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A NOTE ON HYPERSPACES AND THE FIXED POINT PROPERTY

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1. A question (No. 186 of the New Scottish Book, due to B. Knaster) of some interest has been:

If X is a metric continuum with the fixed point property, then does the hyperspace $C(X)$ (metrized with the Hausdorff metric) of all non-empty subcontinua of X also have the fixed point property? (Note that the answer to this question is yes when X is locally connected because then $C(X)$ is an absolute retract [5], p. 191.) In [4] Segal gave a partial answer to this question by showing that the hyperspace of a chainable continuum has the fixed point property.

The purpose of this note* is to show that the answer to this question is *in general* no. We give an example of a metric continuum Y such that Y has the fixed point property and yet its hyperspace $C(Y)$ does not have the fixed point property. Our continuum Y has the additional property of being acyclic.

Recently an example was given (in [3]) of a metric continuum Z whose hyperspace $C(Z)$ does not have the fixed point property. This continuum Z appears in Fig. 2 and, as far as the authors know, is the first such example. Even though Z itself does not have the fixed point property, it and its hyperspace play a crucial role in our investigation which answers Knaster's question.

2. We define Y and Z in polar coordinates. Let $S^1 = \{(r, \theta): r = 1\}$, let $D = \{(r, \theta): r \leqslant 1\}$, and let $\mathscr{S} = \{(r, \theta): \theta \geqslant 0 \text{ and } r = 1+1/(1+\theta)\}$. Let $Y = D \cup \mathscr{S}$ and let $Z = S^1 \cup \mathscr{S}$ (see Figs. 1 and 2, respectively).

It will be convenient (see the proof of the lemma) to consider the metric for Y as the usual Euclidean metric d, i.e., if $(x_1, y_1), (x_2, y_2) \in Y$, then

$$d\big((x_1, y_1), (x_2, y_2)\big) = \big((x_2-x_1)^2+(y_2-y_1)^2\big)^{1/2}.$$

We will consider Z as contained in Y.

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In this paper the symbol $\bar{S}$ means the closure of S.

For each $A \in \overline{(C(Y)-C(D))}$, let $\mu(A) = A \cap Z$.

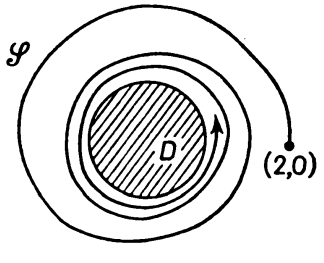
The following lemma will be used in proving our Theorem:

LEMMA. *The function μ is a continuous function, actually a retraction, from $\overline{C(Y)-C(D)}$ into $C(Z)$.*

Proof. First note that

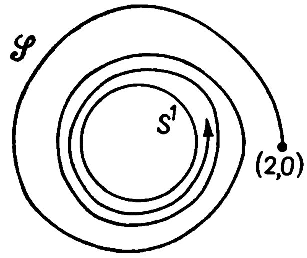
$$\overline{C(Y)-C(D)} = \big(C(Y)-C(D)\big)\cup C(S^1)\cup\mathscr{F},$$
$$\text{where } \mathscr{F} = \{A \in C(D)\colon\ S^1 \subset A\}.$$

To see that the values of μ are subcontinua of Z, let $A \in \overline{(C(Y)-C(D))}$. If $A \in \big(C(Y)-C(D)\big)$, then we have $\mu(A) = \overline{A\cap\mathscr{S}}$ and therefore, since $A\cap\mathscr{S}$ is connected, $\mu(A)\in C(Z)$. If $A \in C(S^1)$, then $A \subset S^1 \subset Z$, so $\mu(A) = A \in C(Z)$. If $A \in \mathscr{F}$, then $\mu(A) = A\cap Z = S^1 \in C(Z)$. It remains to show that μ is continuous on $\overline{C(Y)-C(D)}$. Let H denote the Hausdorff metric for $C(Y)$ obtained from the usual Euclidean metric d for Y (cf. [1], p. 131). It is easy to verify that, for any $A \in \overline{(C(Y)-C(D))}$, (i) $\mu(A) \subset A$ and (ii) $\mu(A) = A$ or $\mu(A) = (A\cap\mathscr{S})\cup S^1$. Using (i) and (ii) together with the fact that a point in $\mathscr{S}$ has its nearest point in D on S^1, it can be verified, by taking the cases indicated by (ii), that $H\big(\mu(A_1), \mu(A_2)\big) \leqslant H(A_1, A_2)$ for any A_1 and A_2 in $\overline{C(Y)-C(D)}$. This completes the proof of the lemma.



Y

Fig. 1



Z

Fig. 2

We now state and prove our main result.

THEOREM. *The hyperspace $C(Y)$, where Y is the disk with a spiral as in Fig. 1, does not have the fixed point property.*

Proof. We define a retraction r of $C(Y)$ onto $C(Z)$. In view of the result mentioned above in [3], this proves the theorem. First we define a retraction g from $C(D)$ onto $C(S^1)$ with special properties. To do this, let $\mathscr{F}$ be defined as above, $\mathscr{F} = \{A \in C(D)\colon S^1 \subset A\}$, and let $f\colon C(S^1)\cup\mathscr{F} \to C(S^1)$ be the restriction of μ to $C(S^1)\cup\mathscr{F}$, i.e., for each $A \in \big(C(S^1)\cup\mathscr{F}\big)$, $f(A) = \mu(A)$. Since $C(S^1)$ is a 2-cell (see, for example, [2]), f can be extended to a continuous function $g\colon C(D) \to C(S^1)$. Clearly, g is a retraction of $C(D)$ onto $C(S^1)$ such that if $A \in \mathcal{F}$, then $g(A) = S^1$. Now we define our retraction r. If $A \in C(Y)$, then let

$$r(A) = \begin{cases} g(A) & \text{for } A \in C(D), \\ \mu(A) & \text{for } A \in \left(\overline{C(Y) - C(D)}\right). \end{cases}$$

Since $\overline{C(Y) - C(D)} = \left(C(Y) - C(D)\right) \cup C(S^1) \cup \mathcal{F}$ and since g has the properties mentioned above, g and μ agree on

$$C(S^1) \cup \mathcal{F} = C(D) \cap \left(\overline{C(Y) - C(D)}\right).$$

Thus, using the Lemma, r is a continuous function from $C(Y)$ into $C(Z)$. Since μ is the identity on $C(Z)$, r is our desired retraction.

Remark. In [4], p. 237, Segal remarks that if $C(X)$ is a quasi-complex, then $C(X)$ has the fixed point property. Thus, the hyperspaces of the continua Y and Z are not quasi-complexes (see the Question in [4], p. 248). In fact, our continuum Y is acyclic and yet $C(Y)$ is not a quasi-complex.

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