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R-Geodesics and R-asymptotic lines

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1. Introduction. An asymptotic line of order p [1] ([1]) has been defined as a curve lying in a subspace whose first p normals relative to the enveloping space lie in the subspace and a geodesic of order p [2] as a curve whose first p curvatures are identically zero. In the present paper we define an R-asymptotic line of order p as a curve lying in a subspace whose last $p+1$ normals relative to the enveloping space lie in the subspace and having coincident last normals relative to the two spaces. An R-geodesic of order p is defined as a curve whose last p curvatures are identically zero. Later we investigate properties of such curves, and some theorems bringing out their relationships have been proved.

Let V_n be a subspace of a Riemannian space V_m and $a_{\alpha\beta}dy^\alpha dy^\beta$ ([2]) and $g_{ij}dx^i dx^j$ the metrics of V_m and V_n respectively.

A curve C in V_n can be regarded as a curve in V_m or in V_n. We denote the $m-1$ curvatures and $m-1$ normals of C in V_m by k_r ($r = 1, 2, \ldots, m-1$) and $\eta^\alpha_{r|}$ ($r = 2, 3, \ldots, m$) while those in V_n by k'_r ($r = 1, 2, \ldots, n-1$) and $\xi^i_{r|}$ ($r = 2, 3, \ldots, n$). The unit tangent to C shall be denoted by $\eta^\alpha_{1|}$ or $\xi^i_{1|}$ according as it is considered as a vector in V_m or in V_n respectively. Other symbols used in this paper are as in Weatherburn [3].

2. A curve C in V_m is defined as an R-geodesic of order p (to be abbreviated to *R_p-geodesic*) of V_m, if at every point of the curve its last p curvatures relative to V_m are all zero, i.e.,

$$k_{m-r} = 0, \quad r = 1, 2, \ldots, p. \tag{2.1}$$

By Frenet's formulae ([3], p. 95) for a curve in V_m,

$$\frac{\delta}{\delta s}\eta^\alpha_{r|} = k_r \eta^\alpha_{r+1|} - k_{r-1}\eta^\alpha_{r-1|}, \quad r = 1, 2, \ldots, m;\ k_0 \equiv k_m \equiv 0,$$

([1]) Numbers in square brackets refer to the references at the end of the paper.

([2]) Greek indices $\alpha, \beta, \gamma, \ldots$ take the values $1, 2, \ldots, m$, while latin indices $i, j, k, \ldots$ take the values $1, 2, \ldots, n$ throughout the paper unless otherwise stated.

the necessary and sufficient conditions for a curve to be an R_p-geodesic are

$$\frac{\delta}{\delta s}\eta^a_{m-r+1|} = 0, \quad r = 1, 2, \ldots, p. \tag{2.2}$$

Also an R_p-geodesic satisfies

$$\frac{\delta}{\delta s}\eta^a_{m-p|} = -k_{m-p-1}\eta^a_{m-p-1|}, \tag{2.3}$$

and we can state

THEOREM 1. *The last p normals of an R_p-geodesic are displaced parallel wise along the curve and the derived vector of $(m-p-1)$-th normal in the direction of an R_p-geodesic is parallel to its $(m-p-2)$-th normal and its magnitude is k_{m-p-1}.*

Remark. For an R_{m-1}-geodesic, we have

$$k_r = 0, \quad r = 1, 2, \ldots, m-1, \tag{2.4}$$

or

$$\frac{\delta}{\delta s}\eta^a_{r|} = 0, \quad r = 1, 2, \ldots, m. \tag{2.5}$$

Thus:

The unit tangent and $m-1$ unit normals of an R_{m-1}-geodesic in V_m are all displaced parallelwise along the curve.

We observe that a geodesic of maximum order and an R-geodesic of maximum order coincide. In a Euclidean 3-space an R_1-geodesic is a plane curve, while an R_2-geodesic is a straight line.

3. A curve C in V_n is defined as an R-asymptotic line of order p $(p < n-1)$, to be abbreviated to R_p-asymptotic line, if it satisfies the following conditions:

(3.1) The last $p+1$ normals of the curve relative to V_m are in V_n,

(3.1′) The last normals of the curve relative to the two spaces coincide.

The second condition is equivalent to

$$\eta^a_{m|} = \xi^i_{n|}y^a_{,i}.$$

Differentiating this along the curve and using Frenet's formulae, we obtain

$$-k_{m-1}\eta^a_{m-1|} = -k'_{n-1}\xi^i_{n-1|}y^a_{,i} + y^a_{;ij}\xi^i_{1|}\xi^j_{n|}.$$

Since by the first condition $\eta^a_{m-1|}$ lies in V_n and the second part on the right of the above equation is orthogonal to V_n,

$$-k_{m-1}\eta^a_{m-1|} = -k'_{n-1}\xi^i_{n-1|}y^a_{,i}$$

or equivalently

(3.2) $$k_{m-1} = k'_{n-1},$$

(3.2′) $$\eta^{a}_{m-1|} = \xi^{i}_{n-1|} y^{a}_{,i},$$

and

(3.2″) $$y^{a}_{;ij}\, \xi^{i}_{1|}\, \xi^{j}_{n|} = 0.$$

Differentiating (3.2′) along the curve, we have

$$k_{m-1}\eta^{a}_{m|} - k_{m-2}\eta^{a}_{m-2|} = (k'_{n-1}\,\xi^{i}_{n|} - k'_{n-2}\,\xi^{i}_{n-2|}) y^{a}_{,i} + y^{a}_{;ij}\,\xi^{i}_{1|}\,\xi^{j}_{n-1|}.$$

From (3.1′) and (3.2), the above equation reduces to

$$-k_{m-2}\eta^{a}_{m-2|} = -k'_{n-2}\,\xi^{i}_{n-2|} y^{a}_{,i} + y^{a}_{;ij}\,\xi^{i}_{1|}\,\xi^{j}_{n-1|}.$$

As before since $\eta^{a}_{m-2|}$ lies in V_n,

$$-k_{m-2}\eta^{a}_{m-2|} = -k'_{n-2}\,\xi^{i}_{n-2|} y^{a}_{,i},$$

or equivalently

(3.3) $$k_{m-2} = k'_{n-2},$$

(3.3′) $$\eta^{a}_{m-2|} = \xi^{i}_{n-2|} y^{a}_{,i},$$

and

(3.3″) $$y^{a}_{;ij}\,\xi^{i}_{1|}\,\xi^{j}_{n-1|} = 0.$$

Repeating this argument p times, we verify the following conditions for an R_p-asymptotic line:

(3.4) $$k_{m-r} = k'_{n-r}, \qquad r = 1, 2, \ldots, p,$$

(3.4′) $$\eta^{a}_{m-r|} = \xi^{i}_{n-r|} y^{a}_{,i}, \qquad r = 1, 2, \ldots, p,$$

(3.4″) $$y^{a}_{;ij}\,\xi^{i}_{1|}\,\xi^{j}_{n-r+1|} = 0, \qquad r = 1, 2, \ldots, p.$$

Each of the above sets of conditions, together with the condition that the last normals w.r.t. V_n and V_m coincide, is necessary and sufficient for a curve to be an R_p-asymptotic line. For if the last normals coincide,

$$\eta^{a}_{m|} = \xi^{i}_{n|} y^{a}_{,i},$$

which, when differentiated along the curve, gives

$$-k_{m-1}\eta^{a}_{m-1|} = -k'_{n-1}\,\xi^{i}_{n-1|} y^{a}_{,i} + y^{a}_{;ij}\,\xi^{i}_{1|}\,\xi^{j}_{n|}.$$

But (3.4″) for $r = 1$ is

$$y^{a}_{;ij}\,\xi^{i}_{1|}\,\xi^{j}_{n|} = 0,$$

and hence from the above

$$\eta^{a}_{m-1|} = \xi^{i}_{n-1|} y^{a}_{,i},$$

which implies that the normal $\eta^a_{m-1|}$ lies in V_n. Repeating this argument p times and using (3.4″), we get

$$\eta^a_{m-r|} = \xi^i_{n-r|} y^a_{,i}, \quad r = 1, 2, \ldots, p,$$

which shows that the last $p+1$ normals relative to V_m lie in the subspace V_n. Hence the curve is an R_p-asymptotic line.

We can demonstrate in a like manner that each of the sets of conditions (3.4) and (3.4′) is necessary and sufficient for a curve to be an R_p-asymptotic line.

Thus we have proved the following:

THEOREM 2. *The necessary and sufficient conditions for a curve in V_n in V_m, whose normals relative to the two spaces coincide, to be an R_p-asymptotic line are any of the following*:

$$k_{m-r} = k'_{n-r}, \quad r = 1, 2, \ldots, p, \tag{3.5}$$

$$\eta^a_{m-r|} = \xi^i_{n-r|} y^a_{,i}, \quad r = 1, 2, \ldots, p, \tag{3.5′}$$

and

$$y^a_{;ij} \xi^i_{1|} \xi^j_{n-r+1|} = 0, \quad r = 1, 2, \ldots, p. \tag{3.5″}$$

4. Consider a curve C in V_n whose last normals relative to the two spaces coincide. Then we have

$$\eta^a_{m|} = \xi^i_{n|} y^a_{,i}. \tag{4.1}$$

Differentiating this along the curve, we get

$$\frac{\delta}{\delta s} \eta^a_{m|} = \frac{\delta}{\delta s} \xi^i_{n|} y^a_{,i} + y^a_{;ij} \xi^i_{1|} \xi^j_{n|}. \tag{4.2}$$

If the curve is an R_p-geodesic ($p < n-1$) of V_m,

$$k_{m-r} = 0, \quad r = 1, 2, \ldots, p, \tag{4.3}$$

or equivalently

$$\frac{\delta}{\delta s} \eta^a_{m-r+1|} = 0, \quad r = 1, 2, \ldots, p. \tag{4.3′}$$

By Frenet's formulae and (4.3′), (4.2) reduces to

$$0 = -k'_{n-1} \xi^i_{n-1|} y^a_{,i} + y^a_{;ij} \xi^i_{1|} \xi^j_{n|}.$$

Since the first part of the right-hand side of this equation is tangential while the second part is normal to V_n, their sum cannot be zero unless they are separately zero. Therefore

$$y^a_{;ij} \xi^i_{1|} \xi^j_{n|} = 0, \tag{4.4}$$

$$k'_{n-1} \xi^i_{n-1|} y^a_{,i} = 0. \tag{4.4′}$$

Since $\xi^i_{n-1|}y^a_{,i}$ are the V_m-components of the unit vector $\xi^i_{n-1|}$ in V_n, (4.4′) yields

(4.4″) $$k'_{n-1} = 0,$$

which, with (4.3) for $r = 1$, gives

(4.4‴) $$k_{m-1} = k'_{n-1}.$$

Hence, from condition (3.5), the curve is an R_1-asymptotic line. Then from (3.5′) we have

(4.5) $$\eta^a_{m-1|} = \xi^i_{n-1|}y^a_{,i}.$$

Again differentiating this along the curve and using (4.3′), (4.4″), we have

$$0 = -k'_{n-2}\,\xi^i_{n-2|}y^a_{i,} + y^a_{;ij}\,\xi^i_{1|}\,\xi^j_{n-1|}.$$

As before

(4.6) $$y^a_{;ij}\,\xi^i_{1|}\,\xi^j_{n-1|} = 0,$$

(4.6′) $$k'_{n-2}\,\xi^i_{n-2|}y^a_{,i} = 0,$$

and

(4.6″) $$k'_{n-2} = 0,$$

which gives

(4.6‴) $$k_{m-2} = k'_{n-2}.$$

The curve is thus an R_2-asymptotic line and

$$\eta^a_{m-2|} = \xi^i_{n-2|}y^a_{,i}.$$

Repeating the above procedure p times we verify that

(4.7) $$k'_{n-r} = 0, \quad r = 1, 2, \dots, p,$$

(4.7′) $$k_{m-r} = k'_{n-r}, \quad r = 1, 2, \dots, p.$$

Equation (4.7) shows that the curve is an R_p-geodesic in V_n while by (4.7′) it is an R_p-asymptotic line.

Conversely, suppose that the curve is an R_p-asymptotic line as well as an R_p-geodesic in V_n. Then

(4.8) $$k_{m-r} = k'_{n-r}, \quad r = 1, 2, \dots, p,$$

(4.8′) $$k'_{n-r} = 0, \quad r = 1, 2, \dots, p,$$

and hence

(4.8″) $$k_{m-r} = 0, \quad r = 1, 2, \dots, p.$$

Thus the curve is also an R_p-geodesic in the enveloping space.

This completes the proof of the following:

THEOREM 3. *A necessary and sufficient condition for a curve which lies in a subspace V_n and whose last normals relative to the two spaces coincide to be an R_p-geodesic $(p < n-1)$ of V_m is that it be both an R_p-geodesic and an R_p-asymptotic line in V_n.*

5. Let us consider a curve C lying in the subspace V_n whose last normals in the two spaces coincide. Thus

$$\eta^a_{m|} = \xi^i_{n|} y^a_{,i}, \tag{5.1}$$

which on differentiating along the curve gives

$$-k_{m-1}\eta^a_{m-1|} = -k'_{n-1}\xi^i_{n-1|}y^a_{,i} + y^a_{;ij}\xi^i_{1|}\xi^j_{n|}. \tag{5.2}$$

Now if $\eta^a_{m-1|}$ is orthogonal to V_n,

$$k'_{n-1}\xi^i_{n-1|}y^a_{,i} = 0,$$

from which

$$k'_{n-1} = 0,$$

i.e., the curve is an R_1-geodesic in V_n.

Conversely, if the curve is an R_1-geodesic in the subspace V_n,

$$k'_{n-1} = 0$$

and (5.2) yields

$$-k_{m-1}\eta^a_{m-1|} = y^a_{;ij}\xi^i_{1|}\xi^j_{n|},$$

which shows that $\eta^a_{m-1|}$ is orthogonal to V_n. Thus we have shown:

THEOREM 4. *A necessary and sufficient condition for a curve lying in a subspace V_n of V_m whose last normals relative to the two spaces coincide, to be an R_1-geodesic in V_n is that the $(m-2)$-th normal of the curve relative to V_m is orthogonal to V_n.*

Suppose next that the curve is an R_p-geodesic in V_n and an R_q-asymptotic line where $q < p$. Then

$$k'_{n-r} = 0, \quad r = 1, 2, \ldots, p, \tag{5.3}$$

and

$$k_{m-r} = k'_{n-r}, \quad r = 1, 2, \ldots, q, \tag{5.4}$$

$$\eta^a_{m-r+1|} = \xi^i_{n-r+1|}y^a_{,i}, \quad r = 1, 2, \ldots, q+1. \tag{5.4'}$$

(5.3) and (5.4) give

$$k_{m-r} = 0, \quad r = 1, 2, \ldots, q. \tag{5.5}$$

Also, since $p > q$,

$$k'_{n-q-1} = 0. \tag{5.6}$$

From (5.4′), for $r = q+1$, we have

$$\eta^a_{m-q|} = \xi^i_{n-q|}y^a_{,i}.$$

Differentiating this along the curve, we get, using (5.3), (5.5) and (5.6),

$$-k_{m-q-1}\eta^{a}_{m-q-1|} = y^{a}_{;ij}\xi^{i}_{1|}\xi^{j}_{n-q|},$$

and we can state

THEOREM 5. *The $(m-q-2)$-th normal relative to V_m of a curve which is an R_p-asymptotic line and an R_p-geodesic ($p > q$) in the subspace V_n of V_m is orthogonal to V_n at all points of the curve and its $(m-q-1)$-th curvature relative to V_m is given by*

$$k_{m-q-1} = -\sum_{r=n+1}^{m}\Omega_{r|ij}\xi^{i}_{1|}\xi^{j}_{n-q|}N^{a}_{r|}\eta_{m-q-1|a}.$$

Remarks.

1. If $q = 0$, $p = 1$, this reduces to the necessary part of Theorem 4.

2. If V_n is a hypersurface, i.e., $m = n+1$, the normal of the hypersurface at all points of the curve is along the $(m-q-2)$-th normal of the curve relative to V_{n+1}.

6. We shall consider in this section an R-asymptotic line of maximum order $n-2$. In this case the last $n-1$ normals $\eta^{a}_{m-r+1|}$ $(r = 1, 2, \ldots, n-1)$ are all tangential to V_n and therefore from Theorem 2,

$$\eta^{a}_{m-r+1|} = \xi^{i}_{n-r+1|}y^{a}_{,i}, \quad r = 1, 2, \ldots, n-1. \tag{6.1}$$

Consequently, the remaining $m-n$ normals $\eta^{a}_{r|}$ $(r = 2, 3, \ldots, m-n+1)$ are all orthogonal to V_n. From Theorem 2 we have

$$k_{m-r} = k'_{n-r}, \quad r = 1, 2, \ldots, n-2, \tag{6.2}$$

and

$$y^{a}_{;ij}\xi^{i}_{1|}\xi^{j}_{m-r+1|} = 0, \quad r = 1, 2, \ldots, n-2. \tag{6.3}$$

Our object in this section is to derive an expression for k_{m-n+1} which depends only on the point and direction of the curve at that point. From (6.1) for $r = n-1$,

$$\eta^{a}_{m-n+2|} = \xi^{i}_{2|}y^{a}_{,i}.$$

Differentiating this along the curve and using (6.2), we get

$$-k_{m-n+1}\eta^{a}_{m-n+1|} = -k'_{1}\xi^{i}_{1|}y^{a}_{,i} + y^{a}_{;ij}\xi^{i}_{1|}\xi^{j}_{2|}. \tag{6.4}$$

Since $\eta^{a}_{m-n+1|}$ is orthogonal to V_n,

$$k'_{1} = 0,$$

and the curve is a geodesic of order one in V_n. Thus [2]

$$k_{1}\eta^{a}_{2|} = y^{a}_{;ij}\xi^{i}_{1|}\xi^{j}_{1|}. \tag{6.5}$$

Also (6.4) reduces to

(6.6) $$-k_{m-n+1}\eta^{a}_{m-n+1|} = y^{a}_{;ij}\xi^{i}_{1|}\xi^{j}_{2|},$$

so that

(6.6′) $$k_{m-n+1} = -y^{a}_{;ij}\xi^{i}_{1|}\xi^{j}_{2|}\eta_{m-n+1|a}.$$

The fundamental tensors of V_m and V_n satisfy

(6.7) $$a_{\alpha\beta} = \sum_{r=1}^{m}\eta_{r|\alpha}\eta_{r|\beta},$$

(6.7′) $$g^{ij} = \sum_{s=1}^{n}\xi^{i}_{s|}\xi^{j}_{s|}.$$

From (6.6)

(6.8) $$(y^{a}_{;ij}\xi^{i}_{1|}\xi^{j}_{2|})\eta_{r|a} = 0, \quad r = 1, 2, \ldots, m-n,\ m-n+2, \ldots, m.$$

Now

$$(k_{m-n+1})^2 = \eta_{m-n+1|\alpha}\eta_{m-n+1|\beta}y^{\alpha}_{;ij}y^{\beta}_{;kl}\xi^{i}_{1|}\xi^{j}_{2|}\xi^{k}_{1|}\xi^{l}_{2|}$$
$$= \sum_{r=1}^{m}\eta_{r|\alpha}\eta_{r|\beta}y^{\alpha}_{;ij}y^{\beta}_{;kl}\xi^{i}_{1|}\xi^{j}_{2|}\xi^{k}_{1|}\xi^{l}_{2|} = a_{\alpha\beta}y^{\alpha}_{;ij}y^{\beta}_{;kl}\xi^{i}_{1|}\xi^{k}_{1|}\Big(g^{jl} - \sum_{\substack{s=1\\ s\neq 2}}^{n}\xi^{j}_{s|}\xi^{l}_{s|}\Big),$$

and by (6.3),

$$(k_{m-n+1})^2 = g^{jl}a_{\alpha\beta}y^{\alpha}_{;ij}y^{\beta}_{;kl}\xi^{i}_{1|}\xi^{k}_{1|} - a_{\alpha\beta}y^{\alpha}_{;ij}y^{\beta}_{;kl}\xi^{i}_{1|}\xi^{j}_{1|}\xi^{k}_{1|}\xi^{l}_{1|},$$

and then by (6.5),

$$(k_{m-n+1})^2 = g^{jl}a_{\alpha\beta}y^{\alpha}_{;ij}y^{\beta}_{;kl}\xi^{i}_{1|}\xi^{k}_{1|} - a_{\alpha\beta}(k_1\eta^{\alpha}_{2|})(k_1\eta^{\beta}_{2|})$$

or

$$(k_{m-n+1})^2 = a_{\alpha\beta}g^{jl}y^{\alpha}_{;ij}y^{\beta}_{;kl}\xi^{i}_{1|}\xi^{k}_{1|} - k_1^2.$$

Thus we have obtained an expression for k_{m-n+1} which depends only on the point and the direction of the curve at that point.

7. Let $e^{i}_{h|}$ be the unit tangents to the n-congruences of an orthogonal ennuple in V_n and let $\xi^{i}_{hk|}$ $(k = 1, 2, \ldots, n)$ be the unit tangent and $n-1$ unit normals at a point of the congruence with the unit tangent $e^{i}_{h|}$. Then $\xi^{i}_{hk|}$ can be expressed as a linear combination of $e^{i}_{h|}$. Thus let

(7.1) $$\xi^{i}_{hk|} = C^{l|}_{hk|}e^{i}_{l|}, \quad h, l, k = 1, 2, \ldots, n.$$

From Theorem 1, the necessary and sufficient conditions for the curve of the congruence $e^{i}_{h|}$ to be an R_p-geodesic are

$$\xi^{i}_{hk|,j}e^{j}_{h|}e_{t|i} = 0, \quad t = 1, 2, \ldots, n;\ k = n-p+1,\ n-p+2, \ldots, n,$$

i.e., are

$$(C^{l|}_{hk|} e^{i}_{l|})_{,j} e^{j}_{,h|} e_{t|i} = 0, \quad t = 1, 2, \ldots, n; \; k = n-p+1, \; n-p+2, \ldots, n,$$

i.e., are

$$C^{t|}_{hk|,j} e^{j}_{h|} + C^{l|}_{hk|} \gamma_{lth} = 0, \quad t = 1, 2, \ldots, n; \; k = n-p+1, \; n-p+2, \ldots, n.$$

If the C's are constants, the conditions reduce to

$$C^{l|}_{hk|} \gamma_{lth} = 0, \quad t = 1, 2, \ldots, n; \; k = n-p+1, \; n-p+2, \ldots, n.$$

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