Classical boundary value problems for integrable temperatures in a $C^1$ domain

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Abstract. We study a Neumann problem for the heat equation in a cylindrical domain with $C^1$-base and data in $h^1$, a subspace of $L^1$. We derive our results, considering the action of an adjoint operator on $B_T MOC$, a predual of $h^1$, and using known properties of this last space.

Introduction. The classical Neumann and Dirichlet problems for the heat equation in the cylindrical domain $D \times (0, T)$ have been studied by E. Fabes and N. Rivière ([5]) using the method of layer potentials. They considered bounded domains of class $C^1$ and $L^p$ data on the boundary $\partial D \times (0, T)$ for $1 < p < \infty$. For $p = 1$ the problems are still open. As for the Laplacian we must introduce a suitable subspace of $L^1$ on which the corresponding results can be proved.

In [3] E. Fabes and C. Kenig studied the Neumann problem for the Laplacian with data in $h^1$, a subspace of $L^1$ whose dual is BMO. They use the fact that the solvability of the Neumann problem with boundary data in $h^1$ is closely related to solvability of the Dirichlet problem with boundary data in BMO.

In order to study the Neumann problem for the heat operator with suitable $h^1$ data, we shall consider the action of an adjoint operator on a subspace $B_T MOC$ of caloric-BMO. The definition of $B_T MOC$ is analogous to that of $B_0 MOC$ ([7]).

As usual we must construct $h^1$ (caloric-$h^1$), a predual of $B_T MOC$, making use of an atomic representation.

Applying the results obtained on $B_0 MOC$ ([7], [8]) we solve the Neumann problem with data in $h^1$.

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§ 1. Definitions and preliminaries. A bounded domain $D \subset \mathbb{R}^n$ is called a $C^1$ domain if for each point $Q \in \partial D$ there exists a ball $B = B(r, Q)$ with center $Q$ and

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radius \( r \) and a coordinate system of \( \mathbb{R}^n \) with \( Q \) as the origin such that with respect to these new coordinates
\[
B \cap D = B \cap \{(x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > \Phi(x')\},
\]
\[
B \cap \partial D = B \cap \{(x', \Phi(x')) : x' \in \mathbb{R}^{n-1}\},
\]
where \( \Phi \in C^1_0(\mathbb{R}^{n-1}) \), \( \Phi(0) = 0 = (\partial \Phi/\partial x_i)(0), i = 1, \ldots, n. \)

We will assume the radius of the ball \( B \) can be chosen independently of \( Q \in \partial D \), and we call it \( r_0 \); it is clearly a geometric constant depending only on the structure of \( D \).

If \( D \) is a bounded \( C^1 \) domain we will let \( N_Q \) denote the unit inner normal to \( \partial D \) at \( Q \). We write
\[
D_+ = D \times \mathbb{R}^+, \quad D_T = D \times (0, T),
\]
\[
\partial D_+ = \partial D \times \mathbb{R}^+, \quad \partial D_T = \partial D \times (0, T), \quad \text{for} \quad 0 < T < \infty.
\]
\( X, Y, \ldots \) are points in \( D \) (or \( \mathbb{R}^n \)), while \( P, Q, \ldots \) are points of \( \partial D \). The letters \( t \) and \( s \) are used for the time variable in \( \mathbb{R}^+ \). We set
\[
\Gamma(X, t) = (\pi t)^{-n/2} \exp(-|X|^2/4t),
\]
the fundamental solution of the heat equation, and
\[
K(X, t) = \langle \nabla_{x} \Gamma(X, t), N_Q \rangle = c_n \frac{\langle X-Q, N_Q \rangle}{t^{n/2+1}} \exp\{-|X-Q|^2/4t\},
\]
the kernel of the double layer potential.

We recall the definition of the space \( BMOC(\partial D_T) \) (\([7]\)):
\( f \in BMOC(\partial D_T) \) if
\[
\|f\|_* = \sup_{\mathcal{D}_l} \|\partial_l^{-1} \int_{\mathcal{D}_l} |f - f_0| dQ_d Q_s ds \| < \infty
\]
where \( \mathcal{D} = \mathcal{D}(P, t) = \{(Q, s) \in \partial D_T : |P - Q| < r, |s - t| < r^2\}, f_0 = |\partial_l|^{-1} \int_{\mathcal{D}_l} f \).

With the identification \( f_1 \sim f_2 \) if \( f_1 - f_2 = \text{constant} \), \( BMOC \) is a complete normed space with norm (1.1).

We say that \( f \in B_0 MOC (\([7]\)) \) if (1.1) is valid and
\[
B_0(f) = \sup_{\mathcal{D}_0} \|\partial_l^0|^{-1} \int_{\mathcal{D}_0} f \| < \infty
\]
where \( \mathcal{D}_0 = \mathcal{D}_0(P_0) = \{(P, t) \in \partial D_T : |P - P_0| < r, 0 < t < r^2\}. B_0 MOC \) is a complete normed space with norm
\[
\|f\|_{0,*} = B_0(f) + \|f\|_*
\]
For our purpose we introduce the space \( B_T MOC \), which is analogous to \( B_0 MOC \), except that the condition (1.2) at \( t = 0 \) is replaced by a condition at \( t = T \) : \( f \in B_T MOC \) if (1.1) is valid and
\[
B_T(f) = \sup_{\mathcal{D}_T} \|\partial_l^T|^{-1} \int_{\mathcal{D}_T} f \| < \infty
\]
where $A^T = A^T_0(P_0) = \{(P, t) \in \partial D_T: |P - P_0| < r, T - r^2 < t < T\}$. $B_T$ MOC is a complete normed space with norm

$$\|f\|_{T, *} = \|f\|_* + B_T(f).$$

As done for $B_0$ MOC ([7]), it is possible to show that (1.5) is equivalent to the norm

$$\|f\|_{p, T, *} = C_{p, T}(f) + \|f\|_{*, p}$$

where

$$C_{p, T}(f) = \sup_{\partial D_T} \{|A^T|^{-1} \int_{\partial D_T} |f|^p\}^{1/p}, \quad \|f\|_{*, p} = \sup_{\partial D_T} \{|A|^{-1} \int_{\partial D_T} |f - f_d|^p\}^{1/p}.$$

We now consider the so-called adjoint of the heat operator, $I^* = A_X + D_t$. Choosing a Banach space $E = E(\partial D_T)$ of functions defined on $\partial D_T$, we can associate to this operator the Dirichlet problem

$$\begin{cases}
I^* u = 0 & \text{in } D_T, \\
u(X, T) = 0, \\
u(X, t)|_{\partial D_T} = f,
\end{cases}$$

where $f$ is in $E$.

If we look for a solution of (1.7) in the form of a double layer potential, we will write it

$$u(X, s) = \int \frac{\langle X - Q, N_Q \rangle}{\delta D} \exp\left\{ -|X - Q|^2/4(t - s) \right\} g(Q, t) dQ dt$$

where $g$ is an unknown function in $E$.

The trace on $\partial D_T$ of $u(X, s)$ gives rise to the operator

$$\Phi(g)(P, s) = \int \frac{\langle P - Q, N_Q \rangle}{\delta D} \exp\left\{ -|P - Q|^2/4(t - s) \right\} g(Q, t) dQ dt.$$

When $E = L^p(\partial D_T)$, $\Phi$ is a singular integral operator. Using, with the obvious modifications, the techniques and results of [5] for the operator

$$J(g)(P, t) = \int \frac{\langle P - Q, N_Q \rangle}{\partial D} \exp\left\{ -|P - Q|^2/4(t - s) \right\} g(Q, s) dQ ds$$

one can show that $\Phi$ is bounded linear operator on $L^p(\partial D_T)$, $1 < p < \infty$. Moreover, if $(a, b) < (0, T)$ then

$$\|\Phi(\chi_{(a,b)} f)\|_{L^p(\partial D \times (a,b))} \leq \omega(b - a) \|f\|_{L^p(\partial D \times (a,b))}$$

where $\omega(\delta) \to 0$ as $\delta \to 0^+$

The solvability of (1.7), with the method of the double layer potential, is related, as for the analogous problem for the heat operator, to the invertibility of the operator $c_n I + \Phi$ where $c_n > 0$ and $I$ is the identity operator.
This can be obtained directly as in [5] using property (1.10), or simply observing that $\Phi$ is the adjoint of $J'$, where

$$J'(y)(P, t) = -\int_0^t \int_{\partial D} \langle P-Q, N_p \rangle \exp\{-|P-Q|^2/4(t-s)\} g(Q, s) dQ ds,$$

and using the results of [5] for this operator.

So the problem (1.7), for $E = L^p(\partial D_T), 1 < p < \infty$, has a unique solution if the convergence of $u(X, \tau)$ to $f$ is understood in the non-tangential sense. When $E = B_T$ MOC the continuity of $\Phi$ in $E$ and the invertibility of $c_n I + \Phi$ in $E$ can be obtained, with only slight modifications, in the same manner as for the operator $J$ in $B_{o}$ MOC (see [7], [8]).

§ 2. The space $h^{1,q}_c(\partial D_T)$. We begin by giving the definition of a "caloric" atom. We say that a function $a = a(P, t)$ is a $(1, q)$ c-atom, $1 < q < \infty$, if

(i) the support of $a$ is contained in $A \subset \partial D_T$, with $A = A_c(P_0, t_0),$

(ii) $\left(\int_A |a(P, t)|^q\right)^{1/q} \leq |A|^{-1 + 1/q},$

(iii) $|\int_A a(P, t)| \leq r/(T-t_0 + r^2)^{1/2}$

When $q = \infty$, (ii) becomes $|a(P, t)| \leq |A|^{-1}$.

The definition of c-atom is a generalization of the classical definition of atom, see [11]; in fact the condition $\int_{\partial D_T} a = 0$ is replaced by the more general condition (iii).

We define $h^{1,q}_c, 1 < q < \infty$, to be the space of functions admitting an atomic decomposition

$$f = \sum_{j=1}^{\infty} \lambda_j a_j$$

where the $a_j$ are $(1, q)$ c-atoms, $\sum_{j=1}^{\infty} |\lambda_j| < \infty$ and the convergence is in the sense of $L^1(\partial D_T)$. The infimum of the numbers $\sum_{j=1}^{\infty} |\lambda_j|$ taken over all such representations will be denoted by $\|f\|_{1,q}$; it turns out to be a norm on $h^{1,q}_c(\partial D_T)$.

**Theorem 2.1.** $B_T$ MOC$_q(\partial D_T)$ is the dual space of $h^{1,q}_c$, with $1/q + 1/q' = 1$, i.e. for each $f \in B_T$ MOC and for all $g = \sum_{j=1}^{\infty} \lambda_j a_j \in h^{1,q}_c$, the mapping $f \rightarrow \langle f, \lambda_j \rangle\int_{\partial D_T} a_j$ is a well defined linear functional on $h^{1,q}_c$. Moreover, each continuous functional has this form.

**Proof.** Fix $f \in B_T$ MOC and let $a$ be a $(1, q)$ c-atom. Then $|\langle f, a \rangle| = |\int f a| \leq |\int (f - \int f) a| + |\int f a| = I' + I''$ By H"{o}lder's inequality we have

$$I' \leq \left( \int |a|^q \right) ^{1/q} \left( \int |f - \int f a|^q \right) ^{1/q'} \leq |A|^{-1} \int |f - \int f a|^q \leq \|f\|_{1,q}.$$ 

In the same manner
\[ I'' \leq \frac{r}{(T-t_0+r^2)^{1/2}} \| \partial_t^{-1} \int_\partial \| \leq \frac{r}{(T-t_0+r^2)^{1/2}} \left\| \Delta \right\|^n \left( \int_\partial \left\| l \right\|^{n+1} \right)^{1/(n+1)} \]

\[ \leq c \left( \| \partial_t^{-1} \|_{\infty} \right)^{1/(n+1)} \| l \|_{q', \tau, *} \leq c \| l \|_{q', \tau, *}, \]

where \( \tilde{r} = (T-t_0+r^2)^{1/2} \), and \( c \) depends only on the dimension of the space. More generally, for any \( f = \sum_{i=1}^n \lambda_j a_j \in h^1_{q'} \) we have \( \langle l, f \rangle = \lim_{n \to \infty} \sum_{j=1}^n \lambda_j \langle a_j, l \rangle \).

By the previous argument we have \( |\langle l, a_j \rangle| \leq c \| l \|_{q', \tau, *} \) for all \( j \). Therefore

\[ |\langle l, f \rangle| \leq c \left( \lim_{n \to \infty} \sum_{j=1}^n |\lambda_j| \right) \| l \|_{q', \tau, *} \leq c \| f \|_{1, q'} \| l \|_{q', \tau, *}, \]

and so \( B_{T, MOC_q} \preceq h^1_{q'} \).

To prove the reverse inclusion we first observe that any function \( f \in \mathcal{F}(\Delta) \) such that \( \| f \|_{L^q} = 1 \) and \( \text{supp}(f) \subseteq \Delta \) can be normalized to be a \( (1, q') \) c-atom. In fact, letting

\[ (2.1) \quad g_1 = (|\Delta|^{1/q'}/|\partial_t^r|) f \]

we have \( \text{supp}(g_1) \subseteq \Delta \) and

\[ \left( \int_\Delta |g_1|^{q'} \right)^{1/q'} \leq |\Delta|^{1/q'} \int_\partial |f| \leq |\Delta|^{-1+1/q'} \]

In addition

\[ \left( \int_\Delta |g_1| \right) \leq \frac{|\Delta|^{1/q'}}{|\partial_t^r|} \int_\Delta |f| \leq \frac{|\Delta|^{1/q'}}{|\partial_t^r|} |\Delta|^{1/q'} \leq c \frac{r}{(T-t_0+r^2)^{1/2}}, \]

so (iii) holds.

Furthermore, besides the caloric \( g_1 \) we can associate to \( f \) another caloric atom

\[ (2.2) \quad g_2 = (f - f_\partial)/(2|\Delta|^{1/2}) \]

having integral zero on \( \partial D_T \).

Suppose now \( L \) is a linear functional on \( h^1_{q'}(\partial D_T) \). Given any \( \Delta \) in \( \partial D_T \) and any \( f \in L^q(\Delta) \), where \( L^q(\Delta) \) is the class of all \( f \in \mathcal{F}(\Delta) \) such that \( \text{supp}(f) \subseteq \Delta, f \) belongs to \( h^1_{q'}(\Delta) \). Assuming, as we may, that \( \| f \|_{L^q(\Delta)} = 1 \) and writing \( f = g_1 \| D_T^r \| |\Delta|^{-1/q'} \), since \( g_1 \) is a \( (1, q') \) c-atom, it follows that

\[ \| f \|_{h^1_{q'}} \leq \| D_T^r \| \| \Delta \|^{1/q'} \]

Therefore, for any \( f \in L^q(\Delta) \)

\[ |L(f)| \leq \| L \| \frac{|D_T^r|}{|\Delta|} \| f \|_{L^q(\Delta)}, \]

J — Annales Polonici Math. 54.1
i.e., $L$ is a bounded linear functional on $L^q_0(\Delta)$. By the Hahn–Banach theorem, $L$ extends continuously to all $f \in L^q(\Delta)$. Thus, by the Riesz representation theorem, there exists $\ell \in B^*(\Delta)$ such that $L(f) = \int_\Delta \ell f$ for all $f \in L^q(\Delta)$. In particular, for $\Delta = \partial D_T$ it follows that $\ell \in B^*(\partial D_T)$.

Recall that, given $f \in L^q(\Delta_T)$ with supp$(f) \subset \Delta_T$ and norm 1, then $g_1$, defined by (2.1), is a $(1, q')$ c-atom. Hence

$$
|L(g_1)| = \left| |\Delta_T|^{1/q} \int_{\Delta_T} f \right| = |L(|\Delta_T|^{-1/q} f)| \leq ||L||.
$$

Moreover, because $g_2$ is defined by (2.2), for all $f \in L^q(\Delta)$

$$
\left| \int_{\Delta} f(l - l_{\Delta}) \right| = \left| \int_{\Delta} f l - \int_{\Delta} f l_{\Delta} \right| = \left| \int_{\Delta} f l - \int_{\Delta} f l |\Delta|^{-1} \int_{\Delta} l \right| = \left| \int_{\Delta} f l - \int_{\Delta} f l_{\Delta} \right| = \left| \int_{\Delta} l(f - f_{\Delta}) \right|.
$$

Hence

$$
|\Delta|^{-1/q} \int_{\Delta} |l| |f - f_{\Delta}| = 2 \int_{\Delta} |f - f_{\Delta}| |\Delta|^{-1/q} = 2 |L(g_2)| \leq 2 ||L||.
$$

Consequently

$$
(\int \Delta|^{-1/q} |l - l_{\Delta}|^q)^{1/q} \leq 2 ||L||.
$$

Combining (2.3) and (2.4) it follows that $\ell \in B^*(\partial D_T)$.

The space $h^{1,q}(\partial D_T)$ is easily shown to be complete, and if $1 < q_1 < q_2 < \infty$ then

$$
h^{1,\infty} \subset h^{1,q_2} \subset h^{1,q_1}.
$$

The spaces $h^{1,q}_c$ defined above are not a particular case of the spaces $H^{1,q}$ associated to a space of homogeneous type, introduced by R. Coifman and G. Weiss in [1].

In fact, in our space $\partial D_T$, there is no distance for which the anisotropic surface disc $\Delta$ is a sphere. In [9] we studied the geometric structure of $\partial D_T$ in relation to the surface disc $\Delta$, and we established some different results. In particular, we have the following

**Theorem 2.2.** For any $1 < q < \infty$, $h^{1,q} = h^{1,\alpha}$, and the two norms are equivalent.

This theorem enables us to define the space $h^{1}_c$ as one of the spaces $h^{1,q}$, $1 < q < \infty$, and to use the most convenient one in the proofs of the theorems. In the following we will use the space $h^{1,2}_c$ and clearly the same results are valid for any $h^{1,q}_c$.

§3. The Neumann problem with data in $h^{1}_c(\partial D_T)$. Given $g \in h^{1}_c(\partial D_T)$, we consider the Neumann problem for the heat equation
\[
\begin{align*}
\left\{ \begin{array}{l}
Lu(X, t) = \Delta_X u - D_t u = 0 \quad \text{for all } (X, t) \in \partial D_T, \\
\lim_{t \to 0^+} u(X, t) = 0 \quad \text{uniformly on compact subsets of } D, \\
\partial_{N^p} u(X, t) \to g(P, z).
\end{array} \right.
\end{align*}
\]

The last statement is that \( \partial_{N^p} u(X, t) \) approaches \( g(P, z) \) in non-tangential sense, i.e.

\[
\lim_{(X, t) \to (P, z)} \langle \nabla_X u(X, t), N_P \rangle = g(P, z) \quad \text{a.e.}
\]

where

\[
\Gamma(P, z) = \{(X, t) : |X - P| + |t - z|^{1/2} < (1 + \beta) \text{dist}(X, \partial D)\} \cap D_T,
\]

\( \beta \) a constant giving the opening of the cone \( \Gamma \).

We look for a solution of (3.1) in the form of a single layer potential

\[
u(X, t) = -2 \int_0^t \int_{\partial D} \frac{\exp\left\{-|X - Q|^2/4(t - s)\right\}}{(t - s)^{n/2}} f(Q, s) \, dQ \, ds
\]

where \( f \) is an unknown function in \( h^1_{\tau}(\partial D_T) \).

The first step is to study the behavior on the boundary of

\[
\langle \nabla_X u(X, t), N_P \rangle = \int_0^t \int_{\partial D} \frac{\langle X - Q, N_P \rangle}{(t - s)^{n/2} + 1} \exp\left\{-|X - Q|^2/4(t - s)\right\} f(Q, s) \, dQ \, ds
\]

whose trace on \( \partial D_T \) is the singular integral operator

\[
J(f) = \lim_{\varepsilon \to 0^+} \int_0^{1 - \varepsilon} \int_{\partial D} \frac{\langle P - Q, N_P \rangle}{(t - s)^{n/2} + 1} \exp\left\{-|P - Q|^2/4(t - s)\right\} f(Q, s) \, dQ \, ds
\]

\[
= \lim_{\varepsilon \to 0^+} \int_0^{1 - \varepsilon} \int_{\partial D} K'(P - Q, t - s) f(Q, s) \, dQ \, ds.
\]

As already remarked, \( J' \) is the adjoint of \( -\Phi \) and is continuous from \( L^p(\partial D_T) \) to \( L^p(\partial D_T), 1 < p < \infty \).

Now we will show that \( J' \) is a continuous operator from \( h^1_{\tau}(\partial D_T) \) to \( h^1_{\tau}(\partial D_T) \). To do this, we need some lemmas.

**Lemma 3.1.** If \( a = a(P, t) \) is a \((1, 2)\) c-atom with support in \( \Delta = \Delta_r(P_0, t_0) \), then

\[
|\int_{\partial D_T} J'(a)| \leq c \frac{r}{(T - t_0 + r^2)^{1/2}}, \quad c \text{ independent of } a.
\]

**Proof.** We consider first the case \( r^2 < T - t_0 \). Recalling that \( \sup_n |J'_n(a)| \in L^2 \) (see [5]), we can write
\[
\lim_{t \to 0^+} J'(a) = \lim_{t \to 0^+} \int_{\partial D_T} J'_t(a) = \lim_{t \to 0^+} \int_{\partial D_T} J'_t(a) \\
= \lim_{t \to 0^+} \int_{\partial D} \int_{0}^{t} \{ \int_{\partial D} K'(P - Q, t - s) a(Q, s) dQ dS \} dP dt \\
= \lim_{t \to 0^+} I_t.
\]

Now performing the change of variable \( z = \frac{|P - Q|^2}{(t - s)} \), we have
\[
I_t = \int_{\partial D} a(Q, s) \{ \int_{\partial D} \frac{\langle P - Q, N_P \rangle}{|P - Q|^n} \left( \int_{\partial D} z^{n/2 - 1} e^{-4z} dz \right) dP \} dQ ds \\
= \int_{\partial D} a(Q, s) \{ \int_{\partial D} \frac{|P - Q|^2 e^{-4z}}{|P - Q|^3 (T - s)} dz dP \} dQ ds \\
- \int_{\partial D} a(Q, s) \{ \int_{\partial D} \frac{|P - Q|^2 (T - t_0)}{dz dP} dQ ds = I'_t - I''
\]

Using the results in [7], we have
\[
\lim_{t \to 0^+} |I'_t| = (c_n/2) \Gamma(n/2) \int_{\partial D} |a| \leq c \frac{r}{(T - t_0 + r^2)^{1/2}}.
\]

Moreover,
\[
I'' = \int_{\partial D} a(Q, s) \{ \int_{\partial D} \frac{|P - Q|^2 (T - t_0)}{dz dP} dQ ds \\
+ \int_{\partial D} a(Q, s) \{ \int_{\partial D} \frac{|P - Q|^2 (T - t_0)}{dz dP} dQ ds = I'_1 + I'_2.
\]

If \( s < t_0 \), observing that \( (T - s)(T - t_0)^{-1} < 2 \), we have
\[
|I'_1| \leq \int_{\partial D} a(Q, s) \left\{ \int_{\partial D} \frac{|P - Q|}{T - s} \frac{1}{(T - t_0)^{n/2 - 1}} \frac{r^2}{T - t_0} e^{-4|P - Q|^2 (T - s)} dP \right\} dQ ds \\
\leq c \frac{r^2}{T - t_0} \int_{\partial D} a(Q, s) \left\{ \int_{\partial D} \frac{|P - Q|}{(T - s)^{n/2 - 1}} \int_{\partial D} \frac{|P - Q|^2}{(T - s)^{1/2}} e^{-4|P - Q|^2 (T - s)} dP \right\} dQ ds.
\]

Since \( \dim(\partial D) = n - 1 \), the term \( \{ \} \) is uniformly bounded in \( T = s \) and \( P \), so that
\[
|I'_1| \leq c \frac{r^2}{T - t_0} \leq c \frac{r}{(T - t_0 + r^2)^{1/2}}.
\]

Reasoning in the same manner, for \( s > t_0 \) we obtain \( |I''_1| \leq c r/(T - t_0 + r^2)^{1/2} \).

Now we split \( I''_2 \) as follows:
\[ I'_2 = \int \int_{|P-Q|<\delta r} \left[ \int_{|P-Q|<\delta r} \int_0^{|P-Q|^2/(T-t_0)} \cdots \left( \int_0^{|P-Q|^2/(T-t_0)} \cdots dz \right) dP \right] dQ ds \]
\[ + \int \int_{|P-Q|>\delta r} \left[ \int_{|P-Q|>\delta r} \int_0^{|P-Q|^2/(T-t_0)} \cdots \left( \int_0^{|P-Q|^2/(T-t_0)} \cdots dz \right) dP \right] dQ ds = A + B. \]

We have
\[ A \leq c \int \int_{|P-Q|<\delta r} \left( \int_{|P-Q|<\delta r} \frac{|P-Q|^2}{(T-t_0)^{n/2}} \right) dP ds \leq c \frac{r^2}{(T-t_0)^{n/2}} \int \int_{|P-Q|<\delta r} a \]
\[ \leq cr/(T-t_0 + r^2)^{1/2}. \]

For estimating \( B \), we split it in two parts:
\[ B = \int \int_{|P-Q|>\delta r} \left[ \int_{|P-Q|>\delta r} \int_0^{|P-Q|^2/(T-t_0)} \cdots \left( \int_0^{|P-Q|^2/(T-t_0)} \cdots dz \right) dP \right] dQ ds \]
\[ + \int \int_{|P-Q|>\delta r} \left[ \int_{|P-Q|>\delta r} \int_0^{|P-Q|^2/(T-t_0)} \cdots \left( \int_0^{|P-Q|^2/(T-t_0)} \cdots dz \right) dP \right] dQ ds = B_1 + B_2. \]

We have
\[ |B_1| \leq c \int \int_{|P-Q|>\delta r} \left( \int_{|P-Q|>\delta r} |P-Q|^{1-n} \frac{|P-P_0||Q-P_0|}{T-t_0} \right)^{(n-2)/2} \]
\[ \times \exp \left\{ -\frac{|P-Q|^2/(T-t_0)^{n/2}}{r/(T-t_0 + r^2)^{1/2}} \right\} dP ds \leq cr/(T-t_0 + r^2)^{1/2}, \]
\[ B_2 = \int \int_{|P-Q|>\delta r} \left( \int_{|P-Q|>\delta r} \left( \frac{\langle P-Q, N_P \rangle}{|P-Q|^n} \frac{\langle P-P_0, N_P \rangle}{|P-P_0|^n} \right)^{\frac{|P-P_0|^2/(T-t_0)}{2}} \cdots \left( \int_0^{|P-Q|^2/(T-t_0)} \cdots dz \right) dP \right\} dQ ds \]
\[ \times dQ ds + \int \int_{|P-Q|>\delta r} \left( \int_{|P-Q|>\delta r} \left( \frac{\langle P-P_0, N_P \rangle}{|P-P_0|^n} \right)^{\frac{|P-P_0|^2/(T-t_0)}{2}} \cdots \left( \int_0^{|P-Q|^2/(T-t_0)} \cdots dz \right) dP \right\} dQ ds \]
\[ \leq B'_2 + B'_2. \]

We can estimate \( B'_2 \) and \( B'_2 \) reasoning as before and we find that both are
\[ \leq cr/(T-t_0 + r^2)^{1/2}. \]

If \( r^2 \geq T-t_0 \), then \( 2^{-1/2} \leq r/(T-t_0 + r^2)^{1/2} \), so it is enough to show that
\[ \left| \int_{\partial D_T} J'(a) \right| < c, \]
where \( c \) is a constant independent of \( a \). This can be obtained by introducing a dyadic decomposition of \( \partial D_T \) as in the following Lemma 3.2, using the \( L^2 \)-continuity of \( J' \) and estimates of \( K'(P-Q, t-s) \) (see [7]), and reasoning as in the last part of case (a) of Lemma 3.3.

**Lemma 3.2.** If \( a = a(P, t) \) is a \((1, 2)\) c-atom such that \( \int_{\partial D_T} a = 0 \), then \( J'(a) \in h^1(\partial D_T) \) and its norm depends only on the dimension of the space.
Proof. We introduce some notations: if $\Delta$ is a surface disc of center $(P_0, t_0)$ and radius $r$ containing the support of $a$, we denote by $\Delta_i$ the surface disc concentric with $\Delta$ of radius $2^i r$, and we set

$$\chi_1 = \chi_{\Delta_1}, \quad \chi_i = \chi_{\Delta_i - \Delta_{i-1}} \quad \text{for } i \geq 2.$$ 

We distinguish two cases: (a) $T-t_0 > r^2$, (b) $T-t_0 \leq r^2$.

For case (a), let $p$ denote the integer such that

$$4^{-p-1}(T-t_0) \leq r^2 < 4^{-p}(T-t_0).$$

We split $J'(a)$ as follows:

$$J'(a) = \left( \chi_1 J'(a) + \frac{\chi_1}{|\Delta_1|} \int_{\partial D_T - \Delta_1} J'(a) \right) + \sum_{i=2}^p \left( \chi_i J'(a) + \frac{\chi_i}{|\Delta_i - \Delta_{i-1}|} \int_{\partial D_T - (\Delta_i - \Delta_{i-1})} J'(a) \right)$$

$$+ \sum_{i > p} \chi_i J'(a) - \frac{\chi_i}{|\Delta_1|} \int_{\partial D_T - \Delta_1} J'(a) - \sum_{i=2}^p \frac{\chi_i}{|\Delta_i - \Delta_{i-1}|} \int_{\partial D_T - (\Delta_i - \Delta_{i-1})} J'(a)$$

$$= M_1 + \sum_{i=2}^p M_i + \sum_{i > p} M_i - R_1 - \sum_{i=2}^p R_i.$$ 

We will show that, up to a multiplicative constant, $2^i M_i$ are $(1, 2)$-c-atoms. We start with $M_1$: by continuity of $J'(a)$ in $L^2(\partial D_T)$ we have

$$||M_1||_2 \leq ||J'(a)||_2 + |\Delta_1|^{-1/2} \int_{\partial D_T - \Delta_1} J'(a) \leq c |\Delta|^{-1/2}.$$ 

Moreover, using Lemma 3.1,

$$\int_{\partial D_T} M_1 \leq \int_{\partial D_T} J'(a) \leq cr/(T-t_0 + r^2)^{1/2}.$$ 

So $M_1$, up to a multiplicative constant, is a $(1, 2)$-c-atom.

For $2 \leq i \leq p$ we have

$$||M_i||^2 \leq c \left\{ \int_{\Delta_i - \Delta_{i-1}} J'(a)^2 + |\Delta_i|^{-1} \left( \int_{\partial D_T - (\Delta_i - \Delta_{i-1})} J'(a)^2 \right) \right\} = c \{ I_1 + I_2 \}.$$ 

Recalling that $\int_{\Delta} a = 0$, if we use the mean value theorem and the estimates on $K'$ we obtain, following the techniques in ([7], p. 303),

$$I_1 = \int_{\Delta_i - \Delta_{i-1}} \left( \int (K'(P - Q, t-s) - K'(P-P_0, t-t_0)) a(Q, s) dQ ds \right) dP dt$$

$$\leq c 2^{-2i} |\Delta_i|^{-1},$$ 

while, from Lemma 3.1 and Hölder’s inequality,

$$I_2 \leq c \{ |\Delta_i|^{-1} \left( \int_{\partial D_T} J'(a)^2 \right) + |\Delta_i|^{-1} \left( \int_{\partial D_T} J'(a)^2 \right) \}$$

$$\leq c |\Delta_i|^{-1} \left( \frac{r^2}{(T-t_0 + r^2)^2} + c \int_{\Delta_i} J'(a)^2 \right) \leq c |\Delta_i|^{-1} \left( \frac{r^2}{(T-t_0 + r^2)^2} + 2^{-2i} \right)$$

$$\leq c 2^{-2i} |\Delta_i|^{-1}$$.
We also have
\[
\left| \int_{\partial D_T} M_i \right| \leq c \cdot \frac{r}{(T-t_0+r^2)^{1/2}} \leq c 2^{-i} \frac{2^i r}{(T-t_0+(2^i r)^2)^{1/2}}
\]
\[
\leq c 2^{-i} \frac{2^i r}{(T-t_0+(2^i r)^2)^{1/2}}.
\]
So \(2^i M_i\), up to a multiplicative constant, is a \((1, 2)\) c-atom.

For \(i > p\), we proceed in the same manner in order to show that \(2^i M_i\) satisfies property (ii). For (iii) we recall that
\[
2^i r/(T-t_0+(2^i r)^2)^{1/2} > 2^{-1/2}.
\]
So
\[
\left| \int_{\partial D_T} M_i \right| \leq \left| \int_{A_{i-1}} \cdots \int_{A_1} J'(a) \right|
\]
\[
\leq \left| \int_{A_{i-1}} \cdots \int_{A_1} \left( \int (K'(P-Q, t-s) - K'(P-P_0, t-t_0)) a(Q, s) dQ ds \right) dP dt \right|
\]
\[
\leq c 2^{-i}.
\]
Next we can write
\[
R_1 + \sum_{i=2}^{p} R_i = \left( \frac{\chi_1}{|A_i|_{\partial D_T-A_i}} \int_{A_i} J'(a) + \frac{\chi_2}{|A_2-A_1|_{\partial D_T-A_1}} \int_{A_1} J'(a) \right)
\]
\[
+ \left( \frac{\chi_2}{|A_2-A_1|_{\partial D_T-A_2}} \int_{A_2} J'(a) + \frac{\chi_3}{|A_3-A_2|_{\partial D_T-A_2}} \int_{A_2} J'(a) \right) + \cdots + \left( \frac{\chi_p}{|A_p-A_{p-1}|_{\partial D_T-A_{p-1}}} \int_{A_{p-1}} J'(a) \right)
\]
\[
= T_1 + \ldots + T_p.
\]
We show that \(2^i T_i, 1 \leq i \leq p\), up to a multiplicative constant, is a \((1, 2)\) c-atom, with support in \(A_{i+1}\), except for \(T_p\) whose support is in \(A_p\). Clearly, \(2^i T_i\) satisfies property (iii). For (ii) we write for \(1 \leq i \leq p\),
\[
T_i = \frac{\chi_i}{|A_i-A_{i-1}|_{\partial D_T-A_i}} \int_{A_i} J'(a) + \frac{\chi_{i+1}}{|A_{i+1}-A_i|_{\partial D_T-A_i}} \int_{A_i} J'(a)
\]
\[
- \frac{\chi_{i+1}}{|A_{i+1}-A_i|_{\partial D_T-A_i}} \int_{A_i} J'(a) = I_1 + I_2 - I_3.
\]
Using Lemma 3.1 and (3.4) we have
\[
\int_{A_{i+1}} I_2^2 \leq c |A_{i+1}|^{-1} (\int_{\partial D_T} J'(a))^2 \leq c 2^{-2(i+1)} |A_{i+1}|^{-1},
\]
\[
\int_{A_{i+1}} I_3^2 \leq c |A_{i+1}|^{-1} \sum_{q > i+1} \int_{A_q-A_{q-1}} J'(a))^2 \leq c 2^{-2i} |A_{i+1}|^{-1}
\]
The same estimate is valid for \(I_1\).
For case (b): \( T-t_0 \leq r^2 \), since \( r/(T-t_0+r^2)^{1/2} > 2^{-1/2} \) we write
\[
J'(a) = \chi_a J'(a) + \sum_{i>2} \chi_i J'(a) = \sum_{i \geq 1} M_i
\]
and reasoning as before it is not hard to show that \( 2^i M_i \) is, up to a multiplicative constant, a \((1, 2)\) c-atom.

**Lemma 3.3.** If \( a = a(P, t) \) is a \((1, 2)\) c-atom with support in \( \Lambda = \Lambda_r(P, t_0) \), then \( J'(a) \in h^1_\beta(\partial D_T) \) and its norm depends only on \( n \).

**Proof.** Write
\[
a = \left(a - \frac{\chi_a}{|\Lambda|} \int_{\Lambda} a\right) + \frac{\chi_a}{|\Lambda|} \int_{\Lambda} a.
\]
Since the function in brackets is an atom with mean value zero, by the previous lemma, we are reduced to the case in which the atom has the form
\[
a = \frac{\chi_a}{|\Lambda|} \frac{1}{(T-t_0+r^2)^{1/2}}.
\]
As usual, we distinguish two cases: (a) \( T-t_0 > r^2 \), (b) \( T-t_0 \leq r^2 \).

**Case (a).** Using the notation of Lemma 3.2, we write
\[
J'(a) = \chi_a a + \sum_{i \geq 2} \chi_i a = \sum_{i \geq 1} M_i.
\]
If \( i \leq p \), the integer of Lemma 3.2, we have
\[
\int_{A_{i+1}} M_i = \frac{1}{|\Lambda|^2} \frac{r}{(T-t_0+r^2)^{1/2}} \int_{A_{i+1}-A_i} (\int K'(P-Q, t-s) dQ dS) dP dt 
\leq c 2^{-i} \frac{2^i r}{(T-t_0+(2^i r)^2)^{1/2}} \leq c 2^{-i} \frac{2^i r}{(T-t_0+(2^i r)^2)^{1/2}},
\]
while
\[
\int_{A_{i+1}} M_i^2 = \frac{1}{|\Lambda|^2} \frac{r^2}{(T-t_0+r^2)^{1/2}} \int_{A_{i+1}-A_i} (\int K'(P-Q, t-s) dQ dS)^2 dP dt 
\leq c |A_{i+1}|^{-1} \frac{r^2}{T-t_0+r^2} \leq c 2^{-2i} |A_{i+1}|^{-1}.
\]
If \( i > p \), since \( 2^{2i} r^2 + t_0 > T \) and \( T > t > s \) we have
\[
\int_{A_{i+1}} M_i^2 = \frac{1}{|\Lambda|^2} \int_{A_{i+1}-A_i} (\int K'(P-Q, t-s) dQ dS)^2 dP dt 
\leq c \frac{|T-t_0|}{2^{2i} r^2} \leq c \frac{2^{2p-2i}}{2^{2i} r^2} \frac{1}{2^{2i} r^2} 
\leq c \frac{2^{2p-2i}}{2^{2i} r^2} \frac{1}{|T-t_0|} \leq c 2^{2(p-i)} |A_{i+1}|^{-1}.
\]
Case (b) is shown by the same reasoning as in the last part of the previous case.

**Corollary 3.1.** The operator $J'$ is continuous from $h^1_\ast(\partial D_T)$ to $h^1_\ast(\partial D_T)$.

Moreover, since $h^1_\ast(\partial D_T)$ is the predual of $B_T MOC$, the invertibility of $c_n I + J'$ follows from the invertibility of $c_n I + \Phi$ on $B_T MOC$ (see §1).

§ 4. Boundary behavior of single layer potential. In the next theorem we study the behavior of the single layer potential near the boundary. Consider the parabolic cone

$$
\Gamma(P, t) = \{(X, z): |X - P| + |z - t|^{1/2} < (1 + \beta) \text{dist}(X, \partial D)\} \cap D_T
$$

where $\beta > 0$ gives the opening of $\Gamma$. For $u$ caloric in $D_T$

$$
N(u)(P, t) = \sup_{(X, s) \in \Gamma(P, t)} |u(X, s)|
$$

is the tangential maximal function of $u$ and for an atom $a(P, t)$, $u_a(X, t)$ is the single layer potential of $a$, i.e.

$$
u_a(X, z) = -2 \int_0^t \int_{\partial D} \frac{\exp\{-|X - Q|^2/4(z - s)^{n/2}\}}{(z - s)^{n/2}} a(Q, s) dQ ds.
$$

Since, given an atom $a = a(P, t)$ with support in $A = A_\epsilon(P_0, t_0)$, we can write

$$
a = \left( a - \frac{\chi_A}{|A|} \int_A a \right) + \frac{\chi_A}{|A|} \int_A a,
$$

the study of $u_a$ near the boundary reduces to the study of two cases:

(a) $a$ is an atom with mean value zero,

(b) $a$ is an atom of the form $a = c(\chi_A/|A|) r/(r^2 + r^2)^{1/2}$

Then we have the two lemmas.

**Lemma 4.1.** If $a$ is an atom of the form (b), then $N(\nabla X u_a) \in L^1(\partial D_T)$ and $||N(\nabla X u_a)||_{L^1} \leq c$, where $c = c_\beta$ is independent of $a$ ($\beta$ gives the opening of $\Gamma$).

**Proof.** As shown in [5], $N(\nabla X u_a)$ is in $L^2(\partial D_T)$ and $||N(\nabla X u_a)||_{L^2(\partial D_T)} \leq c||a||_{L^2(\partial D_T)}$.

If $A_\epsilon = A_\epsilon(P_0, t_0)$ has the same meaning as in Lemma 3.2 we write

$$
\int_{\partial D_T} N(\nabla X u_a) = \int_{D_T} N(\ldots) + \int_{D_T - D_T} N(\ldots) = I_1 + I_2.
$$

We have, from the previous observation,

$$
|I_1| \leq c r^{(n+1)/2} ||a||_{L^2} \leq c.
$$

In order to estimate $I_2$ we distinguish two cases:

$(\omega)$ $T - t_0 > r^2$, $(\psi)$ $T - t_0 \leq r^2$
Case (e). If $p$ is the integer introduced in Lemma 3.2, we split $I_2$ as follows:

$$I_2 = \sum_{i > p} \sum_{d_i - d_{i-1}} N(...) = \sum_{i > p} L_i + \sum_{i > p} L_i.$$

For $(P, t) \in A_{\gamma} - A_{\gamma-1}$, consider the cone $\Gamma(P, t)$. If $(X, z) \in \Gamma(P, t)$ and $(X, z)$ is such that $|X - P| + |z - t|^{1/2} \leq 2^{i-2}r$, we have:

$$|P - Q| \geq 2^{i-1}r \quad \text{implies} \quad |X - Q| = |X - P + P - Q|$$

$$\geq ||P - Q| - |X - P|| \geq c2^{i-1}r,$$

$$|z - s| \geq 2^{2i-2}r^2 \quad \text{implies} \quad |z - s| = |z - t + t - s|$$

$$\geq ||t - s|| - |z - t| \geq c2^{2i-2}r^2.$$

From this, it follows easily, using estimates in (\cite{7}, p. 303), that

$$|\nabla_X u_a| \leq c_\beta 2^{i-1}(n+1)^{1/2} \left(\frac{1}{r_0^2 + r^2}\right)^{1/2}.$$

On the other hand, for $(X, z) \in \Gamma(P, t)$ with $|X - P| + |z - t|^{1/2} \geq 2^{i-2}r$ we have $|X - Q| \geq \text{dist}(X, \partial D) \geq c_\beta (|X - P| + |z - t|^{1/2}) \geq c_\beta 2^{i-2}r$ and then

$$|\nabla_X u_a| \leq c_\beta 2^{i-1}(n+1)^{1/2} \left(\frac{1}{r_0^2 + r^2}\right)^{1/2}.$$

So for $i \leq p$, recalling the observations of Lemma 3.2 on $p$,

$$|L_i| \leq c_\beta \frac{\lambda_i - \lambda_{i-1}}{r_0^2 + 2^{(i-1)(n+1)}(T - t_0 + r^2)^{1/2}} \leq c_\beta 2^{i-1}$$

and for $i > p$ (see once again Lemma 3.2)

$$|L_i| \leq c_\beta \frac{2^{(i-1)(n+1)}(T - t_0)}{r_0^2 + 2^{(i-1)(n+1)}(T - t_0)} \leq c_\beta 2^{2(i - p)}.$$

Collecting these results we have $I_2 \leq C_\beta$.

**Lemma 4.2.** If $a$ is an atom of the form $(\psi)$ then $N(\nabla_X u_a) \in L^1(\partial D_T)$ and $\|N(\cdot)\|_{L^1} \leq c$ where $c = c_\beta$ is independent of $a$ (\beta gives the opening of $\Gamma$).

**Proof.** Splitting as in the previous lemma we have

$$\int_{\partial D_T} N(\nabla_X u_a) = \int_{\partial A_\gamma} N(...) + \int_{\partial D_T - A_\gamma} N(...) = I_1 + I_2.$$

For the same reason as in Lemma 4.1, $|I_1| \leq c$, while to estimate $I_2$, we use the fact that $a$ has mean value zero and we write

$$\nabla_X u_a(X, z) = \int_{\partial \partial D} (K(X - Q, z - s) - K(X - P_0, z - t_0)) \alpha(Q, s) dQ ds$$
where

\[ K(X - Q, z - s) = \frac{X - Q}{(z - s)^{n+2}/2} \exp(-|X - Q|^2/4(z - s)). \]

Recalling the estimates of \( K \) (see [7]), introducing once again the decomposition of Lemma 3.2, and reasoning as in the previous lemma we have \( |I_2| \leq c. \)

**Theorem 4.1.** There exists a constant \( c > 0 \), independent of \( f \), such that for any \( f \in h^1_\circ (\partial D_T), \)

\[ \|N(\nabla_X u_f)\|_{L^1(\partial D_T)} \leq c \|f\|_{h^1_\circ (\partial D_T)}. \]

Moreover, writing \( u = u_f, \)

\[ \langle \nabla_X u(X, t), N_p \rangle \to (c_n I + J')f(P, z) \]

pointwise for almost every \((P, z) \in \partial D_T\) as \((X, t) \to (P, z), (X, t) \in \Gamma (P, z),\) where \( c_n = \omega_n H(0)/2 \) and \( J' \) is the operator defined by (1.11).

**Proof.** Since \( c_n I + J' \) is invertible on \( h^1_\circ (\partial D_T), \) we only have to show the non-tangential convergence of \( \partial_{N_p} u_f \) to \((c_n I + J')f, \) where \( f \in h^1_\circ. \)

We know ([5]) that this convergence holds when \( f \in L^2(\partial D_T) \) and then also when \( f \) is an atom. Using very simple arguments (see [4], p. 7) one can easily show that this convergence holds for any \( f \in h^1_\circ (\partial D_T). \)

In order to solve the Neumann problem (3.1) it remains to show the uniqueness of the solution. To this end, we show

**Lemma 4.3.** Suppose \( Lu = 0, N(\nabla_X u) \in L^1(\partial D_T) \) and \( u(X, t) \to 0 \) as \( t \to 0^+ \) uniformly on compact subsets of \( D \). If \( \partial_{N_p} u = 0 \) on \( \partial D_T, \) then \( u = 0 \) on \( D_T. \)

**Proof.** Using techniques similar to those of Lemma 1.6 of [3], one can show that there exists an integer \( q > 1 \) such that \( N(\nabla_X u) \in B_2 \) and moreover \( u = 0 \) on \( \partial D_T. \) Then the fact that \( u = 0 \) in \( D_T \) is a consequence of Theorem 2.2 and Theorem 2.3 of [5].

**References**


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