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# On the convergence of multistep methods for nonlinear two-point boundary value problems

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**Abstract.** A wide class of multistep methods for nonlinear two-point boundary-value problems is considered. Sufficient conditions for convergence and error estimates are given. The convergence theorems are stated under the assumptions that partial derivatives of the increment function $F$ and $g$ satisfy the Lipschitz or Perron conditions.

**1. Introduction.** We study the application of numerical methods to very general nonlinear two-point boundary value problems of the form

$$y'(t) = f(t, y(t)), \quad t \in I = [a, b], \ a < b, \tag{1}$$

$$g(y(a), y(b)) = 0, \tag{2}$$

where $f\colon I \times \mathbf{R}^p \to \mathbf{R}^p$ is continuous and $g\colon \mathbf{R}^p \times \mathbf{R}^p \to \mathbf{R}^p$. We also associate with the equation (1) the initial condition

$$y(a) = s \in \mathbf{R}^p. \tag{3}$$

Under the assumption that $f$ satisfies the Lipschitz condition with respect to the second variable, the initial value problem (1–3) has a solution $y \in C^1(I)$ ($C^1(I)$ is the space of all functions of class $C^1$ from $I$ into $\mathbf{R}^p$). Now, if $s$ is a root of the equation

$$\Phi(s) \equiv g(s, y(b; s)) = 0, \tag{4}$$

then $y$ is also a solution of the boundary value problem (1–2). It is well known that if in addition the function $f(t, \cdot)$ is of class $C^1$ for fixed $t$, then $\frac{\partial y}{\partial s}(t; s)$ exists and is the fundamental solution of the problem

$$\begin{cases} Y'(t; s) = f_y(t, y(t; s))Y(t; s), & t \in I, \\ Y(a; s) = I_{p \times p}, \end{cases} \tag{5}$$

so $Y(t; s) = \frac{\partial y}{\partial s}(t; s)$. To find a solution $s$ of (4), iterative methods can be used. We apply here the Newton method. For this we have to assume that $g$ is of class $C^1$ to get

(6) $$\Phi'(s) = B_a(y) + B_b(y) Y(b; s),$$

where

(7) $$B_a(y) = \frac{\partial g}{\partial y(a)}(y(a), y(b)), \quad B_b(y) = \frac{\partial g}{\partial y(b)}(y(a), y(b)).$$

Now the Newton iterates are defined by

(8) $$s_{j+1} = s_j - (\Phi'(s_j))^{-1} \Phi(s_j), \quad j = 0, 1, \ldots$$

Our task here is to find a numerical solution $y_h$ of BVP (1–2). The basic idea is to replace the solutions $y$ and $Y$ by the approximations $y_h$ and $Y_h$. We consider a partition of $I$:

$$a = t_0 < t_1 < \ldots < t_N = b, \quad \text{where } t_{n+1} = t_n + h_n, \; n = 0, 1, \ldots, N-1,$$

and $h = \max_n h_n$. Our analysis usually refers to a family of such nets in which $N \to \infty$ while $h \to 0$. At each point $t_n$ we seek vectors $y_n^j = y_h(t_n; s^j)$ and matrices $Y_n^j = Y_h(t_n; s^j)$ which are to approximate $y(t_n; s_j)$ and $Y(t_n; s_j)$, respectively. They may be defined by the quasilinear nonstationary multistep ($k$-step) method of the form

(9) $$\sum_{i=0}^{k} a_i(t_n, h_n) y_{n+i}^j = h_n F(t_n, \ldots, t_{n+k}, h_n, y_n^j, \ldots, y_{n+k}^j)$$
$$\equiv h_n \mathscr{F}(t_n, h_n, y_n^j), \quad a_k = 1,$$

(10) $$\sum_{i=0}^{k} a_i(t_n, h_n) Y_{n+i}^j = h_n \sum_{i=0}^{k} F_i(t_n, \ldots, t_{n+k}, h_n, y_n^j, \ldots, y_{n+k}^j) Y_{n+i}^j$$
$$\equiv h_n \sum_{i=0}^{k} P_n^j(i) Y_{n+i}^j,$$

for $n = 0, 1, \ldots, N-k$, $j = 0, 1, \ldots$, where $y_i^j$ and $Y_i^j$ are given for $i = 0, \ldots, k-1$, $j = 0, 1, \ldots$ Here $F_i$ denotes the partial derivative of $F$ with respect to the $(k+3+i)$th variable. The Newton iterates $\{s^j\}$ are now defined for any $s^0$ by

(11) $$s^{j+1} = s^j - (B_a^j + B_b^j Y_N^j)^{-1} g(s^j, y_N^j), \quad j = 0, 1, \ldots,$$

where $B_a^j = g_1(s^j, y_N^j)$, $B_b^j = g_2(s^j, y_N^j)$, and $g_1$ and $g_2$ denote the partial derivatives of $g$ with respect to the first and second variables, respectively. Actually $s^j$ is a function of $h$ because $y_N^j$ and $Y_N^j$ are computed as functions of $h_n$.

Almost all results known to me concern the convergence of onestep methods to the solution of (1–2) under the assumption that the corresponding increment function satisfies the Lipschitz condition. The centered Euler method is efficient and frequently used.

Numerical methods for BVPs were considered by many authors (see for example [2, 5–9]). Keller in [6–7] obtained his results for isolated solutions of (1–2) ("locally unique").

Let $g(u, v) = B_1 u + B_2 v + B_3$, $B_3 \in \mathbf{R}^p$, where $B_1$ and $B_2$ are $p \times p$ matrices. One-step methods for linear BVPs discussed in [5] are special cases of (9–11). Sufficient conditions for convergence were obtained under the Lipschitz condition on the partial derivatives of the corresponding increment function.

In the present paper we discuss the quasilinear multistep method (9–11) and formulate some sufficient conditions for its convergence. This convergence is obtained under the assumptions that $F_i$ and $g_r$ satisfy the Lipschitz or Perron conditions. Estimates of errors are also given.

**2. Convergence and consistency.** We need some definitions.

DEFINITION 1. We say that the method (9–11) is *convergent* to the solution $\varphi$ of BVP (1–2) if

$$\lim_{\substack{N\to\infty\\ j\to\infty}} \max_{n=0,\ldots,N} \|y_n^j - \varphi(t_n)\| = 0.$$

DEFINITION 2. We say that the method (9–11) is *consistent* with the problem (1–2) on the solution $\varphi$ if there exist functions $\gamma_1, \gamma_2\colon I \times H \to \mathbf{R}_+ = [0, \infty)$, $H = [0, h^*]$, $h^* > 0$, such that

$$\text{(i)}\quad \Big\|h_n F(t_n, \ldots, t_{n+k}, h_n, \varphi(t_n), \ldots, \varphi(t_{n+k})) - \sum_{i=0}^{k} a_i(t_n, h_n)\varphi(t_{n+i})\Big\| \leqslant \gamma_1(t_n, h_n),$$

$$\text{(ii)}\quad \Big\|h_n \sum_{i=0}^{k} F_i(t_n, \ldots, t_{n+k}, h_n, \varphi(t_n), \ldots, \varphi(t_{n+k}))\, Y(t_{n+i}) - \sum_{i=0}^{k} a_i(t_n, h_n)\, Y(t_{n+i})\Big\| \leqslant \gamma_2(t_n, h_n),$$

for $n = 0, 1, \ldots, N-k$ and

$$\text{(iii)}\quad \lim_{h\to 0} \bar{\gamma}_r(h) = 0, \qquad \bar{\gamma}_r(h) = \sum_{i=0}^{N-k} \gamma_r(t_i, h_i), \qquad r = 1, 2,\ h = \max_i h_i,$$

where $Y$ is the bounded solution of (5) with $y = \varphi$.

We say the method (9–11) is *H-consistent* with the problem (1-2) on the solution $\varphi$ if only conditions (i) and (iii) (for $r = 1$) are satisfied.

Remark 1. Because $\varphi$ and $Y$ are solutions of (1–2) and (5), conditions (i), (ii) can also be written in the following form:

$$\text{(i)}'\quad \Big\|h_n F(t_n, \ldots, t_{n+k}, h_n, \varphi(t_n), \ldots, \varphi(t_{n+k})) - \sum_{i=0}^{k} a_i(t_n, h_n)\varphi(t_n) - \sum_{i=1}^{k} a_i(t_n, h_n) \int_{t_n}^{t_{n+i}} f(\tau, \varphi(\tau))\, d\tau\Big\| \leqslant \gamma_1(t_n, h_n),$$

(ii)' $$\Big\|h_n \sum_{i=0}^{k} F_i(t_n, \ldots, t_{n+k}, h_n, \varphi(t_n), \ldots, \varphi(t_{n+k}))Y(t_n) + h_n \sum_{i=1}^{k} F_i(t_n, \ldots, t_{n+k}, h_n, \varphi(t_n), \ldots, \varphi(t_{n+k})) \int_{t_n}^{t_{n+i}} f_y(\tau, \varphi(\tau))Y(\tau)\,d\tau - \sum_{i=0}^{k} a_i(t_n, h_n)Y(t_n) - \sum_{i=1}^{k} a_i(t_n, h_n) \int_{t_n}^{t_{n+i}} f_y(\tau, \varphi(\tau))Y(\tau)\,d\tau\Big\| \leqslant \gamma_2(t_n, h_n).$$

The next theorem deals with the consistency of the method (9–11). Some modification of Theorem 1 of [4] yields the following.

THEOREM 1. *If*

1° $F$: $I^{k+1} \times H \times (\mathbf{R}^p)^{k+1} \to \mathbf{R}^p$ *is continuous and has continuous first order partial derivatives* $F_0, F_1, \ldots, F_k$ *with respect to the last* $k+1$ *variables, i.e.* $F_i(\cdot) \equiv \frac{\partial F}{\partial y_i}(\cdot)$, $i = 0, 1, \ldots, k$,

2° $f$: $I \times \mathbf{R}^p \to \mathbf{R}^p$ *is continuous and bounded and* $f(t, \cdot)$ *is of class* $C^1$ *for fixed* $t$,

3° *there exists a bounded solution* $\varphi \neq \theta$ *of* (1–2), *where* $\theta$ *is the zero vector in* $\mathbf{R}^p$,

4° $a_i$: $I \times H \to \mathbf{R}$, $i = 0, 1, \ldots, k$, $a_k = 1$ *and all* $a_i$ *are bounded*,

*then the method* (9–11) *is consistent with the problem* (1–2) *on* $\varphi$ *if*

(i) $$\lim_{N\to\infty} \sum_{i=0}^{N-k} \Big|\sum_{j=0}^{k} a_j(t_i, h_i)\Big| = 0,$$

(ii) $$\lim_{N\to\infty} \sum_{i=0}^{N-k} \Big\|\sum_{j=1}^{k} a_j(t_i, h_i) \sum_{r=1}^{j} h_{i+r-1} f(t_i, \varphi(t_i)) - h_i F(t_i, \ldots, t_{i+k}, h_i, \varphi(t_i), \ldots, \varphi(t_{i+k}))\Big\| = 0,$$

(iii) $$\lim_{N\to\infty} \sum_{i=0}^{N-k} \Big\|\sum_{j=1}^{k} a_j(t_i, h_i) \sum_{r=1}^{j} h_{i+r-1} f_y(t_i, \varphi(t_i))Y(t_i) - h_i \sum_{j=0}^{k} F_j(t_i, \ldots, t_{i+k}, h_i, \varphi(t_i), \ldots, \varphi(t_{i+k}))Y(t_{i+j})\Big\| = 0.$$

**3. Assumptions.** We introduce

ASSUMPTION $H_1$. Suppose that

1° $F$: $I^{k+1} \times H \times (\mathbf{R}^p)^{k+1} \to \mathbf{R}^p$ is continuous and has continuous first order partial derivatives $F_0, F_1, \ldots, F_k$ with respect to the last $k+1$ variables,

2° $g$: $\mathbf{R}^p \times \mathbf{R}^p \to \mathbf{R}^p$ is of class $C^1$, and $g_2$ is bounded by $K_0$ where $g_i(y_1, y_2) \equiv \frac{\partial g}{\partial y_i}(y_1, y_2)$, $i = 1, 2$, 3° there exist constants $M_r, L_{ir} \geqslant 0$, $i, r = 0, 1, \ldots, k$, and functions $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k$: $I \times H \to \mathbf{R}_+$ such that for $(q_i, h, x_i, \bar{x}_i) \in I \times H \times \mathbf{R}^p \times \mathbf{R}^p$, $i = 0, \ldots, k$, we have

(i) $\quad \|F_r(q_0, q_1, \ldots, q_k, h, x_0, x_1, \ldots, x_k)\| \leqslant M_r, \quad r = 0, 1, \ldots, k,$

(ii) $$\|F_r(q_0, \ldots, q_k, h, x_0, \ldots, x_k) - F_r(q_0, \ldots, q_k, h, \bar{x}_0, \ldots, \bar{x}_k)\| \leqslant \sum_{i=0}^{k} L_{ir}\|x_i - \bar{x}_i\| + \varepsilon_r(q_0, h), \quad r = 0, 1, \ldots, k,$$

and

$$\lim_{h\to 0} \delta_r(h) = 0, \quad \delta_r(h) = \sum_{i=0}^{N-k} h_i \varepsilon_r(t_i, h_i), \quad r = 0, 1, \ldots, k, \; h = \max_i h_i,$$

4° there exist constants $K_{11}, K_{12}, K_{21}, K_{22} \geqslant 0$ such that for $x_1$, $x_2$, $\bar{x}_1$, $\bar{x}_2 \in \mathbf{R}^p$ we have

$$\|g_i(x_1, x_2) - g_i(\bar{x}_1, \bar{x}_2)\| \leqslant K_{1i}\|x_1 - \bar{x}_1\| + K_{2i}\|x_2 - \bar{x}_2\|, \quad i = 1, 2,$$

5° $a_i$: $I \times H \to \mathbf{R}$, $i = 0, 1, \ldots, k-1$, $a_k = 1$ and there exists a nonnegative constant $\tilde{M}$ such that

$$\|A_n\| \leqslant 1 + \tilde{M}h_n, \quad n = 0, 1, \ldots, N-k \text{ (maximum norm)},$$

where

$$A_n = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & 1 \\ a_{0n} & a_{1n} & a_{2n} & & a_{k-1,n} \end{bmatrix}$$

with $a_{in} = -a_i(t_n, h_n)$, $i = 0, 1, \ldots, k-1$, $n = 0, 1, \ldots, N-k$,

6° there exist a constant $\bar{q} \geqslant 0$ and functions $\eta_1, \eta_2$: $H \to \mathbf{R}_+$, $\lim_{h\to 0}\eta_i(h) = 0$, $i = 1, 2$, such that

$$\max_{i=0,\ldots,k-1} \|y_i^j - \varphi(t_i)\| \leqslant \bar{q}\,\|s^j - \varphi(a)\| + \eta_1(h),$$

$$\max_{i=0,\ldots,k-1} \left\|Y_i^j(s^j - \varphi(a)) - (y_i^j - \varphi(t_i))\right\| \leqslant \eta_2(h),$$

for $j = 0, 1, \ldots$

Remark 2. Putting

$$y_0^j = s^j, \quad Y_0^j = I_{p\times p}, \quad j = 0, 1, \ldots,$$

we see that

$$\|y_0^j - \varphi(t_0)\| = \|s^j - \varphi(a)\|, \quad \left\|Y_0^j(s^j - \varphi(a)) - (y_0^j - \varphi(t_0))\right\| = 0.$$

The values of $y_i^j$ and $Y_i^j$ for $i = 1, 2, \ldots, k-1, j = 0, 1, \ldots$, may be obtained by applying any one-step method.

ASSUMPTION $H_2$. Suppose that

1° conditions 1°, 2°, 5° and 6° of Assumption $H_1$ are satisfied,

2° there exist constants $M_r \geqslant 0$ and bounded functions $\Omega_r\colon H \times (\mathbf{R}_+)^{k+1} \to \mathbf{R}_+$, $\varepsilon_r\colon I \times H \to \mathbf{R}_+$, $r = 0, 1, \ldots, k$, such that for $(q_i, h, x_i, \bar{x}_i) \in I \times H \times \mathbf{R}^p \times \mathbf{R}^p$, $i = 0, 1, \ldots, k$, we have

(i) $$\|F_r(q_0, q_1, \ldots, q_k, h, x_0, x_1, \ldots, x_k)\| \leqslant M_r,$$

(ii) $$\|F_r(q_0, \ldots, q_k, h, x_0, \ldots, x_k) - F_r(q_0, \ldots, q_k, h, \bar{x}_0, \ldots, \bar{x}_k)\| \leqslant \Omega_r(h, \|x_0 - \bar{x}_0\|, \ldots, \|x_k - \bar{x}_k\|) + \varepsilon_r(q_0, h),$$

and

$$\lim_{h \to 0} \delta_r(h) = 0, \quad \delta_r(h) = \sum_{i=0}^{N-k} h_i \varepsilon_r(t_i, h_i), \quad r = 0, 1, \ldots, k, \; h = \max_i h_i,$$

3° there exist two bounded functions $\omega_i\colon \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$, $i = 1, 2$, such that for $x_1, x_2, \bar{x}_1, \bar{x}_2 \in \mathbf{R}^p$ we have

$$\|g_i(x_1, x_2) - g_i(\bar{x}_1, \bar{x}_2)\| \leqslant \omega_i(\|x_1 - \bar{x}_1\|, \|x_2 - \bar{x}_2\|), \quad i = 1, 2,$$

4° the functions $\Omega_r$ and $\omega_i$ are continuous with respect to all variables and satisfy the following conditions:

(i) $\Omega_r$ are nondecreasing with respect to each of the last $k+1$ variables,

(ii) $\omega_i$ are nondecreasing with respect to the first and second variables,

(iii) in the interval $[0, \max\{h\colon hM_k < 1\}]$ there exists a continuous and nonnegative solution $\tilde{Z}$ of the inequality

$$\mathscr{A}_h(Z) \leqslant Z,$$

where

$$\mathscr{A}_h(Z) \equiv D\{\omega_1(Z, S_h Z)Z + \omega_2(Z, S_h Z)S_h Z + K_0 c_1 [\eta_2(h) + \tilde{c}(b-a) \sum_{i=0}^{k} \Omega_i(h, S_h Z, \ldots, S_h Z) S_h Z + \tilde{c}\bar{\gamma}_1(h) + \tilde{c} S_h Z \sum_{i=0}^{k} \delta_i(h)]\},$$

and

$$S_h Z \equiv c_1(\bar{q}Z + \eta(h)), \quad D, c_1, \tilde{c} > 0,$$

(iv) in the interval $[0, \max\{\tilde{Z}(h)\colon M_k h < 1\}]$, $Z = 0$ is the unique solution of the equation

$$D\{\omega_1(Z, c_1\bar{q}Z)Z + \omega_2(Z, c_1\bar{q}Z)c_1\bar{q}Z + K_0 c_1 \tilde{c}(b-a) \sum_{i=0}^{k} \Omega_i(0, c_1\bar{q}Z, \ldots, c_1\bar{q}Z)c_1\bar{q}Z\} = Z.$$

**4. The convergence theorem—Lipschitz condition.** The following result is known:

LEMMA 1. *If*

$$0 \leqslant k_{n+1} \leqslant a_n k_n + b_n, \quad a_n, b_n \geqslant 0, n = 0, 1, \ldots, N-k,$$

*then*

$$k_n \leqslant \Big(\prod_{i=0}^{n-1} a_i\Big) k_0 + \sum_{i=0}^{n-1} \Big(\prod_{r=i+1}^{n-1} a_r\Big) b_i, \quad n = 0, 1, \ldots, N-k+1,$$

*where* $\sum_0^{-1} \ldots = 0$, $\prod_0^{-1} \ldots = 1$.

Let

$$0 \leqslant z_{n+1} \leqslant D[Az_n^2 + Bz_n + C], \quad A, B, C, D > 0, \; n = 0, 1, \ldots$$

We will need the following lemma.

LEMMA 2. *Assume that there exists d such that*

$$DB < d < 1, \quad 4p^2AC < 1, \quad \textit{where } p = D/(d-DB).$$

*If* $z_0 \leqslant \varepsilon = DC/(1-d) \leqslant 1/(pA)$ *then*

$$z_n \leqslant d^n \varepsilon + DC(1-d^n)/(1-d), \quad n = 0, 1, \ldots \tag{12}$$

Proof. We can write

$$Q(z) = D[Az^2 + Bz + C] = Dq(z) + dz, \quad \text{where } q(z) = Az^2 - z/p + C.$$

The quadratic function $q$ has distinct positive zeros $z_-$ and $z_+$, where $z_+ > z_- > 0$. The function $Q$ is increasing for $z > 0$ so if $z_0 \leqslant \varepsilon$ then $q(z) \leqslant C$ for $0 \leqslant z \leqslant \varepsilon$ and by induction $z_n \leqslant \varepsilon$ for $n = 0, 1, \ldots$ Now

$$z_{n+1} \leqslant DC + dz_n, \quad n = 0, 1, \ldots,$$

and hence we have (12).

Put

$$\tilde{L} = \sum_{r=0}^{k-1} \sum_{i=0}^{k-1} L_{ir}/2, \quad \tilde{\tilde{L}} = \sum_{r=0}^{k-1} (L_{kr} + L_{rk})/2,$$

$$\max_i (1 - h_i M_k)^{-1} \leqslant \tilde{c}, \quad M = \tilde{M} + \sum_{i=0}^{k-1} M_i,$$

$$c_1 = \exp\big((b-a)(M + M_k \tilde{c})\big), \quad \eta(h) = \eta_1(h) + \tilde{c}\bar{\gamma}_1(h),$$

$$c_2 = \tilde{c}c_1(b-a)(\tilde{L} + \tilde{\tilde{L}} + L_{kk}/2),$$

$$A_0 = c_2(c_1\bar{q})^2, \quad A_{00} = (K_{11} + K_{21}c_1\bar{q})/2, \quad A_{01} = (K_{12} + K_{22}c_1\bar{q})c_1\bar{q}/2,$$

$$B_0(h) = 2c_1^2\bar{q}c_2\eta(h) + \tilde{c}\bar{q}c_1^2 \sum_{r=0}^{k} \delta_r(h),$$

$$B_{00}(h) = K_{21}c_1\eta(h)/2, \qquad B_{01}(h) = (K_{12}+2K_{22}c_1\bar{q})c_1\eta(h)/2,$$

$$C_0(h) = c_2(c_1\eta(h))^2 + c_1\Big[\eta_2(h)+\tilde{c}c_1\eta(h)\sum_{r=0}^{k}\delta_r(h)+\tilde{c}\bar{\gamma}_1(h)\Big],$$

$$C_{01}(h) = K_{22}(c_1\eta(h))^2/2.$$

We consider a family of recurrent equations of order $k$

$$\sum_{i=0}^{k} a_i(t_n, h_n)z_{n+i}^j = c_n^j, \qquad n = 0, 1, \ldots, N-k, \; j = 0, 1, \ldots,$$

where $a_i$: $I \times H \to \mathbf{R}$, $i = 0, 1, \ldots, k$, $a_k(t, h) \equiv 1$. This system may be written as

$$U_{n+1}^j = A_n U_n^j + W_n^j, \qquad n = 0, 1, \ldots, N-k, \; j = 0, 1, \ldots,$$

where

$$U_n^j = [z_n^j, \ldots, z_{n+k-1}^j]^{\mathrm{T}}, \qquad W_n^j = [\theta, \ldots, \theta, c^j]^{\mathrm{T}}.$$

Here $\theta$ is the zero vector in $\mathbf{R}^p$.

Now we can formulate the following theorem.

THEOREM 2. *If Assumption* $\mathrm{H}_1$ *is satisfied and*

1° *there exists a unique solution* $\varphi$ *of* BVP (1–2),

2° *the method* (9–11) *is H-consistent with* BVP (1–2) *on the solution* $\varphi$,

3° *the matrices* $B_a^j + B_b^j Y_N^j$, $j = 0, 1, \ldots$, *are nonsingular and there exists a constant* $D > 0$ *such that*

$$\|(B_a^j + B_b^j Y_N^j)^{-1}\| \leqslant D, \qquad j = 0, 1, \ldots,$$

*then for sufficiently small* $\bar{h}$, $\max_l h_l = h^* \leqslant \bar{h}$ *and* $M_k\bar{h} < 1$ *there exists a constant* $d$ *such that*

$$\text{(13)} \qquad \begin{cases} DB(h) < d < 1, \\ 4p^2(h)AC(h) < 1, \qquad p(h) = D/(d-DB(h)), \\ DC(h)A_p(h) + d \leqslant 1, \end{cases}$$

*for* $h \leqslant \bar{h}$ *and the method* (9–11) *is convergent to the solution* $\varphi$ *of* BVP (1–2) *provided that*

$$\|s^0 - \varphi(a)\| \leqslant u_0(h) = \max_{0 \leqslant h \leqslant \bar{h}} DC(h)/(1-d), \qquad h \leqslant \bar{h}.$$

*Moreover, the estimates*

$$\text{(14)} \qquad \|s^j - \varphi(a)\| \leqslant \dot{u}_j(h),$$

$$\text{(15)} \qquad \max_{n=0,\ldots,N} \|y_n^j - \varphi(t_n)\| \leqslant c_1(\bar{q}u_j(h) + \eta(h)),$$

*hold for* $h \leqslant \bar{h}$ *and* $j = 1, 2, \ldots$, *with*

$$u_j(h) = d^j u_0(h) + DC(h)(1-d^j)/(1-d), \qquad j = 1, 2, \ldots$$

*Here*

$$A = A_{00}+K_0A_0+A_{01}, \quad B(h) = B_{00}(h)+K_0B_0(h)+B_{01}(h),$$
$$C(h) = K_0C_0(h)+C_{01}(h).$$

Proof. We first consider the equation

$$x = A(x),$$

where

$$A(x) \equiv -\sum_{i=0}^{k-1} a_i(t_n, h_n)y_{n+i}^j+h_nF(t_n, \ldots, t_{n+k}, h_n, y_n^j, \ldots, y_{n+k-1}^j, x),$$

for fixed $n$ and $j$. In view of Assumption $H_1$(3°(i)) and the mean value theorem we have for $x_1, x_2 \in \mathbf{R}^p$,

$$\|A(x_1)-A(x_2)\| \leqslant h_nM_k\|x_1-x_2\|,$$

so $A$ is a contraction mapping for $h_n$ so small that $h_nM_k < 1$. This means that (9) has a unique solution with respect to $y_{n+k}^j$. Actually, (10) also has a unique solution with respect to $Y_{n+k}^j$.

Put

$$v_n^j = y_n^j-\varphi(t_n), \quad V_n^j = \max_{r=0,\ldots,k-1} \|v_{n+r}^j\|, \quad z^j = s^j-\varphi(a), \quad Z^j = \|z^j\|,$$

$$C_n^j = h_n\mathscr{F}(t_n, h_n, \varphi(t_n))-\sum_{i=0}^{k} a_i(t_n, h_n)\varphi(t_{n+i}),$$

$$A_n^j(i) = \int_0^1 F_i(t_n, \ldots, t_{n+k}, h_n, \varphi(t_n)+\tau v_n^j, \ldots, \varphi(t_{n+k})+\tau v_{n+k}^j)\,d\tau.$$

We note that for $n = 0, 1, \ldots, N-k$, $j = 0, 1, \ldots$, we have

$$\sum_{i=0}^{k} a_i(t_n, h_n)v_{n+i}^j = \alpha_n^j, \tag{16}$$

where

$$\alpha_n^j = h_n\mathscr{F}(t_n, h_n, y_n^j)-h_n\mathscr{F}(t_n, h_n, \varphi(t_n))+C_n^j.$$

By the mean value theorem we get

$$\alpha_n^j = h_n\sum_{i=0}^{k} A_n^j(i)v_{n+i}^j+C_n^j. \tag{17}$$

Now the relations

$$U_{n+1}^j = A_nU_n^j+W_n^j, \quad W_n^j = [\theta, \ldots, \theta, \alpha_n^j]^{\mathrm{T}},$$

(17) and Assumption $H_1$(3°(i), 5°) yield

$$V_{n+1}^j \leqslant (1+h_nM)V_n^j+h_nM_kV_{n+1}^j+\gamma_1(t_n, h_n),$$

for $n = 0, 1, \ldots, N-k$, $j = 0, 1, \ldots$ By Lemma 1 for $M_k \max_n h_n < 1$ we have

$$(18)\quad V_n^j \leqslant \left(\prod_{i=0}^{n-1} \frac{1+h_i M}{1-h_i M_k}\right) V_0^j + \sum_{i=0}^{n-1} \left(\prod_{r=i+1}^{n-1} \frac{1+h_r M}{1-h_r M}\right) \frac{\gamma_1(t_i, h_i)}{1-h_i M_k}$$

$$\leqslant c_1\big(V_0^j + \tilde{c} \sum_{i=0}^{n-1} \gamma_1(t_i, h_i)\big) \leqslant c_1\big(V_0^j + \tilde{c}\bar{\gamma}_1(h)\big), \qquad n = 0, \ldots, N-k+1.$$

Let

$$\tilde{B}_a^j = \int_0^1 g_1\big(\varphi(a) + \tau z^j,\ \varphi(b) + \tau v_n^j\big)\, d\tau,$$

$$\tilde{B}_b^j = \int_0^1 g_2\big(\varphi(a) + \tau z^j,\ \varphi(b) + \tau v_N^j\big)\, d\tau.$$

By (11) and the mean value theorem we get

$$(19)\quad \begin{aligned} z^{j+1} &= z^j - (B_a^j + B_b^j Y_N^j)^{-1}\big[g(s^j, y_N^j) - g(\varphi(a), \varphi(b))\big] \\ &= (B_a^j + B_b^j Y_N^j)^{-1}\big[(B_a^j + B_b^j Y_N^j) z^j - \tilde{B}_a^j z^j - \tilde{B}_b^j v_N^j\big] \\ &= (B_a^j + B_b^j Y_N^j)^{-1}\big[(B_a^j - \tilde{B}_a^j) z^j + B_b^j (Y_N^j z^j - v_N^j) + (B_b^j - \tilde{B}_b^j) v_N^j\big] \end{aligned}$$

for $j = 0, 1, \ldots$

Put

$$T_n^j = Y_n^j z^j - v_n^j, \qquad n = 0, 1, \ldots, N, \qquad j = 0, 1, \ldots$$

In view of (10) we see that

$$\sum_{i=0}^{k} a_i(t_n, h_n) T_{n+i}^j = h_n \sum_{i=0}^{k} P_n^j(i) Y_{n+i}^j z^j - \sum_{i=0}^{k} a_i(t_n, h_n) v_{n+i}^j.$$

According to (16) and (17) the last equality yields

$$(20)\quad \sum_{i=0}^{k} a_i(t_n, h_n) T_{n+i}^j = h_n \sum_{i=0}^{k} P_n^j(i) T_{n+i}^j + h_n \sum_{i=0}^{k} [P_n^j(i) - A_n^j(i)] v_{n+i}^j - C_n^j,$$

and now by Assumption $H_1(3°, 5°)$ and 2° we infer that

$$\tilde{T}_{n+1}^j = \max_{r=0,\ldots,k-1} \|T_{n+r+1}^j\| \leqslant (1 + h_n M) \tilde{T}_n^j + h_n M_k \tilde{T}_{n+1}^j$$

$$+ h_n \sum_{r=0}^{k} \Big[\sum_{i=0}^{k} (L_{ir}/2) \|v_{n+i}^j\| + \varepsilon_r(t_n, h_n)\Big] \|v_{n+r}^j\| + \gamma_1(t_n, h_n).$$

Hence

$$\tilde{T}_{n+1}^j \leqslant a_n \tilde{T}_n^j + b_n^j/(1 - h_n M_k),$$

with

$$a_n = (1 + h_n M)/(1 - h_n M_k),$$

$$b_n^j = h_n\big[\tilde{L}(V_n^j)^2 + (L_{kk}/2)(V_{n+1}^j)^2 + \tilde{\tilde{L}} V_n^j V_{n+1}^j$$

$$+ \sum_{r=0}^{k-1} \varepsilon_r(t_n, h_n) V_n^j + \varepsilon_k(t_n, h_n) V_{n+1}^j\big] + \gamma_1(t_n, h_n),$$

for $n = 0, 1, \ldots, N-k$, $j = 0, 1, \ldots$ By Lemma 1 we now have

$$\tilde{T}^j_{N-k+1} \leqslant c_1 \{\tilde{T}^j_0 + \tilde{c} \sum_{i=0}^{N-k} b^j_i\}, \quad j = 0, 1, \ldots$$

According to (18) and Assumption $H_1(6^\circ)$ the last relation gives

$$\|T^j_N\| \leqslant \tilde{T}^j_{N-k+1} \leqslant A_0(Z^j)^2 + B_0(h)Z^j + C_0(h), \quad j = 0, 1, \ldots \tag{21}$$

In view of Assumption $H_1(4^\circ, 6^\circ)$ and (18) we get

$$\|B^j_a - \tilde{B}^j_a\| Z^j \leqslant .5(K_{11} Z^j + K_{21} \|v^j_N\|) Z^j \leqslant A_{00}(Z^j)^2 + B_{00}(h) Z^j, \tag{22}$$

$$\begin{aligned} \|B^j_b - \tilde{B}^j_b\| \, \|v^j_N\| &\leqslant .5(K_{12} Z^j + K_{22} \|v^j_N\|) \|v^j_N\| \\ &\leqslant A_{01}(Z^j)^2 + B_{01}(h) Z^j + C_{01}(h), \end{aligned} \tag{23}$$

for $j = 0, 1, \ldots$

Now combining (19), (21–23) and 3° we obtain

$$Z^{j+1} \leqslant D[A(Z^j)^2 + B(h)Z^j + C(h)], \quad j = 0, 1, \ldots \tag{24}$$

The estimates (14) and (15) follow directly from Lemma 2 and (18). The proof is complete.

LEMMA 3. *Assume that*

1° *Assumption* $H_1$ *and condition* 1° *of Theorem* 2 *are satisfied,*

2° *the method* (9–11) *is consistent with* BVP (1–2) *on the solution* $\varphi$,

3° *the matrix* $\tilde{Q} = g_1(\varphi(a), \varphi(b)) + g_2(\varphi(a), \varphi(b)) Y(b)$ *is nonsingular and* $\|\tilde{Q}^{-1}\| \leqslant \beta$,

4° $\max_{i=0,\ldots,k-1} \|Y^j_i - Y(t_i)\| \leqslant \eta_3(h)$, *for sufficiently small* $h$, $M_k h < 1$ *and* $\lim_{h\to 0} \eta_3(h) = 0$.

*Then condition* 3° *of Theorem* 2 *holds if* $s^0$ *is sufficiently close to* $\varphi(a)$.

Proof. Put

$$Q^j = B^j_a + B^j_b Y^j_N, \quad Q(s) = g_1(s, \varphi(b; s)) + g_2(s, \varphi(b; s)) Y(b; s).$$

It is easy to see that

$$\begin{aligned} Q^j - Q(\varphi(a)) &= g_1(s^j, y^j_N) - g_1(\varphi(a), \varphi(b)) \\ &\quad + [g_2(s^j, y^j_N) - g_2(\varphi(a), \varphi(b))]\, Y(b) + g_2(s^j, y^j_N)[Y^j_N - Y(b)], \end{aligned} \tag{25}$$

and by Assumption $H_1(4^\circ)$ we have

$$\begin{aligned} \|Q^j - Q(\varphi(a))\| &\leqslant (K_{11} + K_{12} B_Y) \|s^j - \varphi(a)\| \\ &\quad + (K_{21} + K_{22} B_Y) \|y^j_N - \varphi(b)\| + K_0 \|q^j_N\|, \end{aligned}$$

where $Y$ is bounded by $B_Y$ and

$$q^j_n = Y^j_n - Y(t_n), \quad n = 0, 1, \ldots, N.$$

We note that

$$\sum_{i=0}^{k} a_i(t_n, h_n)q_{n+i}^j = h_n \sum_{i=0}^{k} P_n^j(i)q_{n+i}^j + h_n \sum_{i=0}^{k} [P_n^j(i) - \tilde{P}_n(i)]\, Y(t_{n+i})$$

$$+ h_n \sum_{i=0}^{k} \tilde{P}_n(i)\, Y(t_{n+i}) - \sum_{i=0}^{k} a_i(t_n, h_n)\, Y(t_{n+i}),$$

for $n = 0, 1, \ldots, N-k$ and $j = 0, 1, \ldots,$ where

$$\tilde{P}_n(i) = F_i(t_n, \ldots, t_{n+k}, h_n, \varphi(t_n), \ldots, \varphi(t_{n+k})).$$

Moreover, by Assumption $H_1(3°)$ we have

$$\|P_n^j(i) - \tilde{P}_n(i)\| = \|F_i(t_n, \ldots, t_{n+k}, h_n, y_n^j, \ldots, y_{n+k}^j)$$

$$- F_i(t_n, \ldots, t_{n+k}, h_n, \varphi(t_n), \ldots, \varphi(t_{n+k}))\|$$

$$\leqslant \sum_{r=0}^{k} L_{ri} \|y_{n+r}^j - \varphi(t_{n+r})\| + \varepsilon_i(t_n, h_n), \qquad n = 0, 1, \ldots, N-k,$$

so

$$\tilde{q}_{n+1}^j = \max_{p=0,\ldots,k-1} \|q_{n+p+1}^j\| \leqslant (1 + h_n\tilde{M})\tilde{q}_n^j + h_n \sum_{i=0}^{k-1} M_i \tilde{q}_n^j + h_n M_k \tilde{q}_{n+1}^j$$

$$+ h_n \sum_{i=0}^{k} \Big[\sum_{r=0}^{k} L_{ri} \|y_{n+r}^j - \varphi(t_{n+r})\| + \varepsilon_i(t_n, h_n)\Big] B_Y + \gamma_2(t_n, h_n).$$

Hence we get

$$\tilde{q}_{n+1}^j \leqslant a_n \tilde{q}_n^j + b_n^j/(1 - h_n M_k), \qquad n = 0, 1, \ldots, N-k,$$

with $a_n$ defined in the proof of Theorem 2 and

$$b_n^j = \gamma_2(t_n, h_n) + h_n B_Y \sum_{i=0}^{k} \varepsilon_i(t_n, h_n) + h_n B_Y \sum_{i=0}^{k} \sum_{r=0}^{k} L_{ri} \|v_{n+r}^j\|.$$

By Lemma 1 we have for $j = 0, 1, \ldots,$

$$\tilde{q}_{N-k+1}^j \leqslant c_1 \Big[\tilde{q}_0^j + \tilde{c} \sum_{i=0}^{N-k} b_i^j\Big]$$

$$\leqslant c_1 \Big\{\tilde{q}_0^j + \tilde{c} \sum_{n=0}^{N-k} \gamma_2(t_n, h_n) + \tilde{c} B_Y \sum_{i=0}^{k} \sum_{n=0}^{N-k} h_n \varepsilon_i(t_n, h_n)$$

$$+ \tilde{c} B_Y \sum_{n=0}^{N-k} h_n \sum_{i=0}^{k} \Big[\sum_{r=0}^{k-1} L_{ri} \|v_{n+r}^j\| + L_{ki} \|v_{n+k}^j\|\Big]\Big\}$$

$$\leqslant c_1 [\tilde{q}_0^j + \Gamma(h) + \tilde{c} B_Y (b-a) L c_1 \bar{q} Z^j],$$

where

$$L = \sum_{i=0}^{k} \sum_{r=0}^{k} L_{ri},$$

$$\Gamma(h) = \tilde{c}\bar{\gamma}_2(h) + \tilde{c}B_Y \sum_{i=0}^{k} \delta_i(h) + \tilde{c}B_Y(b-a)Lc_1\eta(h).$$

Now using condition 4° we obtain

$$\tilde{q}^j_{N-k+1} \leqslant \tilde{K}Z^j + \xi(h), \quad j = 0, 1, \ldots,$$

where

$$\tilde{K} = c_1\tilde{c}B_Y(b-a)Lc_1\bar{q}, \quad \xi(h) = c_1[\eta_3(h) + \Gamma(h)].$$

Combining the previous estimates we have

$$\|Q^j - Q(\varphi(a))\| \leqslant \tilde{L}Z^j + \nu(h), \quad j = 0, 1, \ldots,$$

where

$$\tilde{L} = K_{11} + K_{21}c_1\bar{q} + (K_{12} + K_{22}c_1\bar{q})B_Y + K_0\tilde{K},$$

$$\nu(h) = (K_{21} + K_{22}B_Y)c_1\eta(h) + K_0\xi(h).$$

Hence

(26) $$p^j = \|Q^{-1}(\varphi(a))[Q^j - Q(\varphi(a))]\| \leqslant \beta[\tilde{L}Z^j + \nu(h)], \quad j = 0, 1, \ldots$$

Let

$$Z^0 \leqslant \varrho \quad \text{and} \quad \beta\tilde{L}\varrho < 1.$$

Because $\nu(h) \to 0$ as $h \to 0$ there exists $\alpha$ such that

$$\beta[\tilde{L}\varrho + \nu(h)] \leqslant \alpha < 1,$$

for sufficiently small $h$. Now by Lemma 4.4.14 of [9], p. 180, we conclude that the matrix

$$I + Q^{-1}(\varphi(a))[Q^0 - Q(\varphi(a))]$$

is nonsingular. Hence the matrix

$$Q^0 = Q(\varphi(a))\{I + Q^{-1}(\varphi(a))[Q^0 - Q(\varphi(a))]\}$$

is also nonsingular and

(27) $$\|(Q^0)^{-1}\| \leqslant \beta/(1-\alpha).$$

Put

$$u_0(h) = \min(\varrho, \max_{0 \leqslant h \leqslant \bar{h}} DC(h)/(1-d)), \quad D = \beta/(1-\alpha),$$

where $\bar{h}$ is defined in Theorem 2.

Now we need to prove that $Q^j$ is nonsingular and

(28) $$\|(Q^j)^{-1}\| \leqslant \beta/(1-\alpha).$$

Assume that this is true for $j = m \geqslant 0$. By (14) and (26) we have

$$p^{m+1} \leqslant \beta[\tilde{L}u_{m+1}(h)+v(h)] \leqslant \beta[\tilde{L}u_0(h)+v(h)] \leqslant \alpha < 1.$$

Hence by Lemma 4.4.14 of [9] the matrix

$$I+Q^{-1}(\varphi(a))[Q^{m+1}-Q(\varphi(a))],$$

is nonsingular and (28) holds for $j = m+1$. Now (28) follows by induction.

**5. The convergence theorem—Perron condition.** The basic result of this part is

THEOREM 3. *If Assumption* $H_2$ *and conditions* 1°–3° *of Theorem* 2 *are satisfied then for sufficiently small* $\bar{h}$, $\max_i h_i = h^* \leqslant \bar{h}$ *such that* $\bar{h}M_k < 1$ *the method* (9–11) *is convergent to the solution* $\varphi$ *of* BVP (1–2) *provided that*

$$\|s^0-\varphi(a)\| \leqslant \tilde{Z}(h), \quad h \leqslant \bar{h}.$$

*Moreover,*

$$\|s^{j+1}-\varphi(a)\| \leqslant \mathscr{A}_h(\|s^j-\varphi(a)\|), \quad j = 0, 1, \ldots,$$

$$\max_{n=0,\ldots,N} \|y_n^j-\varphi(t_n)\| \leqslant c_1[\bar{q}\,\|s^j-\varphi(a)\|+\eta(h)], \quad j = 0, 1, \ldots$$

Proof. From Theorem 2 and condition 6° of Assumption $H_1$ we have

$$V_n^j \leqslant c_1[\bar{q}Z^j+\eta(h)].$$

Now by Assumption $H_1$ and (20) we get

$$\tilde{T}_{n+1}^j \leqslant (1+h_nM)\tilde{T}_n^j+h_nM_k\tilde{T}_{n+1}^j+\gamma_1(t_n, h_n)+h_n\sum_{i=0}^{k-1}\varepsilon_i(t_n, h_n)V_n^j$$
$$+h_n\varepsilon_k(t_n, h_n)V_{n+1}^j+h_n\sum_{i=0}^{k-1}\Omega_i(h_n, V_n^j, \ldots, V_n^j, V_{n+1}^j)V_n^j$$
$$+h_n\Omega_k(h_n, V_n^j, \ldots, V_n^j, V_{n+1}^j)V_{n+1}^j,$$

or

$$\tilde{T}_{n+1}^j \leqslant a_n\tilde{T}_n^j+b_n^j/(1-h_nM_k), \quad n = 0, 1, \ldots, N-k,\ j = 0, 1, \ldots,$$

with

$$b_n^j = h_n\Big[\sum_{i=0}^{k-1}\Omega_i(h_n, V_n^j, \ldots, V_n^j, V_{n+1}^j)V_n^j+\Omega_k(h_n, V_n^j, \ldots, V_n^j, V_{n+1}^j)V_{n+1}^j\Big]$$
$$+\gamma_1(t_n, h_n)+h_n\sum_{i=0}^{k-1}\varepsilon_i(t_n, h_n)V_n^j+h_n\varepsilon_k(t_n, h_n)V_{n+1}^j.$$

Using Lemma 2 we hence obtain

$$\tilde{T}_n^j \leqslant c_1 \{\tilde{T}_0^j + \tilde{c}[(b-a)c_1(\bar{q}Z^j+\eta(h)) \sum_{i=0}^{k} \Omega_i(h, c_1(\bar{q}Z^j+\eta(h)), \ldots, \tag{29}$$

$$c_1(\bar{q}Z^j+\eta(h))) + \bar{\gamma}_1(h) + c_1(\bar{q}Z^j+\eta(h)) \sum_{i=0}^{k} \delta_i(h)]\},$$

$$n = 0, 1, \ldots, N-k+1, \quad j = 0, 1, \ldots$$

Moreover,

$$\|B_a^j - \tilde{B}_a^j\| \leqslant \omega_1(Z^j, c_1(\bar{q}Z^j+\eta(h))), \tag{30}$$

$$\|B_b^j - \tilde{B}_b^j\| \leqslant \omega_2(Z^j, c_1(\bar{q}Z^j+\eta(h))). \tag{31}$$

Combining (19), (29–31) we infer that $Z^{j+1} \leqslant \mathscr{A}_h(Z^j)$, $j = 0, 1, \ldots$

Let

$$\tilde{Z}^0 = \tilde{Z}, \qquad \tilde{Z}^{j+1} = \mathscr{A}_h(\tilde{Z}^j), \quad j = 0, 1, \ldots$$

We have

$$\tilde{Z}^{j+1} \leqslant \tilde{Z}^j \leqslant \ldots \leqslant \tilde{Z}^0 \quad \text{and} \quad Z^j \leqslant \tilde{Z}^j, \quad j = 0, 1, \ldots$$

This means that the sequence $\{\tilde{Z}^j\}$ has a limit and if also $h \to 0$ then

$$\lim_{\substack{h\to 0\\ j\to\infty}} \tilde{Z}^j = 0.$$

Hence our method is convergent to $\varphi$. The proof is complete.

LEMMA 4. *Let Assumption* $H_2$ *be satisfied with* $\omega_i(0, 0) = 0$, $i = 1, 2$, *and* $\Omega_r(0, 0, \ldots, 0) = 0$, $r = 0, 1, \ldots, k$. *Let condition* 1° *of Theorem* 2 *and conditions* 2°–4° *of Lemma* 3 *be satisfied. Then condition* 3° *of Theorem* 2 *holds if* $s$ *is sufficiently close to* $\varphi(a)$.

Proof. We will skip some details because the proof is similar to the proof of Lemma 3. By Assumption $H_2(3°)$ and (25) we have

$$\|Q^j - Q(\varphi(a))\| \leqslant \omega_1(Z^j, V_n^j) + \omega_2(Z^j, V_n^j) B_Y + K_0 \|q_N^j\|.$$

Moreover,

$$\|P_n^j(i) - \tilde{P}_n(i)\| \leqslant \Omega_i(h, V_n^j, \ldots, V_n^j, V_{n+1}^j) + \varepsilon_i(t_n, h_n),$$

and hence

$$\tilde{q}_{N-k+1}^j \leqslant \bar{L} \sum_{i=0}^{k} \Omega_i(h, S_h Z^j, \ldots, S_h Z^j) + \bar{\nu}(h),$$

with

$$\bar{L} = c_1 \tilde{c} B_Y (b-a), \qquad \bar{\nu}(h) = c_1 [\eta_3(h) + \tilde{c}\bar{\gamma}_2(h) + \tilde{c} B_Y \sum_{i=0}^{k} \delta_i(h)].$$

Now we have

$$p^j \leqslant \beta[\omega_1(Z^j, S_h Z^j) + \omega_2(Z^j, S_h Z^j) B_Y + K_0 \bar{L} \sum_{i=0}^{k} \Omega_i(h, S_h Z^j, \ldots, S_h Z^j) + K_0 \bar{v}(h)].$$

Let $Z^0 \leqslant \varrho$ and

$$\zeta(\varrho) \equiv \beta[\omega_1(\varrho, S_h\varrho) + \omega_2(\varrho, S_h\varrho) B_Y + K_0 \bar{L} \sum_{i=0}^{k} \Omega_i(h, S_h\varrho, \ldots, S_h\varrho)] < 1.$$

Hence also $\zeta(\varrho) + \beta K_0 \bar{v}(h) \leqslant \alpha < 1$ for sufficiently small $h$, and by Lemma 4.4.14 of [9] the condition (27) is satisfied.

The reest of the proof is obvious.

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