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How to prove Fefferman's theorem without use of differential geometry

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Abstract. A simpler proof of Fefferman's theorem, which says that every biholomorphic mapping between strictly pseudoconvex domains with smooth boundaries can be extended to a diffeomorphism between theirs closures, is given. The properties of Bergman kernel function, essential for the proof of this theorem are pointed out.

In the second part of this paper these properties are stated in more general form. It is also proved that they hold for strictly starlike complete circular domains and are invariant under Cartesian product.

Hence the Fefferman's theorem is valid for some classes of domains with non-smooth boundaries.

The Fefferman's Theorem says that if D and G are strictly pseudoconvex domains with smooth boundaries and h is a biholomorphic mapping from D onto G, then h can be extended to a diffeomorphism between $\bar{D}$ and $\bar{G}$. The original proof of this theorem [2] consists of two parts; first, containing the proof of (local) boundary estimates of the Bergman kernel function $K_D(z, t)$ of strictly pseudoconvex domain D with smooth boundary, and second, in which the Fefferman's theorem is proved with use of these estimates and rather complicated machinery of differential geometry. In further versions of this proof the second part of it was slightly shortened by using some more powerful theorems from differential geometry.

The aim of the present paper is to show that the whole machinery of differential geometry can be avoided, and the Fefferman's theorem can be obtained in a quite elementary way as a consequence of a "good boundary behaviour" of the Bergman kernel function. It should be mentioned that the proof of the bondary estimates of the Bergman function is not elementary and it is difficult (see [1]). In fact, our result shows that these estimates form the most essential part of the proof of the Fefferman's theorem.

The structure of our proof is as follows: We point out the properties of the Bergman kernel function which are essential for the proof of the Fefferman's theorem (Theorem 1) and then we check that the Bergman kernel function of strictly pseudoconvex domain D with smooth boundary has these properties. We shall use the results from [1], although the local version due to Fefferman is quite sufficient for our purpose.

At the end of this paper we shall give several remarks in order to generalize Theorem 1 and to indicate some situations in which this generalization can be useful.

This approach to Fefferman's theorem was suggested to the author by the paper of Skwarczyński [3], and by Skwarczyński himself.

DEFINITION 1. Let D be a domain in C^n. We say that a function f on $\bar{D}$ is of *class* C^k on $\bar{D}$ if f is of class C^k on D, and every derivative of f up to the k-th order can be extended to a continuous function on $\bar{D}$. We say that the function f is *smooth* (i.e., C^∞) on $\bar{D}$ if it is smooth on D, and every derivative of f can be extended to a continuous function on $\bar{D}$.

DEFINITION 2. Let D and G be domains in C^n and denote by h a homeomorphism from $\bar{D}$ onto $\bar{G}$. We say that h is a *diffeomorphism between* $\bar{D}$ *and* $\bar{G}$ if h is smooth on $\bar{D}$, and h^{-1} is smooth on $\bar{G}$ in the sense of Definition 1.

Remark. If f is a function on D which can be extended to a function of class C^k on $\bar{D}$ in the sense of Definition 1, then we shall often identify f with its extension, and say that f is of *class* C^k *on* $\bar{D}$. It should also be mentioned that if $\bar{D}$ and $\bar{G}$ have smooth boundaries, then the diffeomorphism in the sense of Definition 2 is a diffeomorphism between manifolds with boundaries in the usual sense.

We can now formulate three properties characterizing the "good boundary behaviour" of the Bergman kernel function in a bounded domain D.

PROPERTY A. *The function* $K_D(z,t)$ *is smooth on* $D\times\bar{D}$. *It means that every derivative of* $K_D(z,t)$ *can be extended to a continuous function on* $D\times\bar{D}$.

PROPERTY B. *For every* $t_0\in\partial D$

$$\lim_{\substack{z,t\to t_0\\ z,t\in D}} |K_D(z,t)| = \infty$$

and for each $t_1\in\partial D$ *different from* t_0

$$\overline{\lim_{\substack{z\to t_1\\ t\to t_0}}} |K_D(z,t)| < \infty .$$

PROPERTY C. *For every* $t\in\partial D$ *there exists* $z_0\in D$ *such that* $K_D(z_0,t)\neq 0$ *and*

$$\det\left[\frac{\partial^2 \ln K_D(z,t)}{\partial z_i\,\partial\bar{t}_j}\right]_{z=z_0} \neq 0 .$$

Note that the functions in the matrix are well defined on the set $D\times\bar{D}\setminus\{(z,t),\, K_D(z,t)=0\}$.

THEOREM 1. *Let G and D be bounded domains in C^n. Assume that the Bergman kernel functions $K_D(z,t)$ and $K_G(w,s)$ have Properties* A, B, C *in D and G, respectively. Then every biholomorphic mapping $h\colon D\to G$ from D onto G can be extended to a diffeomorphism between $\bar{D}$ and $\bar{G}$.*

Proof. From the results of M. Skwarczyński [3] follows that if $K_D(z,t)$ and $K_G(w,s)$ have Properties A, and B, then every biholomorphic mapping h from D onto G can be extended to a homeomorphism between $\bar{D}$ and $\bar{G}$. We shall identify h with this extension. We must show that for every $t_0\in\partial D$ there exists a neighbourhood V of t_0 in $\bar{D}$ such that every derivative of h can be extended to a continuous function on V. Since $t_0\in\partial D$, it follows that $s_0=h(t_0)$ belongs to ∂G. Since $K_D(z,t)$ and $K_G(w,s)$ satisfy Properties A and B, there exists an open set $U\subset D$ such that $t_0\in\bar{U}$, $\bar{U}$ is a neighbourhood of t_0 in $\bar{D}$, $K_D(z,t)\neq 0$ on $U\times\bar{U}$, and $K_G(w,s)\neq 0$ on $h(U)\times\overline{h(U)}$.

Let $a\in U$; then the functions $K_D(z,t)/K_D(a,t)$ and $K_G(w,s)/K_G\big(h(a),s\big)$ are smooth on $U\times\bar{U}$ and $h(U)\times\overline{h(U)}$, respectively. For $(z,t)\in U\times U$ we have

$$c\,\frac{K_D(z,t)}{K_D(a,t)}\,Jh(z)^{-1}=\frac{K_G\big(h(z),h(t)\big)}{K_G\big(h(a),h(t)\big)}$$

and

$$\begin{aligned}(1)\qquad cJh(z)^{-1}\frac{\partial}{\partial\bar{t}_i}\frac{K_D(z,t)}{K_D(a,t)}&=\frac{\partial}{\partial\bar{t}_i}\frac{K_G\big(h(z),h(t)\big)}{K_G\big(h(a),h(t)\big)}\\&=\sum_{j=1}^{n}\frac{\partial}{\partial\bar{s}_j}\frac{K_G(w,s)}{K_G\big(h(a),s\big)}\bigg|_{\substack{w=h(z)\\ s=h(t)}}\left(\overline{\frac{\partial h_j}{\partial t_i}}\right).\end{aligned}$$

Here $Jh(z)$ denotes the Jacobi determinant of the mapping h at z, and $c=Jh(a)$.

We have

$$\frac{\partial}{\partial\bar{s}_j}\frac{K_G(w,s)}{K_G\big(h(a),s\big)}=\frac{K_G(w,s)}{K_G\big(h(a),s\big)}\left[\frac{\partial}{\partial\bar{s}_j}\ln K_G(w,s)-\frac{\partial}{\partial\bar{s}_j}\ln K_G\big(h(a),s\big)\right].$$

Note that $K_G(w,s)\neq 0$ on $h(U)\times\overline{h(U)}$, so the function $\dfrac{\partial}{\partial\bar{s}_j}\ln K_G(w,s)$ is well defined and smooth on $h(U)\times\overline{h(U)}$, and the function $\dfrac{\partial}{\partial\bar{s}_j}\ln K_G\big(h(a),s\big)$ is well defined and smooth on $\overline{h(U)}$.

Dividing both sides of (1) by $\dfrac{K_G(w,s)}{K_G\big(h(a),s\big)}\bigg|_{\substack{w=h(z)\\ s=h(t)}}$, we obtain

$$(2)\quad cJh(z)^{-1}\frac{\partial}{\partial\bar{t}_i}\frac{K_D(z,t)}{K_D(a,t)}\frac{K_G(h(a),h(t))}{K_G(h(z),h(t))}$$
$$=\sum_{j=1}^{n}\left[\frac{\partial}{\partial\bar{s}_j}\ln K_G(w,s)\Big|_{\substack{w=h(z)\\ s=h(t)}}-\frac{\partial}{\partial\bar{s}_j}\ln K_G(h(a),s)\Big|_{s=h(t)}\right]\left(\overline{\frac{\partial h_j}{\partial t_i}}\right).$$

Let us consider the point $s_0 = h(t_0) \in \partial G$. From Property C it follows that there exists $w_0 \in G$ such that

$$\det\left[\frac{\partial^2 \ln K_G}{\partial w_i\,\partial\bar{s}_j}(w_0, s_0)\right] \neq 0.$$

We can assume that $w_0 \in h(U)$. It implies that the functions of the variable w,

$$g_j(w, s_0) = \frac{\partial}{\partial\bar{s}_j}\left[\ln K_G(w,s) - \ln K_G(h(a),s)\right]\Big|_{s=s_0},$$

$j = 1, 2, \ldots, n$, from a local coordinate system in a neighbourhood of w_0. We can find points $w_1, w_2, \ldots, w_n$ from this neighbourhood such that $\det[g_j(w_k, s_0)] \neq 0$. The functions $g_j(w_k, s)$ considered as functions of s are smooth on $\overline{h(U)}$, so there exists an open set $W \subset h(U)$ such that $\bar{W}$ is a neighbourhood of s_0 in $\bar{G}$ and $\det[g_i(w_k, s)] \neq 0$ for $s \in W$.

Denote by $F_i(z, t)$ the left-hand side of (2), and by V the set $h^{-1}(W)$. Let $b_k = h^{-1}(w_k)$. It follows from (2) that the functions $\partial h_j/\partial t_i$ on V form the solution of the system of n linear equations

$$\overline{F_i(b_k, t)} = \sum_{j=1}^{n} \overline{g_j(h(b_k), h(t))}\,\partial h_j/\partial t_i,$$

$k = 1, 2, \ldots, n$; i fixed. Since $\det[g_j(h(b_k), h(t))] \neq 0$ on V, we can (using Cramer formulae), express the functions $\partial h_j/\partial t_i$ by $\overline{F_i(b_k, t)}$ and $\overline{g_j(h(b_k), h(t))}$. The functions $F_i(b_k, t)$ and $g_j(h(b_k), h(t))$ are continuous on $\bar{V}$ in view of Condition A and the fact that h is a homeomorphism between $\bar{V}$ and $\bar{W}$. It follows that the functions $\partial h_j/\partial t_i$ can be extended to continuous functions on $\bar{V}$, and the mapping h is of class C^1 on $\bar{D}$.

Assume now that h is of class C^{q-1} on $\bar{D}$. We want to show that h is of class C^q on $\bar{D}$. Consider a multiindex α such that $|\alpha| = q$. It is easy to prove that

$$Jh(z)^{-1}\frac{\partial^{|\alpha|}}{\partial\bar{t}^{\alpha}}\frac{K_D(z,t)}{K_D(a,t)} = G_\alpha(z,t) + \sum_{j=1}^{n}\frac{\partial}{\partial\bar{s}_j}\frac{K_G(w,s)}{K_G(h(a),s)}\Big|_{\substack{w=h(z)\\ s=h(t)}}\left(\overline{\frac{\partial^{|\alpha|}h_j(t)}{\partial t^{\alpha}}}\right)$$

on $U \times U$, where $G_\alpha(z,t)$ is a sum of products of the functions

$$\left(\overline{\frac{\partial^{|\beta|}h_j}{\partial t^{\beta}}}\right),\quad |\beta| \leqslant q-1 \quad\text{and}\quad \frac{\partial^{|\gamma|}}{\partial\bar{s}^{\gamma}}\frac{K_G(w,s)}{K_G(h(a),s)}\Big|_{\substack{w=h(z)\\ s=h(t)}}\quad |\gamma| \leqslant q.$$

Since h is of class C^{q-1}, it follows that $G_\alpha(z,t)$ is continuous on $\bar{D}$. It follows that we have essentially the same situation as for $q = 1$, and the functions $\partial^{|\alpha|}h_j/\partial t^\alpha$, $j = 1, 2, \ldots, n$, can be expressed in terms of functions which are continuous on $\bar{D}$. Therefore h is of class C^q on $\bar{D}$. By induction on q it follows that h is of class C^∞ on $\bar{D}$. In the same way we can prove that h^{-1} is of class C^∞ on $\bar{G}$. Hence h is a diffeomorphism between $\bar{D}$ and $\bar{G}$. The proof is completed.

We shall now use Theorem 1 to simplify the second part of the proof in the Fefferman theorem. Note that in order to prove the Fefferman theorem it is enough to show that in every strictly pseudoconvex domain with smooth boundary the Bergman function $K_D(z,t)$ has Properties A, B and C. To achieve this we shall use the following theorem proved by Boutet de Monvel and J. Sjöstrand in [1].

THEOREM 2. *Consider a strictly pseudoconvex domain* $D \subset C^n$ *given by* $D = \{z \in C^n\colon \varrho(z) < 0\}$, *where* ϱ *is a smooth function on* C^n, *strictly plurisubharmonic in a neighbourhood of* ∂D, *and* $\operatorname{grad} \varrho \neq 0$ *on* ∂D. *There exists a smooth function* $\psi(z,t)$ *on* $C^n \times C^n$ *with the following properties*:

1° $\psi(t,t) = (1/i)\varrho(t)$,

2° $\psi(z,t) = -\overline{\psi(t,z)}$,

3° $\bar{\partial}_z\psi$ *and* $\partial_t\psi$ *have zeros of infinite order along the diagonal* $\{(t,t), t \in C^n\}$, *so the Taylor development of* ψ *at the point* $(t,t) \in C^n \times C^n$ *has the form*

$$\psi(t+u, t+v) = \frac{1}{i}\sum_{\alpha,\beta}\frac{\partial^{|\alpha|+|\beta|}\varrho}{\partial z^\alpha \bar{z}^\beta}(t)\frac{u^\alpha}{\alpha!}\frac{\bar{v}^\beta}{\beta!},$$

4° $\operatorname{Im}\psi(z,t) > c[\operatorname{dist}(z,\partial D)+\operatorname{dist}(t,\partial D)+|z-t|^2]$ *on* $\bar{D}\times\bar{D}$.

Moreover,

$$K_D(z,t) = \frac{1}{(-i\psi(z,t))^{n+1}}F(z,t)+G(z,t)\ln(-i\psi),$$

where $F(z,t)$ *and* $G(z,t)$ *are smooth functions on* $\bar{D}\times\bar{D}$, *and* $F(t,t) \neq 0$ *for* $t \in \partial D$.

Properties A and B of $K_D(z,t)$ are immediate consequences of this theorem. Only Property C requires some explanation. We shall prove a stronger fact:

PROPOSITION 1. *If* D *is a strictly pseudoconvex domain with smooth boundary, then for each* $t_0 \in \partial D$

$$\lim_{z,t\to t_0}\det\left[\frac{\partial^2}{\partial z_i\,\partial\bar{t}_j}\ln K_D(z,t)\right] = \infty.$$

Proof. We can express $K_D(z,t)$ in the form

$$K_D(z,t) = \frac{1}{\psi(z,t)^{n+1}}\left(\frac{-1}{i}\right)^{n+1}[F(z,t)+G(z,t)(-i\psi)^{n+1}\ln(-i\psi)]$$
$$= \frac{1}{\psi(z,t)^{n+1}}\theta(z,t), \qquad \theta(t_0,t_0)\neq 0.$$

Note that since the function $w^{n+1}\ln w$ is of class C^n on the closed halfplane $\operatorname{Re} w \geqslant 0$, the function

$$-i\psi^{n+1}(z,t)\ln(-i\psi(z,t))$$

is of class C^n on $\bar{D}$ and $\theta(z,t)$ is of class C^n on $\bar{D}$.

We have

$$\det\left[\frac{\partial^2}{\partial z_i\partial \bar{t}_j}\ln K_D(z,t)\right] = \det\left[-(n+1)\frac{\partial^2 \ln\psi}{\partial z_i\partial \bar{t}_j}+\frac{\partial^2\ln\theta}{\partial z_i\partial\bar{t}_j}\right]. \tag{3}$$

The last determinant can be expressed as a sum of 2^n determinants. One of these determinants has the form

$$\big(-(n+1)\big)^n\det\left[\frac{\partial^2\ln\psi}{\partial z_i\partial\bar{t}_j}\right] = \big(-(n+1)\big)^n\det\left[\frac{1}{\psi}\frac{\partial^2\psi}{\partial z_i\partial\bar{t}_j}-\frac{1}{\psi^2}\frac{\partial\psi}{\partial z_i}\frac{\partial\psi}{\partial\bar{t}_j}\right]$$
$$= \big(-(n+1)\big)^n\frac{1}{\psi^n}\det\left[\frac{\partial^2\psi}{\partial z_i\partial\bar{t}_j}\right]+$$
$$+\sum_{K=1}^{n}\det\begin{bmatrix}\frac{1}{\psi}\frac{\partial^2\psi}{\partial z_1\partial\bar{t}_1} & \cdots & \frac{-1}{\psi^2}\frac{\partial\psi}{\partial z_1}\frac{\partial\psi}{\partial\bar{t}_K} & \cdots & \frac{1}{\psi}\frac{\partial^2\psi}{\partial z_1\partial\bar{t}_n}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \frac{1}{\psi}\frac{\partial^2\psi}{\partial z_n\partial\bar{t}_1} & \cdots & \frac{-1}{\psi^2}\frac{\partial\psi}{\partial z_n}\frac{\partial\psi}{\partial\bar{t}_K} & \cdots & \frac{1}{\psi}\frac{\partial^2\psi}{\partial z_n\partial\bar{t}_n}\end{bmatrix}$$
$$= \frac{(-1)^n(n+1)^n}{\psi^{n+1}}\det\begin{bmatrix}\psi & \frac{\partial\psi}{\partial\bar{t}_1}, & \cdots, & \frac{\partial\psi}{\partial\bar{t}_n}\\ \frac{\partial\psi}{\partial z_1} & & & \\ \vdots & & \frac{\partial^2\psi}{\partial z_i\partial\bar{t}_j} & \\ \frac{\partial\psi}{\partial z_n} & & & \end{bmatrix}.$$

The limit of the last determinant as $z,t\to t_0$ is equal to $(-i)^n L(\varrho)$, where

$$L(\varrho) = \det\begin{bmatrix}\varrho & \frac{\partial\varrho}{\partial\bar{z}_1}, & \cdots, & \frac{\partial\varrho}{\partial\bar{z}_n}\\ \frac{\partial\varrho}{\partial z_1} & & & \\ \vdots & & \frac{\partial^2\varrho}{\partial z_i\partial\bar{z}_n} & \\ \frac{\partial\varrho}{\partial z_n} & & & \end{bmatrix}_{z=t_0}$$

as it can easily be seen from Property 3 of the function ψ. Observe that $L(\varrho)$ is the so-called *Levi determinant* of the function ϱ at $t_0 \in \partial D$.

We shall supply here a proof for the fact that if ϱ is strictly plurisubharmonic at z, and $\operatorname{grad} \varrho \neq 0$ at z, then $L(\varrho)(z) < 0$. We have

$$\det \begin{bmatrix} 0 & \dfrac{\partial \varrho}{\partial \bar{z}_1}, & & , \dfrac{\partial \varrho}{\partial \bar{z}_n} \\ \dfrac{\partial \varrho}{\partial z_1} & & \dfrac{\partial^2 \varrho}{\partial z_i \partial \bar{z}_j} & \\ \dfrac{\partial \varrho}{\partial z_n} & & & \end{bmatrix}$$

$$= \sum_{j=1}^{n} (-1)^j \frac{\partial \varrho}{\partial \bar{z}_j} \sum_{i=1}^{n} (-1)^{i+1} \det_{\substack{k \neq i \\ l \neq j}} \left[\frac{\partial^2 \varrho}{\partial z_k \partial \bar{z}_l} \right] \frac{\partial \varrho}{\partial z_i}$$

$$= -\det \left[\frac{\partial^2 \varrho}{\partial z_i \partial \bar{z}_j} \right] \sum_{i,j=1} a_{ij} \frac{\partial \varrho}{\partial z_i} \frac{\partial \varrho}{\partial \bar{z}_j},$$

where

$$a_{ij} = (-1)^{i+j} \det_{\substack{k \neq i \\ l \neq j}} \left[\frac{\partial^2 \varrho}{\partial z_k \partial \bar{z}_l} \right] \Big/ \det \left[\frac{\partial^2 \varrho}{\partial z_i \partial \bar{z}_j} \right]$$

are the elements of the matrix inverse to the matrix $\overline{[\partial^2 \varrho / \partial z_i \partial \bar{z}_j]}$. Since ϱ is strictly plurisubharmonic the last matrix is positive definite. Therefore the matrix a_{ij} is also positive definite, and it is enough to note that, since $\operatorname{grad} \varrho \neq 0$,

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial \varrho}{\partial z_i} \frac{\partial \varrho}{\partial \bar{z}_j} > 0.$$

Therefore the proof that $L(\varrho) < 0$ is completed.

Now we should consider the remaining $2^n - 1$ determinants which arise in the representation of (3) as a sum of 2^n determinants. Observe that each of the remaining determinants has at least one column of the type $\partial^2 \ln \theta / \partial z_i \partial \bar{t}_j$, $i = 1, 2, \ldots, n$. Note that the functions $\partial^2 \ln \theta / \partial z_i \partial \bar{t}_j$ are continuous on a neighbourhood of $(t_0, t_0) \in \bar{D} \times \bar{D}$. On the other hand, if a column has the type $\partial^2 \ln \psi / \partial z_i \partial \bar{t}_j$, $i = 1, 2, \ldots, n$, then it is a sum of two columns:

(i) $$\frac{1}{\psi} \frac{\partial^2 \psi}{\partial z_i \partial \bar{t}_j} \quad (i = 1, 2, \ldots, n);$$

(ii) $$-\frac{1}{\psi^2} \frac{\partial \psi}{\partial z_i} \frac{\partial \psi}{\partial \bar{t}_j} \quad (i = 1, 2, \ldots, n).$$

Note that for two values of j two corresponding columns (ii) are always linearly dependent. This implies that our determinant can be expressed as a sum of determinants, and that each determinant in the sum has at least one column of the type $\partial^2 \ln \theta/\partial z_i \partial\bar{t}_j$ ($i = 1, 2, \ldots, n$), at most one column of the type $\frac{-1}{\psi^2}\frac{\partial\psi}{\partial z_i}\frac{\partial\psi}{\partial\bar{t}_j}$, and the other columns of the type $\frac{1}{\psi}\frac{\partial^2\psi}{\partial z_i \partial\bar{t}_j}$ ($i = 1, 2, \ldots, n$). Such determinant is equal to $\frac{1}{\psi^s} r(z, t)$, where $s \leqslant n$ and $r(z, t)$ is continuous on a neighbourhood of $(t_0, t_0) \in \bar{D} \times \bar{D}$. It follows that

$$\lim_{z,t \to t_0} \psi^{n+1} \det\left[\frac{\partial^2 \ln K_D(z, t)}{\partial z_i \partial\bar{t}_j}\right] = L(\varrho)(t_0) \neq 0.$$

This completes the proof of Proposition 1, and also the proof of Fefferman's theorem.

Remark 1 (generalization of Theorem 1). Theorem 1 remains valid if we replace condition B by the following weaker condition

B_0. For each t_1, $t_2 \in \partial D$, $t_1 \neq t_2$ the functions $K_D(z, t_1)$ and $K_D(z, t_2)$ are linearly independent as functions of the variable z.

Theorem 1 remains also valid if we replace condition A by condition A_k, $k = 2, \ldots$ (or $k = \infty$)

A_k. The Bergman function $K_D(z, t)$ is of class C^k on $D \times \bar{D}$ ($k \geqslant 2$).

THEOREM 1'. *Let D and G be bounded domains in C^n, $n > 1$. If the Bergman functions $K_D(z, t)$, $K_G(z, t)$ satisfy conditions A_k, B_0, and C (in D and G, respectively), then every biholomorphic mapping $h\colon D \to G$ from D onto G can be extended to a diffeomorphism of class C^k between $\bar{D}$ and $\bar{G}$.*

Proof. From conditions A_k and B_0 and Theorem 6.18 of [3] it follows that $\bar{D} = \hat{D}$, and $\bar{G} = \hat{G}$, where $\hat{D}$ and $\hat{G}$ denote the invariant compactification of D and G, respectively. This compactification (invariant under biholomorphic mappings) is constructed in [3]. It follows that h can be extended to a homeomorphism between $\bar{D}$ and $\bar{G}$. From Property B_0 it follows that $K_D(z, t_0) \not\equiv 0$ and $K_G(h(z), h(t_0)) \not\equiv 0$ for every fixed $t_0 \in \partial D$. Since these functions are R-analytic in z, we can find $a \in D$ such that $K_D(a, t_0) \neq 0$ and $K_G(h(a), h(t_0)) \neq 0$. Then there exist a neighbourhood V of a and an open domain $U \subset D$ such that $\bar{U}$ is a neighbourhood of t_0 in $\bar{D}$ and $K_D(z, t) \neq 0$ on $V \times \bar{U}$, and $K_G(h(z), h(t_0)) \neq 0$ on $h(V) \times \overline{h(U)}$. So we can consider the functions $K_D(z, t)/K_D(a, t)$ and $K_G(w, s)/K_G(h(a), s)$ on $V \times \bar{U}$, and $h(V) \times \overline{h(U)}$, respectively (instead of considering then on $U \times \bar{U}$ and $h(U) \times \overline{h(U)}$ as in the proof of Theorem 1). The rest of proof is exactly the same as in the case of Theorem 1.

Note that Property B_0 is essentially weaker than Property B. It is easy to see that if $K_D(z,t)$ has Properties A and B, then $K_D(z,t)$ has Property B_0. However, from Property B follows pseudoconvexity of D, and we shall show in the next Remark that there exist domains which are not pseudoconvex and their Bergman functions have Properties A, B_0, and C.

Remark 2 (the case of circular domain). Let D be a bounded circular domain in C^n such that $\bar{D} \subset \lambda D$ for each $\lambda > 1$. We shall show that the Bergman kernel function $K_D(z,t)$ has Properties A, B_0, and C. The proof is the same as the proof of Theorem 6.3 in [3]. The Bergman kernel function can be written in the form

$$K_D(z,t) = \sum_{k=0} \sum_{|\alpha|=k} \frac{\varphi_{k,\alpha}(z)\overline{\varphi_{k,\alpha}(t)}}{\beta_{k,\alpha}},$$

where $\varphi_{k,\alpha}(z)$ is a homogeneous polynomial of degree k. The series converges almost uniformly. For each $\lambda > 1$ the formula

$$K_\lambda(z,t) = \sum_{k=0} \sum_{|\alpha|=k} \frac{\varphi_{k,\alpha}(\lambda z)\overline{\varphi_{k,\alpha}(t/\lambda)}}{\beta_{k,\alpha}}.$$

gives the extension of $K_D(z,t)$ from $\frac{D}{\lambda}\times D$ to $\frac{D}{\lambda}\times \lambda D$ as an analytic function. It follows that $K_D(z,t)$ has Property A.

Suppose now that $t_1, t_2 \in \partial D$ and for some complex numbers a and b

$$aK_D(z,t_1) = bK_D(z,t_2)$$

as the functions of z on D.

Since $K_D(0,t) = K_D(0,0) > 0$, it follows that $a = b$. Assume that $a \neq 0$. Then from the uniqueness of Taylor development of holomorphic function at $0 \in D$ we conclude that

$$\varphi_{k,\alpha}(t_1) = \varphi_{k,\alpha}(t_2) \quad \text{for every } k \text{ and } \alpha,$$

and in particular for $k = 1$, $\varphi_{1,\alpha} = \varphi_i$, $i = 1,2,\ldots,n$. Since D is bounded, the linear mapping $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is non-degenerate. It follows that $t_1 = t_2$. This completes the proof of Property B_0. Now we shall prove Property C. On $D\times D\backslash\{(z,t), K_D(z,t) = 0\}$ we have

$$\frac{\partial^2 \ln K_D(z,t)}{\partial z_i \partial \bar{t}_j} = \frac{1}{K_D(z,t)} \frac{\partial^2 K_D(z,t)}{\partial z_i \partial \bar{t}_j} - \frac{1}{K_D^2(z,t)} \frac{\partial K_D(z,t)}{\partial z_i} \frac{\partial K_D(z,t)}{\partial \bar{t}_j}$$

$$= \frac{1}{K_D(z,t)} \sum_{k=0} \sum_{|\alpha|=k} \frac{\partial}{\partial z_i} \varphi_{k,\alpha} \left(\overline{\frac{\partial}{\partial t_j} \varphi_{k,\alpha}}\right) -$$

$$- \frac{1}{K_D^2(z,t)} \left(\sum_{k=0} \sum_{|\alpha|=k} \frac{\partial}{\partial z_i} \varphi_{k,\alpha}(z)\overline{\varphi_{k,\alpha}(t)}\right)\left(\sum_{k=0} \sum_{|\alpha|=k} \varphi_{k,\alpha}(z) \overline{\frac{\partial}{\partial t_j} \varphi_{k,\alpha}(t)}\right).$$

Note that $\varphi_{k,\alpha}(z)\left(\overline{\frac{\partial}{\partial t_j}\varphi_{k,\alpha}(t)}\right) = 0$, at every point $(0, t)$, $t \in D$, for every k and α. Also $\frac{\partial}{\partial z_i}\varphi_{k,\alpha}(z)\left(\overline{\frac{\partial}{\partial t_j}\varphi_{k,\alpha}(t)}\right) = 0$, at every point $(0, t)$, $t \in D$, provided that $k \neq 1$. Therefore we have

$$\det\left[\frac{\partial^2 \ln K_D(z, t)}{\partial z_i \partial \bar{t}_j}\right]\Bigg|_{\substack{z=0\\ t=t_0}} = \frac{1}{K_D^n(0, t_0)} \det\left[\frac{\partial^2 \sum_{i=1}^{n} \varphi_i(z)\overline{\varphi_i(t)}}{\partial z_i \partial \bar{t}_j}\right]\Bigg|_{\substack{z=0\\ t=t_0}}.$$

Note that $K_D(0, t_0) = K_D(0, 0) > 0$. Also the determinant is a constant, and can be computed for $z = 0$ and $t = 0$. Since the linear functions φ_i, $i = 1, 2, \ldots, n$, are linearly independent, the determinant is non-zero. Therefore Property C is valid.

Observe that there exist circular domains which are not pseudoconvex. It follows that there exist domains which are not pseudoconvex and for which the Bergman kernel function has Properties A, B_0, C. In particular, we obtain the following fact.

If D and G are bounded complete circular domains in C^n such that for each $\lambda > 1$, $\bar{D} \subset \lambda D$, $\bar{G} \subset \lambda G$ and if h: $D \to G$ maps biholomorphically D onto G, then h can be extended to a diffeomorphism between $\bar{D}$ and $\bar{G}$.

Remark 3 (the smoothness of the boundary and biholomorphic mappings). We assume as before that D and G are bounded domains in C^n such that K_D and K_G have Properties A, B_0, C. Suppose now that $\bar{D}$ satisfies the assumptions of Whitney's extension theorem (see [4], Chapter VI, § 3, Theorem 5), that means that the boundary ∂D can be locally represented as a graph of some function φ: $R^{2n-1} \to R$ satisfying the Lipschitz condition. Assume that h maps biholomorphically D onto G. In view of Theorem 1' the mapping h can be extended to a diffeomorphism between $\bar{D}$ and $\bar{G}$, and we can use Whitney's theorem to extend h to a smooth mapping h from C^n into C^n. It is easy to see that the Jacobian Jh does not vanish on $\bar{D}$. Hence h defines a diffeomorphism of some neighbourhood of $\bar{D}$ onto some neighbourhood of $\bar{G}$. It implies that the mapping h preserves the regularity of the boundary. If ∂D is of class C^k in a neighbourhood of t_0, then ∂G is of class C^k in a neighbourhood of $h(t_0)$. In particular, a circular domain with non-smooth boundary cannot be biholomorphically equivalent to a circular domain with smooth boundary. If a domain D for which K_D satisfies Properties A, B_0, C is biholomorphically equivalent to a strictly pseudoconvex domain with smooth boundary, then D must be a strictly pseudoconvex domain with smooth boundary.

Remark 4 (the local version of Theorem 1'). Let D and G be bounded domains in C^n, and assume that h maps biholomorphically D onto G. Let

$U \subset D$ be a domain such that $K_D(z,t)|_{U\times U}$ and $K_G(w,s)|_{h(U)\times h(U)}$ satisfy conditions A_k, B_0, C. Then the mapping $h|_U\colon U \to h(U)$ can be extended to the homeomorphism of the class C^k from $\bar{U}$ onto $\overline{h(U)}$ such that its inverse h^{-1} is of class C^k on $\overline{h(U)}$.

To prove this fact we must construct the compactification $\tilde{U}$ as follows: Since $K_D(z,t)$ satisfies Conditions A_k and B_0 on $U\times U$, there exists a finite coverings U_j of U, and points $a_j \in U$, such that $K_D(a_j,t) \neq 0$ on U_j, and the family of functions $\varphi_t(z) = K_D(z,t)/K_D(a_j,t)$ is locally bounded on U. So we can now repeate word by word the construction of compactification $\hat{D}$ from [3], and obtain a compactification $\tilde{U}$. In the same way we can construct a compactification $\widetilde{h(U)}$. These compactifications are invariant under the restriction of biholomorphic mapping of D to the domain U. It follows that $h|_U$ can be extended to a homeomorphism from $\tilde{U}$ onto $\widetilde{h(U)}$. On the other hand, it follows from Properties A_k and B_0 that $\tilde{U} = \bar{U}$ and $\widetilde{h(U)} = \overline{h(U)}$. The proof is the same as the proof of 6.18 in [3]. Hence $h|_U$ extends to a homeomorphism from $\bar{U}$ onto $\overline{h(U)}$. Now we can simply repeat the proof of Theorem 1′ to obtain our statement.

As an illustration we shall give the following

EXAMPLE. Let

$$D = \{(z_1,z_2)\in C^2,\ |z_1|^2+|z_2|^2\,|1-z_1^2| < 1\}.$$

Hence

$$\bar{D} = \{(z_1,z_2)\in C^2,\ |z_1|^2+|z_2|^2\,|1-z_1^2|_{|z_2|\leqslant 1} \leqslant 1\}.$$

Let $h(z_1,z_2) = (z_1, z_2\sqrt{1-z_1^2})$. It is a biholomorphic mapping from D onto the unit ball $B(0,1)$ in C^2. The inverse mapping is given by $h^{-1}(w_1,w_2) = (w_1, w_2/\sqrt{1-w_1^2})$. The Bergman function of D can be calculated explicitly:

$$K_D(z,t) = \frac{n!\sqrt{1-z_1^2}\,\overline{\sqrt{1-t_1^2}}}{\pi^n(1-z_1\bar{t}_1-z_2\bar{t}_2\sqrt{1-z_1^2}\,\overline{\sqrt{1-t_1^2}})^3}.$$

Observe that $K_D(z,t)$ satisfies Conditions A, B_0, C on a neighbourhood of $t = (t_1,t_2)\in\partial D$ if and only if $t_1 \neq 1$ and $t_1 \neq -1$. It is easy to check that in this case h is a homeomorphism, and a diffeomorphism of a neighbourhood of t in $\bar{D}$. If $t_1 = 1$, or $t_1 = -1$, then $K_D(z,t) \equiv 0$ and Conditions A, B_0, C fail. Nevertheless h is continuous on $\bar{D}$. We have $h(1,t_2) = (1,0)$ and $h(-1,t_2) = (-1,0)$ for all t_2.

Remark 5 (biholomorphic mappings on product domains). Let $D = D_1\times D_2$, $D_1 \subset C^n$, $D_2 \subset C^m$ be a bounded domain in C^{n+m}. We shall prove that the Bergman kernel function $K_D(z,t)$ has Properties A, B_0, C if and only if both the functions $K_{D_1}(z_1, t_1)$ and $K_{D_2}(z_2, t_2)$ have these properties.

By a theorem of Bremermann,

$$K_D(z, t) = K_{D_1}(z_1, t_1) K_{D_2}(z_2, t_2)$$

and it is easy to see that K_D has Property A if and only if both K_{D_1} and K_{D_2} have this property.

Assume now that for both K_{D_1} and K_{D_2} Property B_0 holds. If

$$aK_D(z, t^{(1)}) = bK_D(z, t^{(2)}), \quad z \in D,$$

then on the non-empty open set $U = \{(z_1, z_2), K_{D_1}(z_1, t_1^{(2)}) \neq 0, K_{D_2}(z_2, t_2^{(1)}) \neq 0\}$ we have

$$a \frac{K_{D_1}(z_1, t_1^{(1)})}{K_{D_1}(z_1, t_1^{(2)})} = b \frac{K_{D_2}(z_2, t_2^{(2)})}{K_{D_2}(z_2, t_2^{(1)})} = \text{const.}$$

Therefore $a = 0$ and $b = 0$, since otherwise we would get a contradiction with Property B_0 in D_1 or D_2. Conversely, assume that for one of the functions K_{D_1} and K_{D_2}, say for $K_{D_1}(z_1, t_1)$ Property B_0 does not hold. Hence there exist $t_1^{(1)}, t_1^{(2)} \in \partial D_1$ such that

$$aK_{D_1}(z_1, t_1^{(1)}) = bK_{D_1}(z_1, t_1^{(2)}), \quad z_1 \in D_1,$$

where $a \neq 0$ or $b \neq 0$. Fix an arbitrary point $t_2 \in D_2$, and set $t^{(1)} = (t_1^{(1)}, t_2)$, $t^{(2)} = (t_1^{(2)}, t_2)$. Then

$$\begin{aligned} aK_D(z, t^{(1)}) &= aK_{D_1}(z_1, t_1^{(1)}) K_{D_2}(z_2, t_2) \\ &= bK_{D_1}(z_1, t_1^{(2)}) K_{D_2}(z_2, t_2) = bK_D(z, t^{(2)}) \end{aligned}$$

and therefore Property B_0 fails for $K_D(z, t)$.

Let us now consider Property C. We have locally on $D \times D \setminus \{(z, t), K_D(z, t) = 0\}$

$$\ln K_D(z, t) = \ln K_{D_1}(z_1, t_1) + \ln K_{D_2}(z_2, t_2) + c,$$

where the constant c depends on the choice of the branches of logarithm. Hence

$$\det\left[\frac{\partial^2 \ln K_D(z, t)}{\partial z_i \partial \bar{t}_j}\right] = \det\left|\frac{\partial^2 \ln K_{D_1}(z_1, t_1)}{\partial (z_1)_i \partial (\bar{t}_1)_j}\right| \det\left|\frac{\partial^2 \ln K_{D_2}(z_2, t_2)}{\partial (z_2)_i \partial (\bar{t}_2)_j}\right|.$$

This equality holds also on $D \times \bar{D} \setminus \{(z, t), K_D(z, t) = 0\}$ so Property C holds for $K_D(z, t)$ if and only if it is valid for both $K_{D_1}(z_1, t_1)$ and $K_{D_2}(z_2, t_2)$. This completes the proof of our statement.

From Theorem 1 and Remark 3 we obtain

THEOREM. *Let D and G denote Cartesian products of strictly pseudoconvex domains with smooth boundaries. Let h be a biholomorphic mapping from D*

onto G. Then h can be extended to a diffeomorphism between some neighbourhood of $\bar{D}$ and some neighbourhood of $\bar{G}$.

The thecrem holds if D and G are Cartesian products of complete circular or strictly pseudoconvex domains H_i such that $\lambda H_i \supset \bar{H}_i$ for $\lambda > 1$, and each H_i satisfies assumptions the of the Whitney extension theorem.

COROLLARY. *A strictly pseudoconvex domain cannot be biholomorphically equivalent to a non-trivial Cartesian product of strictly pseudoconvex domains.*

Added in proof. During the preparation of this paper to print, a remarkable progress was made in the study of the boundary regularity of biholomorphic mappings.

S. Webster in his paper *Biholomorphic mappings and the Bergman kernel off the diagonal*, Invent. Math. 51 (1979), p. 155–169, independently found a similar proof of the Fefferman's theorem and stated Conditions A and C for the Bergman function. It follows from his proof that our Conditions B and B_n in Theorems 1 and 1′ are superfluous. He proved also that in the case of domains with smooth boundary Condition A alone is sufficient to obtain a smooth extension of biholomorphic mappings to an open dense subset of the boundary.

In a paper E. Ligocka, *Some remarks on extension of biholomorphic mappings*, Proc. 7-th Conf. on Anal. Funct. Kozubnik 1979, Springer Lecture Notes (to appear), Condition C was put in more general form:

$$\text{(C}'\text{)} \qquad \forall\, z \in \partial D\ \exists\, a_0, \ldots, a_n \in D \det \begin{bmatrix} K_D(z, a_j) \\ \dfrac{\partial K_D}{\partial z_i}(z, a_j) \end{bmatrix} \neq 0$$

which together with the fundamental Bell's lemma (S. Bell, *Non-vanishing of the Bergman kernel function at boundary points ...*, Math. Ann. 244 (1979), p. 69–74) permits to show that in the case of domains with smooth boundary, Conditions A and C′ (and hence the Fefferman's theorem) are direct consequences of a continuity of the Bergman projector in $C^\infty(\bar{D})$-Fréchet topology. They hold in particular for bounded pseudoconvex domains with real analytic boundary. The full and detailed proof of it was given in S. Bell, E. Ligocka, *A simplification and extension of Fefferman's theorem on biholomorphic mappings*, Invent. Math. (in press).

Recently S. Bell in *Biholomorphic mappings and the Bergman projection* (preprint) proved that if D is a bounded domain with smooth boundary such that its Bergman projector is continuous in $C^\infty(\bar{D})$-topology, G is a bounded pseudoconvex domain with smooth boundary and h maps biholomorphically D onto G, then the Jacobi determinant Jh of h and the mapping $h\,Jh$ extend smoothly to D. It leads to an another proof of Fefferman's theorem.

If D is a bounded strictly pseudoconvex domain with a boundary of class C^{k+4}, then it is possible to follow Kerzman–Stein construction for Szegö projector (Duke Math. J. 45 (1978), p. 197–223) and to obtain an integral operator $Gf(z) = \int_D G(z, t) f(t) dV_t$, with holomorphic in z kernel $G(z, t)$ which has an explicitly written singularity, such that the Bergman projector P can be expressed as $P = \big(I+(G^*-G)\big)^{-1} G = G\big(I+(G-G^*)\big)^{-1}$.

It permits to get in a relatively easy way estimates for P in Hölder norms. These estimates and a suitable modification of Bell's lemma implies that conditions $A_{k+1/2}$ and (C′) hold for $K_D(z, t)$ if $k \geqslant 2$. It yields the following

THEOREM. *If D and G are bounded strictly pseudoconvex domains with boundaries of class at least $k+4$, $k \geqslant 2$, then each biholomorphic mapping from D onto G extends to a diffeomorphism of class $C^{k+1/2}$ between $\bar{D}$ and $\bar{G}$.*

Thus the Fefferman's theorem is a consequence of Hörmander's L^2-estimates on $\bar{\partial}$-problem and of elementary estimates on singular integrals due to Henkin and to Kranz.

Details will be given in the paper E. Ligocka, *On the continuity of Bergman projector in Hölder spaces and the boundary regularity of biholomorphic mappings on strictly pseudoconvex domains with boundaries of class* C^m (in preparation).

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