $H(q, p, t) := H(q, p)$ – for every large enough positive integer N, each mapping $v_j \to (q^H_{j+1}(v_j), p_j)$ is a diffeomorphism.

IN THE SEQUEL, WE CONSIDER ONLY SUCH N's, AND USE THE NOTATION OF (1.2) – WITH $H(q, p, t) := H(q, p)$ – EXCEPT THAT WE WRITE F^H INSTEAD OF $\tilde{F}^H$.

By our choice of N, the 1-periodic solutions of (1′) are those c_v such that v is a critical point of f^H. *Applying Corollary* (2.1) *to* $F = F^H$, *we shall prove that, if N is large enough, then f^H has a critical point v with* $f^H(v) < 0$; by the last remark in (2.2), this will imply Viterbo's theorem, since $f^H(v) = I(c_v)$.

LEMMA A. *For every large enough N, the following hold true*:

(i) F^Q *is a non-degenerate quadratic form on W, the index of which is* $n(N-2)$.

(ii) *Denoting by $y \to y^\#$ the isomorphism of $(\boldsymbol{R}^n)^*$ onto $\boldsymbol{R}^n$ associated to the standard scalar product, the restriction of F^Q to*

$$W^+ = \{(x, y) \in W\colon\ x_j = y_j^\# \ \ for\ j \geqslant 1\}$$

is positive definite.

Proof. For $0 \leqslant s \leqslant 3\pi/2$, let $Q(s)$ be the quadratic form $v \to s|v|^2$ on $T^*\boldsymbol{R}^n$. If we identify $T^*\boldsymbol{R}^n$ to $\boldsymbol{C}^n$ by the isomorphism $(q, p) \to q + ip^\#$, then,

$$g^t_{Q(s)}(v) = e^{2ist}\, v \quad \text{for every } t \in \boldsymbol{R} \text{ and every } v \in \boldsymbol{C}^n, \tag{8}$$

hence in particular

$$q^{Q(s)}_{j+1}(v_j) + i\big(p^{Q(s)}_{j+1}(v_j)\big)^\# = e^{iu(s)}(q_j + ip_j^\#),$$

where

$$u(s) = 2s/(N+1).$$

Therefore, setting $r_j = \sum_0^j x_k$ for $0 \leqslant j \leqslant N$, $p_0 = y_0$ and $p_j = y_0 + y_j$ for $1 \leqslant j \leqslant N$, straight-forward calculations (see also the proof of Lemma B(ii) below) yield

$$F^{Q(s)} = F^0 + R(s), \quad \text{where } R(s)(x, y) = \frac{1}{2}\sum_0^N \Big(\tan u(s)(|r_j|^2 + |p_j|^2) + \Big(\frac{2}{\cos u(s)} - 2\Big)p_j r_j\Big), \tag{9}$$

which makes sense for every s when N is large enough. Now, $F^0(x, y) = \sum_1^N |x_j|^2$ on W^+ and, if N is large enough, every $R(s)$ with $s \neq 0$ is positive definite, hence (ii).

Moreover, by (8), for $N > 5$, the mappings $v_j \to (q_{j+1}^{Q(s)}(v_j), p_j)$ are automorphisms of $T^* \boldsymbol{R}^n$. Therefore, as in (1.2), the critical set of the function $F^{Q(s)}$ is isomorphic to $\{v \in T^* \boldsymbol{R}^n: g_{Q(s)}^1(v) = v\}$. Thus, by (8), the quadratic form $F^{Q(s)}$ is non degenerate for $0 \neq s \neq \pi$, and its critical set is a $2n$-dimensional vector subspace of W for $s = 0$ and $s = \pi$. Now, by (9), the derivative $R'(s_0)$ is a positive definite quadratic form for $s_0 = 0$ and $s_0 = \pi$. Therefore, writing $F^{Q(s)}$ as $F^{Q(s_0)} + \int_{s_0}^{s} R'(u)\,du$ and restricting it to a maximal subspace on which $F^{Q(s_0)}$ is non-negative, (resp. negative), we see that the index of $F^{Q(s)}$ decreases by $2n$ after s has crossed the values 0 and π. Since these are the only values of $s \in \left[0, \frac{3\pi}{2}\right]$ for which the index can change, (i) is proven, for $F^0(x, y) = \sum_1^N y_j x_j$. ■

LEMMA B. (i) *F^H is a quadratic phase function on $\{0\}$. More precisely, $d(F^H - F^Q)$ is bounded.*

(ii) *If N is large enough, then $W^- = \{(x, y) \in W\colon x_j = -y_j^\# \text{ for } j \geqslant 0$ and $x_0 = 0\}$ contains a euclidean sphere S centered at 0, satisfying* $\max F^H(S) < 0$ *and* $\max F^H(S) < \inf F^H(W^+)$.

Proof. Setting $v_j = (q_j, p_j) = h^H(x, y)$ and $v_j' = (q_j', p_j) = h^Q(x, y)$, (3) yields

$$d(F^H - F^Q)(x, y)$$
$$= \sum_{j=0}^{N} \left(p_{j+1}^H(v_j) - p_{j+1}^Q(v_j')\right) d\sum_0^j x_k + (q_0 - q_0')\,dy_0 + \sum_1^N (q_j - q_j')\,d(y_0 + y_j);$$

from this, (i) is clear, since $H - Q$ has compact support.

To prove (ii), notice that

$$F^H(x, y) - F^0(x, y) = \sum_0^N \left(\int_{I_j} H(c_v(t))\,dt + \oint_{T_j} p\,dq\right), \tag{10}$$

where $I_j = \left[\frac{j}{N+1}, \frac{j+1}{N+1}\right]$ and T_j is as in Figure 2.

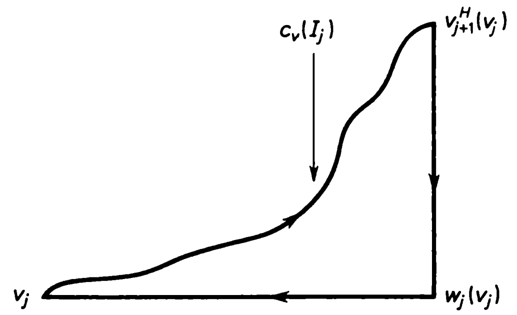

Fig. 2

As T_j is a loop, we have that

$$\oint_{T_j} pdq = \oint_{T_j} \frac{pdq-qdp}{2} = -\frac{1}{2}\int_{I_j} dH(c_v(t))\cdot c_v(t)\,dt$$
$$+\frac{1}{2}(p_{j+1}^H(v_j)-p_j)q_{j+1}^H(v_j)+\frac{1}{2}p_j(q_j-q_{j+1}^H(v_j)).$$

Therefore, since

$$q_j-q_{j+1}^H(v_j) = \int_{I_j}\frac{\partial H}{\partial p}(c_v(t))\,dt \quad \text{and} \quad p_{j+1}^H(v_j)-p_j = \int_{I_j}\frac{\partial H}{\partial q}(c_v(t))\,dt,$$

we have

$$\int_{I_j} H(c_v(t))\,dt+\oint_{T_j} pdq = \int_{I_j}\left(H(c_v(t))+\frac{1}{2}dH(c_v(t))\cdot(w_j(v_j)-c_v(t))\right)dt,$$

where $w_j(v_j) = (q_{j+1}^H(v_j), p_j)$. As H is constant along each $c_v|_{I_j}$, this and Taylor's formula yield

$$\text{(11)} \quad \int_{I_j} H(c_v(t))\,dt+\oint_{T_j} pdq = \frac{1}{2(N+1)}\big(H(v_j)+H(w_j(v_j))\big)$$
$$-\frac{1}{2}\int_{I_j}\left(\int_0^1(1-u)D^2H\big(c_v(t)+u(w_j(v_j)-c_v(t))\big)du\right)\cdot(w_j(v_j)-c_v(t))^2\,dt.$$

Now, since $H-Q$ has compact support, there exists a compact subset K of $T^*\boldsymbol{R}^n$ such that, for every positive integer N and every $v_j \in T^*\boldsymbol{R}^n\setminus K$, all the values of H and its derivatives involved in (11) coincide with the corresponding values of Q and its derivatives, hence, by (8) and straight-forward calculations,

$$\text{(12)} \qquad \int_{I_j} H(c_v(t))\,dt+\oint_{T_j} pdq > 0 \quad \text{if } v_j \text{ does not lie in } K.$$

Moreover, $D^2 H$ is bounded, and, by the definition of c_v, $(N+1)|w_j(v_j)-c_v(t)|$ is uniformly bounded with respect to $v_j \in K$, N and $t \in I_j$; therefore, by (11), since H is non-negative, there exists a positive constant C such that we have $\int_{I_j} H(c_v(t))\,dt + \oint_{T_j} p\,dq \geqslant -C/(N+1)^3$ for every N and every $v_j \in K$, hence, by (12), for every N and every $v_j \in T^* \boldsymbol{R}^n$, hence, by (10), $F^H - F^0 \geqslant -C/(N+1)^2$. In particular,

(13) F^H is bounded below by $-C/(N+1)^2$ on W^+.

Let $r > 0$ be small enough for the ball $\{|v| \leqslant r\}$ to be contained in $H^{-1}(0)$. Clearly, if no $|v_j|$ is greater than r, then

$$f^H(v) = f^0(v) \quad \text{and} \quad (h^H)^{-1}(v) = (h^0)^{-1}(v).$$

Moreover, since $F^0(x, y) = -\sum_1^N |x_j|^2$ on W^-, the set

$$S = \{(x, y) \in W^- : F^0(x, y) = -r^2/2N\}$$

is a euclidean sphere centered at 0 in W^- and, for every $(x, y) \in S$, $v = h^0(x, y)$ satisfies

$$|v_j|^2 = |q_j|^2 + |p_j|^2 = |\sum_1^j x_k|^2 + |x_j|^2 \leqslant 2\big(\sum_1^N |x_k|\big)^2 \leqslant 2N \sum_1^N |x_k|^2 = r^2$$

for $1 \leqslant j \leqslant N$, hence $(x, y) = (g^H)^{-1}(v)$ and $f^H(v) = f^0(v)$; therefore, we have proved

(14) $$F^H(S) = \{-r^2/2N\}.$$

By (13) and (14), if N is large enough, we have $\max F_H(S) = -\dfrac{r^2}{2N} < -\dfrac{C}{(N+1)^2} \leqslant \inf F^H(W^+)$, hence our lemma. ∎

Proof of Viterbo's theorem. Assume N large enough for the conclusions of our two lemmas to hold. By Lemma A, there exists a subspace W_1^+ of W, containing W^+ as a hyperplane, and such that $F^Q|_{W_1^+}$ is positive definite. For each positive number M, let B_M denote the ball $\{(x, y) \in W_1^+ : F^Q(x, y) \leqslant M\}$, and let S_M be its boundary in W_1^+. By Lemma B(i), if M is large enough, then $F^H|_{S_M}$ is positive. Given such an M, let B denote the intersection of B_M with one of the two closed half-spaces of W_1^+ bounded by W^+, and let $j: B \to W$ denote the restriction of the inclusion of W_1^+ (!). If M is large enough, then, the two spheres $j(\partial B)$ and S are linked in W, and, since ∂B consists of one half of S_M (on which F^H is positive) and a piece of W^+, Lemma B(ii) yields

$$\max F^H(S) < \min F^H(j(\partial B)) \quad \text{and} \quad \max F_H(S) < 0.$$

Therefore, by Corollary 2.1, F^H admits a negative critical value, hence Viterbo's theorem. ∎

Note. The above proof follows the lines of (HZ 86). What is much simpler here is the analytical tools we use; moreover, the computation of the index in Lemma A(i) might be used for further purposes. On the other hand, the estimates (13)–(14) in the proof of Lemma B(ii) demand more work than the corresponding part of (HZ 86). However, we hope that our approach will be the ignorant's delight...

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