ANNALES
POLONICI MATHEMATICI
XXII (1969)

Separately analytic functions and envelopes of holomorphy of some lower dimensional subsets of C^n

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Introduction. Osgood proved in [20], [21] that if $f(z) = f(z_1, \ldots, z_n)$ is a function defined in a domain D in the space C^n of n complex variables $z_k = x_k + iy_k$ $(k = 1, \ldots, n)$ and f is locally bounded in D and analytic in each variable z_k separately when the other variables are given arbitrary fixed values, then f is analytic in D. According to the famous theorem of Hartogs [10] the assumption of the local boundedness is superfluous. A new elegant proof of the Hartogs theorem may be found in [12].

If $u(x) = u(x_1, \ldots, x_n)$ is a function defined in a domain D in the space R^n of n real variables x_k $(k = 1, \ldots, n)$ and u is analytic in each variable x_k separately, then u is not, in general, an analytic function in D, even if we in addition assume that $u \in C^\infty(D)$. A corresponding example is given by

$$u(x_1, x_2) = x_1 x_2 \exp\left[-\frac{1}{x_1^2 + x_2^2}\right], \quad u(0, 0) = 0, \quad (x_1, x_2) \in R^2.$$

It is, however, possible to generalize Hartogs' theorem for the following important class of function of n real variables. Let E be a subset of R^n. We identify R^n with the subset $\{z \in C^n\colon y_k = 0,\ k = 1, \ldots, n\}$ of C^n. Then E may be considered as a subset of C^n. Let D be a domain in R^n. Let L_D denote the class of all the functions f defined in D so that for every $x^0 \in D$ there exists a polydisc $P(x^0, r) = \{z \in C^n\colon |z_k - x_k^0| < r_k,$ $k = 1, \ldots, n\}$ such that for fixed ξ_k, where $x_k^0 - r_k < \xi_k < x_k^0 + r_k$ $(k = 1, \ldots, n)$, $k \neq j$, the function $f(\xi_1, \ldots, \xi_{j-1}, x_j, \xi_{j+1}, \ldots, \xi_n)$, $x_j^0 - r_j < x_j < x_j^0 + r_j$, is continuable to an analytic function in the disc $|z_j - x_j^0| < r_j$ $(j = 1, \ldots, n)$. Every function $f \in L_D$ is analytic in each variable x_j separately.

Theorem 7.1 of this paper gives as a special case the following

(I) *If D is a domain in R^n, then every function $f \in L_D$ is analytic in D.*

Corollary. *If $h(x, u) = h(x_1, \ldots, x_p, u_1, \ldots, u\ )$ is a function defined in a domain $D \subset R^{p+q}$ and harmonic with respect to x and y separately, then h is harmonic in D.*

Theorem (I) generalizes results concerning separate analyticity of real functions due to Lelong [19] and Browder [4] (see also [5]), where the analyticity has been proved for those functions $f \in L_D$ which are assumed to satisfy some boundedness conditions. The result formulated in Corollary has been first proved in [19].

The problem of analyticity of the functions belonging to L_D is a special case of the following Problem 1. Let D and G be domains in the space C^m and C^n, respectively. Let E and F be relatively closed subsets of D and G, respectively. Put

$$X = (D \times F) \cup (E \times G) .$$

We say that a function $f(z, w) = f(z_1, \ldots, z_m, w_2, \ldots, w_n)$ defined in X is *separately analytic in* X, if

(i) $f(z, w^0)$ is analytic in D for each fixed $w^0 \in F$,

(ii) $f(z^0, w)$ is analytic in G for each fixed $z^0 \in E$.

PROBLEM 1. *Characterize the subsets E of D and F of G for which every function f separately analytic in $X = (D \times F) \cup (E \times G)$ may be continued to a function $\tilde{f}$ analytic in an open neighborhood of X.*

An answer to this problem gives also an answer to a problem of Hukuhara [11]. For the statement and solution of the Hukuhara problem see [24] and [28].

The following problem is a natural complement to Problem 1.

PROBLEM 2. *Determine the envelope of holomorphy of X, i.e. the maximal domain Ω with the property that $X \subset \Omega$ and every function f analytic in a neighborhood of X admits an analytic continuation into Ω.*

A partial solution to these problems is presented in § 6 and § 7. To get our solution we prove at first (a) a generalization (see Theorem 1.2) of a polynomial lemma due to Leja [14] (see also [7]), (b) a generalization of the Fundamental Lemma of Hartogs (see § 2), (c) a version of the Two Constants Theorem for plurisubharmonic functions (see § 3) and (d) an Approximation Lemma (see § 5). To prove the Approximation Lemma we interpolate separately analytic functions in nodes which are suitably chosen extremal points of Fekete-Leja type (see § 4). By the way we give a contribution to the theory of interpolation and approximation by rational functions developed in [29], Chapter VIII.

§ 1. A polynomial condition. Let E be a subset of C^n, $n \geqslant 1$. We say that E satisfies the polynomial condition (L_0) at a point $a \in C^n$ if for every family $\mathcal{F}$ of polynomials $P(z)$ in n complex variables $z = (z_1, \ldots, z_n)$ such that

$$\mathcal{M}_{\mathcal{F}}(z) = \sup \{|P(z)|:\ P \in \mathcal{F}\} < \infty , \qquad z \in E ,$$

and for every $\varepsilon > 0$ there exist two positive numbers $M = M(a, \varepsilon)$ and $\delta = \delta(a, \varepsilon)$ such that

$$|P(z)| \leqslant M \exp(\varepsilon \deg P), \quad \|z-a\| < \delta, \ P \in \mathcal{F},$$

where $\deg P$ denotes the largest sum of exponents occurring in a monomial term of P.

We say that a set $E \subset C^n$ satisfies the polynomial condition (L) at a point $a \in C^n$ if for every $r > 0$ the set $E_r = \{z \in E\colon \|z-a\| \leqslant r\}$ satisfies the condition (L_0) at a. If E satisfies (L) at each $a \in E$ we write $E \in (L)$.

By induction with respect to n one may easily prove the following (compare with [26])

THEOREM 1.1. *If E_k is a subset of the complex z_k-plane satisfying (L) at $z_k^0 \in E_k$ $(k = 1, \ldots, n)$, then the set $E = E_1 \times \ldots \times E_n$ satisfies (L) at $z^0 = (z_1^0, \ldots, z_n^0)$.*

The following lemma is implicitly contained in [14].

POLYNOMIAL LEMMA I (Leja, [14]). *Let E be a subset of C and let a be a limit point of E. If there exists a positive number ϱ and a subset S of the interval $(0, \varrho)$ with the Lebesgue measure $m(S) = \varrho$ such that for every $r \in S$ the circle $\{z\colon |z-a| = r\}$ intersects E, then E satisfies (L) at a.*

This lemma and Theorem 1.1 imply the following

POLYNOMIAL LEMMA II. *A sufficient condition that a set $E \subset C^n$ satisfy (L) at $z^0 \in E$ is that there exist continuum E_k in the z_k-plane $(k = 1, \ldots, n)$ such that $z^0 \in E_1 \times \ldots \times E_n \subset E$.*

Let E be a compact set in C with positive transfinite diameter $d(E)$. Let $b(z)$ be a real bounded function defined on E. Denote by $\mathcal{F}(E, b)$ the family of all the polynomials P in z such that

$$|P(z)| \leqslant \exp\big(\deg P\, b(z)\big), \quad z \in E.$$

The function Φ defined by

$$\Phi(z) \equiv \Phi(z, E, b) = \sup\{|P(z)|^{1/\deg P}\colon P \in \mathcal{F}(E, b)\}, \quad z \in C,$$

is called the *extremal function* of E with respect to b.

If $b(z) = 0$ we write $L(z, E)$ instead of $\Phi(z, E, 0)$. So

$$L(z, E) = \sup\{|P(z)|^{1/\deg P}\colon P \in \mathcal{F}_0\}, \quad z \in C,$$

where $\mathcal{F}_0$ is the family of all the polynomials P in z such that $|P(z)| \leqslant 1$ on E.

It is known that (see [25], [18]):

(1.1) $\operatorname{Log} \Phi$ is harmonic in $C-E$ and continuous at every point $a \in E$ at which E satisfies (L).

(1.2) If $E \subset \{z \in C: |z-a| < 1\}$, $a \in E$, E satisfies (L) at a, and $b(z) = |z-a|$, then $V(z) = \mathrm{Log}\, \Phi(z, E, b)$ is harmonic in $C-E$, $V(z) > 0$ for $z \neq a$ and $\lim_{z\to a} V(z) = 0$.

(1.3) $\mathrm{Log}\, L(z, E)$ is the Green's function of D_∞ with the pole at ∞, where $D_\infty = D_\infty(E)$ denotes the unbounded component of $C-E$. $L(z, E)$ is continuous at every regular boundary point of D_∞.

We shall need the following lemma which is implicitly contained in Choquet [6] (see also [3] and [13]).

LEMMA 1.1. *If E is a compact set in C and if $\{E_n\}$ is an increasing sequence of compact subsets of E such that $E = \bigcup E_n$, then*

$$\lim d(E_n) = d(E)\,. \tag{1.4}$$

Proof. It is known ([6] and [22]) that given any two compact sets A and B we have

$$d(A \cup B)d(A \cap B) \leqslant d(A)d(B)\,.$$

Hence, using the monotonicity property of the transfinite diameter and the induction with respect to n, we get successively

$$d(X \cup A \cup B)d(X) \leqslant d(X \cup A)d(X \cup B)$$

and

$$d\Big(\bigcup_1^n E_i\Big)\cdot\prod_1^n d(e_i) \leqslant d\Big(\bigcup_1^n e_i\Big)\cdot\prod_1^n d(E_i)\,, \qquad n = 1, 2, \ldots, \tag{1.5}$$

where X, A, B, E_i, e_i are arbitrary compact sets and $e_i \subset E_i$ $(i = 1, \ldots, n)$.

To prove (1.4) it is enough to show that $\lim d(E_n) \geqslant d(E)$. If $d(E) = 0$, then (1.4) is obvious. If $d(E) > 0$, then there exists k such that $d(E_n) > 0$ for $n \geqslant k$ (see [13]). Therefore without loss of generality we may assume that $d(E_n) > 0$ for $n = 1, 2, \ldots$

Given $\varepsilon > 0$ there is an open set G_i such that $E_i \subset G_i$ and

$$d(\bar{G}_i) \leqslant d(E_i)\exp(\varepsilon/2^i)\,, \qquad i = 1, 2, \ldots$$

Since $E \subset \bigcup_1^\infty G_i$, there is k such that $E \subset \Omega = \bigcup_1^k G_i$. Therefore by (1.5)

$$d(\bar{\Omega})\cdot\prod_1^k d(E_i) \leqslant d(E_k)\cdot\prod_1^k d(\bar{G}_i)\,,$$

whence

$$d(E) \leqslant d(\bar{\Omega}) \leqslant d(E_k)\prod_1^k \frac{d(\bar{G}_i)}{d(E_i)} \leqslant d(E_k)e^{\varepsilon}\,.$$

So $d(E) \leqslant d(E_n)e^\varepsilon$, $n \geqslant k$. Hence $\lim d(E_n) \geqslant d(E)$. The proof is concluded.

By the way we want to remark that the proof of Lemma 2.10 in [13] is faulty. The reason of the faultiness lies in the simple fact that the function $\exp(-1/t)$ is convex and not concave in the interval $0 < t < 1/2$. In particular, inequality 2.2.12 in [13] does not hold for the transfinite diameter.

THEOREM 1.2. *A compact set $E \subset C$ satisfies condition (L) at $a \in E$ if and only if each component of $C-E$ containing a on its boundary is regular at a with respect to the Dirichlet problem.*

Proof. Necessity. Let E satisfy (L) at a. Then by (1.2) there exists a function $V(z)$ which is harmonic in $C-E$, $V(z) > 0$ for $z \neq a$ and $\lim_{\to a} V(z) = 0$. So $V(z)$ is a barier for every component ω of $C-E$ such that $a \in \partial\omega$. Therefore ([3], [13]) every such component is regular at a.

Sufficiency. We have to consider two cases.

Case 1. There is no component ω of $C-E$ containing a in its boundary. In this case for every $\varrho > 0$ the set $C-D_\infty(E_\varrho)$ contains a component G such that $a \in G$. The component G is a simple connected domain with boundary $\partial G \subset E_\varrho$. Let $\mathcal{F}$ be a family of polynomials with $\mathcal{M}_{\mathcal{F}}(z) < \infty$ in E_ϱ. By the Polynomial Lemma II and by the Borel-Lebesgue covering theorem for every $\varepsilon > 0$ there are $\delta > 0$ and $M > 0$ such that

$$|P(z)| \leqslant M \exp(\varepsilon \deg P)\,, \qquad P \in \mathcal{F}\,, \qquad \operatorname{dist}(z, \partial G) < \delta\,.$$

By the maximum principle we have $|P(z)| \leqslant M \exp(\varepsilon \deg P)$, $P \in \mathcal{F}$, $z \in G$. So E satisfies (L) at a.

Case 2. Let ω be a component of $C-E$ and let the point a be a regular boundary point of ω. Take $r > 0$ so small that the circle $|z-a| = r$ intersects ω. Then a belongs to the boundary of the unbounded component D_∞ of $C-E_r$. Since the regularity is a local property, the domain D_∞ is regular at a. In particular, the function $L(z, E_r)$ is continuous at a and $\lim_{z\to a} L(z, E_r) = 1$.

Let $\mathcal{F}$ be an arbitrary family of polynomials such that $\mathcal{M}_{\mathcal{F}}(z) < \infty$ in E_r. Put

$$K_n = \{z \in E_r\colon\ \mathcal{M}_{\mathcal{F}}(z) \leqslant n\}\,, \qquad n = 1, 2, \ldots$$

Since $\mathcal{M}_{\mathcal{F}}$ is lower-semicontinuous and $\mathcal{M}_{\mathcal{F}}(z) < \infty$ in E_r, the set K_n is closed and $K_n \subset K_{n+1}$, $E_r = \bigcup_1^\infty K_n$. So by Lemma 1.1 $d(K_n) \nearrow d(E_r)$. By the definition of $L(z, E)$ we have $L(z, E_r) \leqslant L(z, K_n)$. The function $U_n(z) = \operatorname{Log}[L(z, E_r)/L(z, K_n)]$ is harmonic in D_∞ and $U_n(\infty) = \operatorname{Log}[d(K_n)/$

$/d(E_r)]$. So by the Harnack's principle $L(z, K_n)\searrow L(z, E_r)$ for $z \in D_\infty$. Accordingly, given $\varepsilon > 0$ and $z \in D_\infty$ one can find $n_0 = n_0(z, \varepsilon)$ such that

$$L(z, K_n) < L(z, E_r)e^{\varepsilon}, \quad n \geqslant n_0 .$$

Since $L(z, E_r)$ is continuous at a, there is $\varrho > 0$ such that

$$L(z, E_r) < e^{\varepsilon}, \quad z \in H, \quad H = \{z \in D_\infty\colon |z-a| < \varrho\}$$

By the definition of $L(z, K_n)$ and of K_n we have

$$|P(z)| \leqslant nL^{\deg P}(z, K_n), \quad z \in C, \ P \in \mathcal{F}, \ n = 1, 2, \ldots$$

Conseqnently,

$$|P(z)| \leqslant n_0 \exp(2\varepsilon \deg P), \quad z \in H, \ P \in \mathcal{F}.$$

Applying now the Polynomial Lemma I to the family of all the polynomials $Q(z) = P(z)\exp(-2\varepsilon \deg P)$, where $P \in \mathcal{F}$, we can find constants $\delta > 0$ and $M > 0$ such that

$$|P(z)| \leqslant M \exp(3\varepsilon \deg P), \quad |z-a| < \delta, \ P \in \mathcal{F}.$$

By the arbitrariness of $\mathcal{F}$, $r > 0$ and $\varepsilon > 0$ we conclude that E satisfies (L) at a.

From the proof we derive easily the following sufficient condition:

If $a \in E$ and there is a component ω of $C-E$ such that $a \in \partial\omega$ and ω is regular at a, then E satisfies (L) at a.

Theorem 1.2 generalizes the Polynomial Lemma II as well as another result also due to Leja [17].

Using the notion of the thin set (see [3]) Theorem 1.2 may be reformulated in an equivalent way as follows.

THEOREM 1.2.a. *A compact set $E \subset C$ satisfies (L) at $a \in E$ if and only if E is not thin at a.*

We shall further need two following remarks.

Remark 1.1. Let E be a subset of C^n satisfying (L) at $z^0 \in E$. Let f be an analytic function in a ball $\|z-z^0\| < R$ such that $f(z) = 0$ for $z \in E$. Then $f(z) \equiv 0$.

Proof. Without loss of generality we may put $z^0 = 0$. Let $0 < r < \min(1/2, R)$. Then $|f(z)| \leqslant K = \text{const}$ in $\|z\| \leqslant r$. Expand f into the series

$$f(z) = \sum_{0}^{\infty} Q_s(z), \quad \|z\| < R, \tag{1.8}$$

of homogeneous polynomials Q_s of respective degrees s. By homogeneity of Q_s and by the Cauchy inequalities we have

$$|Q_s(a)| \leqslant Kr^{-s}, \quad s = 0, 1, \ldots, \ a \in C^n, \ \|a\| = 1,$$

whence

$$|Q_s(z)| \leqslant K2^{-s}, \quad \|z\| \leqslant r/2, \ s = 1, 2, \ldots \tag{1.9}$$

$P_j(z) = \sum_0^j Q_s(z)$ is a polynomial in n complex variables of degree at most j. By (1.8) and (1.9)

$$|P_j(z)| \leqslant |f(z)| + K2^{-j}, \quad \|z\| \leqslant r/2, \ j = 0, 1, \ldots,$$

whence

$$|P_j(z)| \leqslant K2^{-j}, \quad \|z\| \leqslant r/2, \ z \in E, \ j = 0, 1, \ldots$$

Since E satisfies (L) at $z^0 = 0$, so for every $\varepsilon > 0$ there exist positive numbers δ and M such that

$$|P_j(z)| \leqslant MK2^{-j}e^{j\varepsilon}, \quad \|z\| < \delta, \ j = 0, 1, \ldots$$

If $0 < \varepsilon < \frac{1}{2}\operatorname{Log} 2$, then $|P_j(z)| \leqslant MK2^{-j/2}$, whence $f(z) = \lim_{j\to\infty} P_j(z) = 0$ as $\|z\| < \delta$. Hence $f \equiv 0$.

Remark 1.2. Let $E = E_1 \times \ldots \times E_n$, where E_j $(j = 1, \ldots, n)$ is a compact infinite set in the z_j-plane. Let f be a function analytic in a domain $D \subset C^n$ such that $E \subset D$. If $f = 0$ on E, then $f = 0$ in D. (Proof by induction with respect to n.)

§ 2. A generalization of the Fundamental Lemma of Hartogs.

The following theorem will play a basic role in our further study.

THEOREM 2.1. *Assume that*: (a) *G is an open set in C^n*; (b) *E is a compact subset of G, $E \in (L)$*; (c) *$\{\lambda_\nu\}$ is a sequence of positive real numbers*; (d) *T is an arbitrary non-empty set of arbitrary elements*; *and* (d) *for every $t \in T$ $\{f_\nu(z, t)\}$ is a sequence of analytic functions in G such that*

(i) $\sup_{t\in T} \frac{1}{\lambda_\nu} \operatorname{Log}|f_\nu(z, t)| \leqslant K = \text{const}, \ z \in G, \ \nu \geqslant 1,$

(ii) $\limsup_{\nu\to\infty} \sup_{t\in T} \frac{1}{\lambda_\nu} \operatorname{Log}|f_\nu(z, t)| \leqslant A = \text{const}, \ z \in E.$

Then for every $\varepsilon > 0$ there exists a positive number $M = M(\varepsilon)$ and an open subset $U = U(\varepsilon)$ of G such that $E \subset U$ and

(iii) $|f_\nu(z, t)| \leqslant M \exp[(A + \varepsilon)\lambda_\nu], \ z \in U, \ t \in T, \ \nu \geqslant 1.$

This theorem follows immediately from the following

LEMMA 2.1. *If the set E satisfies* (L) *at a fixed point $z^0 \in E$ and* (i) *and* (ii) *are satisfied, then for every $\varepsilon > 0$ there exist positive numbers $M = M(z^0, \varepsilon)$ and $\delta = \delta(z^0, \varepsilon)$ such that*

(iv) $|f_\nu(z, t)| \leqslant M \exp[(A+\varepsilon)\lambda_\nu]$, $\|z - z^0\| < \delta$, $t \in T$, $\nu \geqslant 1$.

Proof. Without loss of generality we may assume that $z^0 = 0$. Let $0 < r < \operatorname{dist}(E, \partial G)$ and let for every $t \in T$

$$f_j(z, t)e^{-A\lambda_j} = \sum_{s=0}^{\infty} Q_{js}(z, t), \quad \|z\| \leqslant r, \tag{2.1}$$

be the expansion of $f_j \exp(-A\lambda_j)$ into the series of homogeneous polynomials of respective degrees s. By (i)

$$|f_j e^{-A\lambda_j}| \leqslant e^{(K-A)\lambda_j} \quad \text{for } \|z\| \leqslant r, \ t \in T, \ j \geqslant 1 .$$

Hence by the Cauchy inequalities

$$|Q_{js}(a, t)| \leqslant r^{-s} e^{(K-A)\lambda_j}, \quad a \in C^n, \ \|a\| = 1, \ s \geqslant 0, \ j \geqslant 1, \ t \in T,$$

and consequently

$$\sum_{s=[\lambda_j]+1}^{\infty} |Q_{js}| \leqslant e^{(K-A)\lambda_j} \frac{(\varrho/r)^{[\lambda_j]+1}}{1-\varrho/r}, \quad \|z\| \leqslant \varrho < r, \ t \in T, \tag{2.2}$$

where $[\lambda_j]$ denotes the integer such that $\lambda_j - 1 < [\lambda_j] \leqslant \lambda_j$.

By (ii) and (i) for $\varepsilon > 0$ and $z \in E$ there exists a positive number $H = H(z, \varepsilon)$ such that for $t \in T$ and $j \geqslant 1$

$$|f_j(z, t)| e^{-A\lambda_j} \leqslant H e^{\varepsilon\lambda_j} . \tag{2.3}$$

For every fixed $t \in T$, $P_j(z, t) = \sum_{s=0}^{[\lambda_j]} Q_{js}(z, t)$ is a polynomial in $z_1, \ldots, z_n$ of degree at most $[\lambda_j]$. By (2.1), (2.2) and (2.3)

$$|P_j(z, t)| \leqslant H(z, \varepsilon)e^{\varepsilon\lambda_j} + e^{(K-A)\lambda_j} \frac{(\varrho/r)^{[\lambda_j]+1}}{1-\varrho/r}$$

for $z \in E$, $\|z\| \leqslant \varrho < r$, $t \in T$, $j \geqslant 1$, whence

$$|P_j(z, t)e^{-\varepsilon\lambda_j}| \leqslant H(z, \varepsilon) + (e^{K-A}\varrho/r)^{\lambda_j} \frac{r}{r-\varrho} .$$

Therefore

$$|P_j(z, t)e^{-\varepsilon\lambda_j}| \leqslant H(z, \varepsilon) + 2 < \infty, \quad z \in E, \ \|z\| \leqslant \varrho_1, \ t \in T, \ j \geqslant 1,$$

where $0 < \varrho_1 < \min(r/2, re^{A-K})$. Since E satisfies (L) at $z^0 = 0$, so there are positive numbers M_1 and δ_1 such that

$$|P_j(z, t)| \leqslant M_1 e^{(\lambda_j + [\lambda_j])\varepsilon} \leqslant M_1 e^{2\varepsilon\lambda_j}, \quad \|z\| < \delta_1, \ t \in T, \ j \geqslant 1.$$

Hence in virtue of (2.1) and (2.2)

$$|f_j e^{-A\lambda_j}| \leqslant M_1 e^{2\varepsilon\lambda_j} + (e^{K-A}\varrho_1/r)^{\lambda_j} \frac{r}{r-\varrho_1} \leqslant (M_1+2) e^{2\varepsilon\lambda_j},$$

for $\|z\| < \delta = \min(\varrho_1, \delta_1)$, $t \in T$, $j \geqslant 1$. By the arbitrariness of $\varepsilon > 0$ the proof is concluded.

COROLLARY 2.1 (Hartogs Fundamental Lemma, [2]). *Let* $g_\mu(z) = g_{\mu_1 \ldots \mu_m}(z_1, \ldots, z_n)$ $(\mu_1, \ldots, \mu_m = 0, 1, \ldots)$ *be an m-fold sequence of analytic functions uniformly bounded on every compact subset of an open set $G \subset C^n$. Let*

$$\limsup_{|\mu|\to\infty} \sqrt[|\mu|]{|g_\mu(z)| R^\mu} \leqslant 1, \quad z \in G,$$

where $|\mu| = \mu_1 + \ldots + \mu_m$, $R^\mu = R_1^{\mu_1} \ldots R_m^{\mu_n}$, $R_k = \mathrm{const} > 0$ $(k = 1, \ldots, m)$.

Then for every compact subset Q of G and for every $\varepsilon > 0$ there exists a positive number $M = M(Q, \varepsilon)$ such that

$$|g_\mu R^\mu| \leqslant M e^{|\mu|\varepsilon}, \quad z \in Q, \quad |\mu| \geqslant 0.$$

An extension of the Hartogs lemma (for $n = 1$) of the type given by Theorem 2.1 was first offered (using a little bit different language) by Leja ([15] and [16]). The reasoning used by us to prove Theorem 2.1 is a modification of the reasoning used by Leja in [15] or in [16].

Let us also remark that Theorem 2.1 is very akin to Theorem 10 in Lelong's paper [19].

Proof of Corollary 2.1. Every point of Q belongs to a compact polycilinder contained in G. Then by Polynomial Lemma II we may assume that $Q \in (L)$. We can also assume that $\{g_\mu R^\mu\}$ is uniformly bounded in G. Arrange all the m-tuples $\mu = (\mu_1, \ldots, \mu_m)$ into a sequence $\mu^j = (\mu_{1j}, \ldots, \mu_{mj})$ $(j = 1, 2, \ldots)$ without repetitions. The direct application of Theorem 2.1 to the sequence

$$f_j(z) = g_{\mu^j}(z) R^{\mu^j}, \quad j = 1, 2, \ldots$$

with $\lambda_j = |\mu^j|$, $A = 0$ and $K = \sup_\mu \big(\sup_{z \in G} |g_\mu(z) R^\mu|\big)$ and with E replaced by Q gives the corollary.

§ 3. A version of the Two Constants Theorem for plurisubharmonic functions. Let G be a domain in the space C^n and let F

be a compact subset of G. Denote by $\mathfrak{M} = \mathfrak{M}(G, F)$ the family of all the functions $U(w)$ plurisubharmonic ($=$ plsh.) in G such that

$$(3.1) \qquad U(w) \leqslant 0 \text{ on } F \quad \text{and} \quad U(w) \leqslant 1 \text{ on } G .$$

Put

$$(3.2) \qquad h(w) = h_G(w, F) = \limsup_{w' \to w} \sup \{U(w'):\ U \in \mathfrak{M}\}, \qquad w \in G .$$

The function h is plsh. in G as an upper envelope of a uniformly bounded family of plsh. functions (see [8]). Moreover, if V is an arbitrary plsh. function in G such that $V \leqslant m$ on F and $V \leqslant M$ on G, then

$$(3.3) \qquad V(w) \leqslant m + (M-m)h(w), \qquad w \in G .$$

Indeed, if $m \geqslant M$, then $m + (M-m)h(w) = m\big(1-h(w)\big) + M \geqslant M$ and (3.3) is obvious. If $m < M$, then $(M-m)^{-1}\big(V(w)-m\big)$ is a member of $\mathfrak{M}$, whence (3.3) again follows.

We may treat (3.3) as a version of the Two Constants Theorem for plsh. functions.

EXAMPLE 3.1. Let G be a domain in the complex z-plane and let F be a compact subset of G. Denote by $\hat{F} = \hat{F}_G$ the union of F and of all the components of $G-F$ which are relatively compact in G. Then $h_G(z, F)$ is harmonic in $G - \partial\hat{F}$, and in every component of $G - \partial\hat{F}$ the function h_G is identical with the solution of the Dirichlet problem with boundary values equal to 0 on $\partial\hat{F}$ and to 1 on ∂G.

EXAMPLE 3.2. Let G_k be a domain in the complex w_k-plane regular with respect to the Dirichlet problem. Let F_k be a compact subset of G_k such that $\partial\hat{F}_k \in (L)$ $(k = 1, \ldots, n)$. Put

$$(3.4) \qquad \Omega = \{w \in G_1 \times \ldots \times G_n:\ h(w) < 1\}, \qquad h(w) = h_{G_1}(w_w, F_1) + \ldots + h_{G_n}(w_n, F_n) .$$

Then $h(w) = h_\Omega(w, F)$, where $F = F_1 \times \ldots \times F_n$.

Indeed, the function h is plsh. in the domain Ω, continuous in its closure and $h(w) = 0$ on F, $h(w) = 1$ on $\partial\Omega$. It is enough to show that given a plsh. function U in Ω such that $U(w) \leqslant 0$ on F and $U(w) \leqslant 1$ in Ω we have

$$(3.5) \qquad U(w) \leqslant h(w) \quad \text{in } \Omega .$$

If $n = 1$ inequality (3.5) is obvious. Assume that (3.5) holds in the case of $n-1$ variables and observe that the set

$$E = \bigcup_1^n \{w \in \Omega:\ w_k \in F_k\}$$

is closed in Ω. Moreover, the function h is harmonic in $\Omega - E$. It is obvious that (3.5) holds on $\partial\Omega$. Therefore, by the maximum principle for subharmonic functions, it is sufficient to prove that (3.5) holds on E. Let w^0 be a fixed point of E. Then there exists j $(1 \leqslant j \leqslant n)$ such that $w_j^0 \in F_j$. We may assume that $j = 1$. The function $U_1 = U(w_1^0, w_2, \ldots, w_n)$ is plsh. in

$$\Omega_1 = \{(w_2, \ldots, w_n) \in G_2 \times \ldots \times G_n \colon h_2(w_2) + \ldots + h_n(w_n) < 1\}$$

and $U_1 \leqslant 0$ on $F_2 \times \ldots \times F_n$, $U_1 \leqslant 1$ in Ω_1. By the induction assumption

$$U_1 \leqslant h_2(w_2) + \ldots + h_n(w_n) \quad \text{in } \Omega_1 .$$

In particular, $U(w^0) \leqslant h_1(w_1^0) + \ldots + h_n(w_n^0) = h(w^0)$, because $h_1(w_1^0) = 0$. The proof is concluded.

Observe that the domain Ω given by (3.4) is a union of all the polydomains

$$\{w \in G_1 \times \ldots \times G_n \colon h_{G_k}(w_k, F_k) < \theta_k , \ k = 1, \ldots, n\} ,$$

where $\theta_k > 0$, $\theta_1 + \ldots + \theta_n = 1$.

Condition (A_0). Let G be a domain in C^n and let F be a compact subset of G. We say that the pair (G, F) satisfies the condition (A_0), and write $(G, F) \in (A_0)$, if for every σ $(0 < \sigma < 1)$ the set

$$G_\sigma = \{w \in G \colon h_G(w, F) < \sigma\}$$

is a relatively compact subset of G and $F \subset G_\sigma$.

Condition (A). We say that the pair (G, F) satisfies condition (A) if there is a sequence of domains G_s, $s = 1, 2, \ldots$ such that $G_s \Subset G$ (i.e. G_s is relatively compact in G), $F \subset G_s \subset G_{s+1}$, $(G_s, F) \in (A_0)$ and $G = \bigcup G_s$. For (G, F) satisfying (A) we put

$$H_G(w, F) = \lim_{s \to \infty} h_{G_s}(w, F) , \quad w \in G . \tag{3.6}$$

The function H_G is plsh. in G as a limit of decreasing sequence of plsh. functions.

Remark 3.1. If (G, F) satisfies (A), then G is a domain of holomorphy. In particular, Ω given by (3.4) is a domain of holomorphy.

Indeed, $G_{s\sigma} = \{w \in G_s \colon h_{G_s}(w, F) < \sigma\}$ $(0 < \sigma < 1)$ is a domain of holomorphy by Theorem 13.6 in [8], whence by Behnke-Stein theorem (see [8], p. 122) also G_s is a domain of holomorphy and finally, again by the Behnke-Stein theorem, G is a domain of holomorphy.

Remark 3.2. If G is a plane domain and F is a compact subset of G, then 1° $(G, F) \in (A_0)$ if and only if $(\partial\hat{F} \cup \partial G) \in (L)$, 2° $(G, F) \in (A)$ if and only if $\partial\hat{F} \in (L)$. Here $H_G(z, F) = h_G(z, F)$.

§ 4. Extremal points and extremal functions. Let E be a compact subset of C with the positive transfinite diameter $d(E)$. Let $b(z)$ be a real bounded lower-semicontinuous function defined on E.

Given any system $z^{(n)} = \{z_0, \ldots, z_n\}$ of $n+1$ distinct points of E we put

$$(4.1)\qquad V(z^{(n)}) = \prod_{0\leqslant j<k\leqslant n} |z_k - z_j|\,,$$

$$(4.2)\qquad L^{(j)}(z, z^{(n)}) = \prod_{\substack{k=0\\(k\neq j)}}^{n} \frac{z-z_k}{z_j-z_k}\,,\qquad \Phi^{(j)}(z, z^{(n)}, b) = L^{(j)}(z, z^{(n)})e^{nb(z_j)}\,,$$

$j = 0, \ldots, n$.

Every system

$$\eta^{(n)} = \{\eta_{n0}, \eta_{n1}, \ldots, \eta_{nn}\}\,,\qquad n = 1, 2, \ldots,$$

of $n+1$ points of E such that

$$(4.3)\qquad V(z^{(n)})\exp\Big[-n\sum_{k=0}^{n} b(z_k)\Big] \leqslant V(\eta^{(n)})\exp\Big[-n\sum_{k=0}^{n} b(\eta_{nk})\Big] \quad \text{for } z^{(n)}\subset E\,,$$

is called an *n-th system of extremal points* of E with respect to b.

It is easy to check that (4.3) implies

$$(4.4)\qquad |\Phi^{(j)}(z, \eta^{(n)}, b)| \leqslant \exp\big(nb(z)\big)\,,\qquad z\in E\,,\ j = 0, \ldots, n\,.$$

We shall denote by E^* the set of all the limit points of a fixed triangular sequence $\{\eta_{nj}\}$ ($j = 0, \ldots, n$; $n = 1, 2, \ldots$) of extremal points of E with respect to b.

The extremal function Φ defined in § 1 satisfies the following properties (see [18], [25]):

$$(4.5)\qquad \Phi(z, E, b) = \lim_{n\to\infty}\big[\max_{0\leqslant j\leqslant n} |\Phi^{(j)}(z, \eta^{(n)}, b)|^{1/n}\big]\,,\qquad z\in C\,,$$

the convergence being uniform on every compact subset of $C-E^*$. If Φ is continuous in C, the convergence is uniform on every compact subset of C;

$$(4.6)\qquad \Phi(z, E, b) \leqslant \exp\big(b(z)\big)\,,\qquad z\in E;$$

$$(4.7)\qquad \Phi(z, E, b) = \exp\big(b(z)\big)\,,\qquad z\in E^*;$$

$$(4.8)\qquad \Phi(z, E, b+c) = e^c\Phi(z, E, b)\quad \text{in } C\,,\ \text{if } c = \text{const};$$

$$(4.9)\qquad \Phi(z, E, b_1) \leqslant \Phi(z, E, b_2)\quad \text{in } C\,,\ \text{if } b_1(z)\leqslant b_2(z) \text{ on } E;$$

(4.10) $\operatorname{Log}\Phi$ is harmonic in $C-E^*$ and $\lim_{z\to\infty}\big(\Phi(z)/|z|\big)$ exists;

(4.11) Φ is continuous at every point $a\in E$ at which E satisfies (L);

(4.12) If $p(z)$ is an arbitrary polynomial of degree $\leqslant n$ and $|p(z)| \leqslant M\exp\big(nb(z)\big)$ on E, then $|p(z)|\leqslant M\Phi^n(z,E,b)$ in C.

LEMMA 4.1 ([9]). *Let E be a compact set with $d(E)>0$. Let $a_1,\ldots,a_{k-1}$ be finite points in $C-E$. Let $b(z)=(1/k)\operatorname{Log}|p(z)|$, where $p(z)=(z-a_1)\ldots\ldots(z-a_{k-1})$ (we put $p(z)\equiv 1$, if $k=1$). Then*

$$\Phi(z,E,b)=|p(z)|^{1/k},\qquad z\in E. \tag{4.13}$$

Proof. Observe that $|p(z)|^n\equiv\exp\big(knb(z)\big)$. So $p^n(z)\in\mathcal{F}(E,b)$ and $\deg p^n=kn$. Hence $|p(z)|^{1/k}\leqslant\Phi(z,E,b)$. This inequality along with (4.6) gives (4.13).

Condition (r_0). We say that a bounded plane domain D satisfies condition (r_0) (and write $D\in(r_0)$) if (1) D consists of a finite number of disjoined Jordan curves $\Gamma_0,\ldots,\Gamma_{k-1}$, the interior of Γ_0 containing all the other curves, (2) there exists a positive number r_0 such that for every point $z^0\in\partial D$ there exists an open disc $\Delta=\Delta(a,r)$ with center a and radius $r\geqslant r_0$ such that $\Delta\subset C-D$ and $\bar\Delta\cap\bar D=\{z^0\}$.

LEMMA 4.2. *Let D be a bounded domain satisfying condition (r_0). Let E be a compact subset of D. Let a_j be a fixed point in the interior of Γ_j $(j=1,\ldots,k-1)$. Put*

$$b_\lambda(z)=(1/k)\operatorname{Log}|p(z)|+\lambda b(z),$$

where $p(z)=(z-a_1)\ldots(z-a_{k-1})$ and $b(z)=0$ on E, $b(z)=1$ on ∂D. Then

(i) *there exists a positive number λ_0 such that $\Phi(z,E\cup\partial D,b_\lambda)=\exp b_\lambda(z)$, for $z\in E\cup\partial D$, $0<\lambda<\lambda_0$,*

(ii) *$h_D(z,E)=\big(1/(k\lambda)\big)\log[\Phi^k/|p(z)|]$ in $D-\partial\hat E$, $0<\lambda<\lambda_0$, where $h_D(z,E)$ is the subharmonic function in D defined by* (3.2).

Proof. Ad (i). Assume at first that $E\in(L)$. Put $F=E\cup\partial D$. By Polynomial Lemma II also $F\in(L)$. So $\Phi(z,F,b)$ is continuous in C. By Lemma 4.1

$$\Phi(z,F,b_0)=|p(z)|^{1/k}\quad\text{in } F.$$

By (4.9) and (4.6) $\Phi(z,F,b_0)/\sqrt[k]{|p(z)|}\leqslant\Phi(z,F,b_\lambda)/\sqrt[k]{|p(z)|} =\Phi(z,E,b_\lambda)e^{-b_\lambda(z)}\leqslant 1$ in E. Therefore

$$\Phi(z,F,b_\lambda)\exp\big(-b_\lambda(z)\big)=1\quad\text{in } E.$$

Put $4r = \min\{r_0, \operatorname{dist}(E, \partial D), \min_{1\leqslant j\leqslant k-1} \operatorname{dist}(a_j, \Gamma_j)\}$ and

$$S = \{z \in C\colon \operatorname{dist}(z, \partial D) = r\,,\ z \in C-D\}\,.$$

Then for every $a \in S$ the disc $\Delta(a, r)$ is contained in $C-D$ and $\overline{\Delta(a, r)} \cap \overline{D} = \{z^0\}$, where z^0 depends on a. Since $\operatorname{dist}(E, S) > r$, there is an integer $m > 0$ such that

$$e(r/|z-a|)^m \leqslant 1 \quad \text{for } z \in E \text{ and } a \in S\,.$$

Put

$$\mu = \inf_{z\in S_r} \operatorname{Log}[\Phi(z, F, b_0)/|p(z)|^{1/k}]\,, \qquad S_r = \bigcup_{a\in S} \Delta(a, r/2)\,.$$

We claim that $\mu > 0$. Indeed, $b_0(z) = (1/k)\operatorname{Log}|p(z)|$. So by Lemma 4.1

$$U(z) = \operatorname{Log}[\Phi(z, F, b_0)/|p(z)|^{1/k}] = 0 \quad \text{in } F\,.$$

Moreover, the function U is harmonic in $C-F$ and $\lim U(z) = \infty$ as z tends to ∞ or to a_j $(j = 1, \ldots, k-1)$. Therefore, by the maximum principle, $U(z) > 0$ in $C-\overline{D}$. The set S_r is a compact subset of $C-[\overline{D} \cup \{a_1, \ldots, a_{k-1}, \infty\}]$. So $\mu > 0$.

Let z^0 be a fixed point of ∂D and let a be a point of S such that $|z^0-a| = r$. Put $B_\lambda(z) = (1/k)\operatorname{Log}|p(z)| + \lambda \operatorname{Log}[e(r/|z-a|)^m]$. Then $B_\lambda(z^0) = b_\lambda(z^0)$, $\exp B_\lambda(z) = |p(z)|^{1/k}[e(r/|z-a|)^m]^\lambda$ and $|p(z)|^{1/k}[e(r/R)^m]^\lambda \leqslant \exp B_\lambda(z) \leqslant \exp b_\lambda(z)$, $z \in F$, $R = \max\{|z-a|\colon z \in F\,,\ a \in S\}$. Therefore by (4.8) and (4.9)

$$(4.14)\qquad \Phi(z, F, b_0)[e(r/R)^m]^\lambda \leqslant \Phi(z, F, B_\lambda) \leqslant \Phi(z, F, b_\lambda)\,, \qquad z \in C\,.$$

Since $F \in (L)$, the function $H_\lambda(z) = \operatorname{Log}\Phi(z, F, B_\lambda) - B_\lambda(z)$ is harmonic in $G = C-[F^* \cup \{a_1, \ldots, a_{k-1}, \infty\} \cup \Delta(a, r/2)]$ and continuous everywhere except the points $a_1, \ldots, a_{k-1}, \infty, a$. By (4.7) we have $H_\lambda(z) = 0$ in F^*. Further, $\lim H_\lambda(z) = +\infty$ as z tends to ∞ or to a_j $(j = 1, \ldots, k-1)$. Let z belong to the boundary of $\Delta(a, r/2)$. Then by (4.14)

$$\begin{aligned} H_\lambda(z) &\geqslant \operatorname{Log}[\Phi(z, F, b_0)/|p(z)|^{1/k}] + \lambda \operatorname{Log}[e(r/R)^m] - \lambda \operatorname{Log}[e(r/|z-a|)^m] \\ &\geqslant \mu + \lambda \operatorname{Log}[e(r/R)^m] - \lambda \operatorname{Log}(2^m e) = \mu + m\lambda \operatorname{Log}(r/2R)\,. \end{aligned}$$

So $H_\lambda(z) \geqslant 0$ on $\partial\Delta(a, r/2)$ if

$$(4.15)\qquad 0 < \lambda \leqslant \mu/[m \operatorname{Log}(2R/r)] = \lambda_0\,.$$

Therefore, by the maximum principle, $H_\lambda(z) \geqslant 0$ in G for every λ satisfying (4.15). But $(F-F^*) \subset G$, so $H_\lambda(z) \geqslant 0$ on F. By (4.6) $H_\lambda(z) \leqslant 0$ on F. This implies that $H_\lambda(z^0) = 0$, i.e. $\Phi(z^0, F, B_\lambda) = \exp B_\lambda(z^0) = \exp b_\lambda(z^0)$. By the second inequality in (4.14) and in view of (4.6) we have $\Phi(z^0, F, b_\lambda) = \exp b_\lambda(z^0)$. By the arbitrariness of $z^0 \in \partial D$ we have proved (i) under the additional assumption that $E \in (L)$.

Suppose now E is an arbitrary compact subset of D. Take $\delta > 0$ so small that $E_\delta = \bigcup_{a \in E} \{z: |z-a| \leqslant \delta\}$ is contained in D. Then $E_\delta \in (L)$. So $\Phi(z, E_\delta \cup \partial D, b_\lambda) = \exp b_\lambda(z)$, $z \in E_\delta \cup \partial D$, for every sufficiently small positive λ. Hence by (4.6)

$$\exp b_\lambda(z) = \Phi(z, E_\delta \cup \partial D, b_\lambda) \leqslant \Phi(z, E \cup \partial D, b_\lambda) \leqslant \exp b_\lambda(z) \quad \text{on } E \cup \partial D ,$$

whence the result follows.

Ad (ii). It follows from the maximum principle that $\Phi(z, E \cup \partial D, b_\lambda) = \Phi(z, \partial\hat{E} \cup \partial D, b_\lambda)$. By (4.10), (4.11) and by (i) the function $U(z) = \frac{1}{\lambda} \operatorname{Log}[\Phi^k(z, E \cup \partial D, b_\lambda)/|p(z)|]$ is continuous in D, harmonic in $D - \partial\hat{E}$, equal to 1 on ∂D and to 0 on E. By (4.11) the function U is continuous at every point $z \in \partial\hat{E}$ at which $\partial\hat{E}$ satisfies (L). Since the set

$$\{z \in \partial\hat{E}: \partial\hat{E} \text{ does not satisfy } (L) \text{ at } z\}$$

is polar (see Theorem 1.2.a and [3], chapters III and VII), so

$$U^*(z) = \limsup_{a \to z} U(a) = h_D(z, E) , \quad z \in D .$$

The proof of (ii) is concluded.

Put

$$E_\sigma = \{z \in D: h_D(z, E) = \sigma\} , \tag{4.16}$$

$$D_\sigma = \{z \in D: h_D(z, E) < \sigma\} . \tag{4.17}$$

If $\partial\hat{E} \in (L)$, then for every σ $(0 < \sigma < 1)$ the set E_σ is a union of finitely many Jordan curves and E_σ is a boundary of D_σ.

If E is of positive transfinite diameter, not necessarily satisfying (L), then there exists σ_0 $(0 < \sigma_0 < 1)$ such that E_σ is a union of finitely many piecewise analytic Jordan curves bounding D_σ for every $\sigma \in (\sigma_0, 1)$.

The last two lemmas and property (4.5) of the extremal function Φ imply

THEOREM 4.1. *Assume that:* (1) *D is a plane domain satisfying condition* (r_0), (2) *E is a compact subset of D,* (3) $b_\lambda(z) = (1/k) \operatorname{Log}|p(z)| + \lambda b(z)$, *where* $\lambda > 0$, $b(z) = 0$ *on* E, $b(z) = 1$ *on* ∂D *and* $p(z) = (z-a_1)\ldots(z-a_{k-1})$ (*we put* $p(z) \equiv 1$ *if* $k = 1$), a_j $(j = 1, \ldots, k-1)$ *being a fixed point in the interior of* Γ_j, (4) $\eta^{(k\nu)} = \{\eta_0, \ldots, \eta_{k\nu}\}$ *is a* $(k\nu)$*-the extremal points system of* $E \cup \partial D$ *with respect to* b_λ.

Then there exists $\lambda > 0$ *such that*

$$\lim_{\nu \to \infty} \operatorname{Log}[\max_{0 \leqslant j \leqslant k\nu} |\Phi^{(j)}(z, \eta^{(k\nu)}, b_\lambda)|^{1/\nu}/|p(z)|] = k\lambda h_D(z, E) , \quad z \in D - \partial\hat{E} , \tag{4.18}$$

the convergence being uniform on every compact subset of $D - \partial\hat{E}$. *If, moreover,* $\partial E \in (L)$, *then* (4.18) *holds uniformly in* D.

§ 5. Interpolation of separately analytic functions in extremal points. The main result of this section is given by the following

APPROXIMATION LEMMA. *Let D be a k-connected domain in the complex z-plane satisfying (r_0). Let E be a compact subset of D with $d(E) > 0$. Let U be an open set in C^n and let F be a compact subset of U. Suppose $f(z, w) = f(z, w_1, \ldots, w_n)$ is a function defined and separately analytic in $X = (D \times F) \cup (E \times U)$.*

Then there exists a positive number λ (depending only on D and E) and a sequence $\{Q_\nu(z, w)\}$ of analytic functions in $D \times U$ such that:

(a) *The series $\sum Q_\nu$ converges to f in $D \times F$;*

(b) *For every subdomain G of U such that $F \subset G$ and $|f| \leqslant M =$ const in $(E \times G) \cup (D \times F)$ the following inequalities are satisfied*

$$|Q_\nu| \leqslant 2(k\nu+1)Mc \exp\big(-k\lambda\nu[\sigma-\varepsilon-\tau-(\sigma-\varepsilon)h_G(w, F)]\big),$$
$$z \in D_\tau, \quad w \in G, \quad \nu \geqslant \nu_0 = \nu_0(\varepsilon, \sigma),$$

where 1° $c = c(\sigma, \tau)$ depends on σ and τ but not on ν nor on (z, w), 2° $\varepsilon, \sigma, \tau$ are arbitrary real numbers satisfying the conditions $\varepsilon > 0$, $0 \leqslant \sigma_0 < \tau < \sigma < 1$, 3° σ_0 is the smallest number with the property that $0 \leqslant \sigma_0 < 1$ and for every σ $(\sigma_0 < \sigma < 1)$ the set $E_\sigma = \{z \in D\colon h_D(z, E) = \sigma\}$ is a compact subset of $D - E$;

(c) *If $F \in (L)$ and f is bounded in $E \times U$, then there is an open neighborhood V of $D \times F$ such that the series $\sum Q_\nu$ is uniformly convergent on every compact subset of V.*

Observe that (a) and (c) give an analytical continuation of f into V.

Proof. Let Γ_j $(j = 1, \ldots, k-1)$ be the components of ∂D. Let $p(z) = (z-a_1)\ldots(z-a_{k-1})$, $b_\lambda(z)$ and $\eta^{(k\nu)} = \{\eta_0, \ldots, \eta_{k\nu}\}$ $(\nu = 1, 2, \ldots)$ have the same meaning as in Theorem 4.1. Let $\lambda > 0$ be so small that (4.18) is satisfied.

Enumerate the points of $\eta^{(k\nu)}$ in such a way that $\eta_0, \ldots, \eta_{l_\nu} \in E$ and the remaining points of $\eta^{(k\nu)}$ lie in ∂D. Put

$$f_\nu(z, w) = \sum_{j=0}^{l_\nu} f(\eta_j, w) L^{(j)}(z, \eta^{(k\nu)})[p(\eta_j)]^\nu [p(z)]^{-\nu}, \tag{5.1}$$
$$z \in D, \quad w \in U, \quad \nu \geqslant 1.$$

We shall prove that the sequence $\{Q_\nu\}$ defined by

$$Q_1 = f_1, \quad Q_\nu = f_\nu - f_{\nu-1}, \quad \nu = 2, 3, \ldots \tag{5.2}$$

satisfies all the required properties.

Put

$$r_\nu(z) = (z-\eta_0)\dots(z-\eta_{k\nu})[p(z)]^{-\nu}, \qquad \nu = 1, 2, \dots$$

and let σ_0 be the smallest number such that for every σ ($\sigma_0 < \sigma < 1$) the set D_σ given by (4.17) contains E in its interior. Let us orientate E_σ ($\sigma_0 < \sigma < 1$), defined by (4.16), positively with respect to the interior of D_σ. Then by the residue theorem (comp. with [29], p. 186)

$$(5.3) \qquad f_\nu(z, w) = \frac{1}{2\pi i}\int_{E_\sigma} \frac{r_\nu(\zeta)-r_\nu(z)}{r_\nu(\zeta)}\frac{f(\zeta, w)}{\zeta - z}d\zeta, \qquad z \in D_\sigma,\ w \in F,$$

and

$$(5.4) \qquad f(z, w)-f_\nu(z, w) = \frac{1}{2\pi i}\int_{E_\sigma} \frac{r_\nu(z)}{r_\nu(\zeta)}\frac{f(\zeta, w)}{\zeta - z}d\zeta, \qquad z \in D_\sigma,\ w \in F.$$

Observe that

$$\frac{|r_\nu(z)|}{|r_\nu(\zeta)|} = \frac{|\Phi^{(j)}(z)|}{|\Phi^{(j)}(\zeta)|}\frac{|z-\eta_j|}{|\zeta-\eta_j|}\frac{|p(\zeta)|^\nu}{|p(z)|^\nu}, \qquad j = 0, 1, \dots, k\nu,$$

where $\Phi^{(j)}(z) = \Phi^{(j)}(z, \eta^{(k\nu)}, b_\lambda)$ is defined in accordance with (4.2).

By (4.4), (4.12) and by Theorem 4.1 we have

$$|p(z)|^{-\nu}|\Phi^{(j)}(z)| \leqslant \exp\big(k\lambda\nu h_D(z, E)\big), \qquad z \in D,\ j = 0, \dots, k\nu,\ \nu \geqslant 1.$$

Again by Theorem 4.1, given $\varepsilon > 0$ and σ ($\sigma_0 < \sigma < 1$), we have

$$\max_{0\leqslant j\leqslant k\nu} |\Phi^{(j)}(\zeta)|\,|p(\zeta)|^{-1} \geqslant e^{\nu(k\lambda\sigma-\varepsilon)}, \qquad \zeta \in E_\sigma,\ \nu \geqslant \nu_0 = \nu_0(\varepsilon, \sigma).$$

Therefore for $z \in D_\tau$ ($\sigma_0 < \tau < \sigma < 1$), $w \in F$ and $\nu \geqslant \nu_0 = \nu_0(\varepsilon, \sigma)$ we have

$$(5.5) \qquad |f-f_\nu| \leqslant M_\sigma(w)|D|\Lambda(E_\sigma)\varrho(E, E_\sigma)\varrho(D_\sigma, E_\tau)\exp\big(\nu[k\lambda\tau-(k\lambda\sigma-\varepsilon)]\big),$$

where $M_\sigma(w) = \sup\{|f(\zeta, w)|:\ \zeta \in E_\sigma\}$, $|D| = \sup\{|a-b|:\ a, b \in D\}$, $\Lambda(E_\sigma) = \int_{E_\sigma}|d\zeta|$, $\varrho(E, E_\sigma) = \sup\{|a-b|^{-1}:\ a \in E, b \in E_\sigma\}$, $\varrho(D_\sigma, E_\tau) = \sup\{|a-b|^{-1}:\ a \in E_\tau, b \in D_\sigma\}$. The number $\varepsilon > 0$ being arbitrary we may substitute $k\lambda\varepsilon$ for ε in (5.5). So we have proved that given $\varepsilon > 0$, σ and τ ($\sigma_0 < \tau < \sigma < 1$) there exist positive numbers $c_1 = c_1(\sigma, \tau)$ and $\nu_0 = \nu_0(\varepsilon, \sigma)$ such that

$$(5.5') \qquad |f-f_\nu| \leqslant M_\sigma(w)c_1(\sigma, \tau)e^{-k\lambda\nu(-\tau+\sigma-\varepsilon)}, \qquad z \in D_\tau,\ w \in F,\ \nu \geqslant \nu_0.$$

Given τ ($\sigma_0 < \tau < 1$) we may chose σ and ε in such a way that $\sigma_0 < \tau < \sigma < 1$ and $\sigma-\tau-\varepsilon > 0$, whence it follows that $f_\nu \to f$ in $D \times F$.

The proof of (a) is concluded.

We proceed to the proof of (b). Observe that (5.5′) implies

$$(5.6) \qquad |Q_\nu| \leqslant M_\sigma(w)\cdot c(\sigma, \tau)e^{-k\lambda\nu(-\tau+\sigma-\varepsilon)}, \qquad z \in D_\tau,\ w \in F,\ \nu \geqslant \nu_0,$$

where $\sigma_0 < \tau < \sigma < 1$, $\varepsilon > 0$, $c(\sigma, \tau) = c_1(\sigma, \tau)(1+e^{k\lambda})$.

Next observe that

$$|L^{(j)}(z,\eta^{(k\nu)})[p(\eta_j)]^{\nu}| = |\Phi^{(j)}(z,\eta^{(k\nu)},b_\lambda)|\,, \quad j=0,1,\ldots,l_\nu\,,$$

whence, after taking into account (5.1), (4.4) and (4.12), we get

$$|f_\nu| \leqslant M_0(w)(k\nu+1)\Phi^{k\nu}|p(z)|^{-\nu}\,, \quad z\in D\,,\ w\in U\,,\ \nu\geqslant 1\,,$$

where $M_0(w) = \sup\{|f(z,w)|:\ z\in E\}$. So, by Theorem 4.1,

$$|f_\nu| \leqslant M_0(w)(k\nu+1)e^{k\lambda\tau\nu}\,, \quad z\in D_\tau\,,\ w\in U\,,\ \nu\geqslant 1\,.$$

Hence

$$(5.7)\qquad |Q_\nu| \leqslant M_0(w)(k\nu+1)(1+e^{-k\lambda\tau})e^{k\lambda\tau\nu}\,, \quad z\in D_\tau\,,\ w\in U\,,\ \nu\geqslant 1\,.$$

Let G be an arbitrary subdomain of U such that $F\in G$, $(G,F)\in(A_0)$ and $|f|\leqslant M = \text{const}$ in $(D\times F)\cup(E\times G)$. Then (5.7) and (5.6) may be written in the form

$$(5.8)\qquad |Q_\nu| \leqslant 2M(k\nu+1)e^{k\lambda\tau\nu}\,, \quad z\in D_\tau\,,\ w\in G\,,\ \nu\geqslant 1\,,$$

$$(5.9)\qquad |Q_\nu| \leqslant Mc(\sigma,\tau)\exp[-k\lambda\nu(\sigma-\varepsilon-\tau)]\,, \quad z\in D_\tau\,,\ w\in F\,,\ \nu\geqslant\nu_0\,.$$

Without loss of generality we may assume that $c>1$, $M>1$. Given any fixed $z\in D_\tau$ and $\nu\geqslant\nu_0$, the plsh. function $V(w) = \mathrm{Log}[|Q_\nu|e^{-k\lambda\tau\nu}/(k\nu+1)]$ is bounded by $\mathrm{Log}(2M)$ in G and by $-k\lambda\nu(\sigma-\varepsilon)+\mathrm{Log}(Mc)$ in F. Hence by the Two Constants Theorem (after obvious transformations) we get

$$|Q_\nu| \leqslant 2cM(k\nu+1)e^{-k\lambda\nu[(\sigma-\varepsilon)-\tau-(\sigma-\varepsilon)h_G(w,F)]}\,, \quad z\in D_\tau\,,\ w\in G\,.$$

This concludes the proof of (b).

To show (c) put $g_\nu(z,w) = Q_\nu(z,w)e^{-k\lambda\tau\nu}/(k\nu+1)$ and observe that by (5.6)

$$\limsup_{\nu\to\infty}\sqrt[\nu]{\max_{z\in D_\tau}|g_\nu(z,w)|} \leqslant e^{-k\lambda}\,, \quad w\in F\,.$$

Next, by (5.7) the sequence $\{g_\nu\}$ is uniformly bounded in $D_\tau\times U$. Hence by Theorem 2.1 for every $\varepsilon>0$ there is an open neighborhood V_ε of F and there is a positive number M_ε (depending also on τ) such that

$$|Q_\nu| \leqslant M_\varepsilon(k\nu+1)e^{-k\lambda\nu(1-\tau-\varepsilon)}\,, \quad z\in D_\tau\,,\ w\in V_\varepsilon\,,\ \nu\geqslant 1\,.$$

Given τ $(\sigma_0<\tau<1)$ we may take $\varepsilon>0$ so small that $1-\tau-\varepsilon>0$. Hence the series $\sum Q_\nu$ is uniformly convergent in $D_\tau\times V_\varepsilon$. By the arbitrariness of τ this concludes the proof of (c).

Remark. Inequality (5.5) may be treated as a contribution to the theory of interpolation and approximation by rational functions presented in Chapter IX of [29].

Since every plane domain may be approximated by an increasing sequence of relatively compact subdomains satisfying (r_0), so (c) implies the following

COROLLARY 5.1. *Let D be a domain in the complex z-plane and let U be an open set in the space C^n of variables $w = (w_1, \ldots, w_n)$. Let E and F be compact subsets of D and G, respectively, such that $d(E) > 0$ and $F \in (L)$. Then every function f which is separately analytic in $X = (D \times F) \cup (E \times U)$ and bounded on every compact subset of X may be continued (uniquely) to a function $\tilde{f}$ analytic in a neighborhood V of $D \times F$.*

Indeed, let $\{D_j\}$ be an increasing sequence of relatively compact subdomains of D such that $D_j \in (r_0)$ and $D = \bigcup_1^\infty D_j$. By (c) there is a function f_j analytic in $D_j \times V_j$, where V_j is a neighborhood of F and $f_j = f$ in $D_j \times F$. We may assume that $V_{j+1} \subset V_j$ and that every component of V_j $(j = 1, 2, \ldots)$ contains a point of F. The set

$$V = \bigcup_1^\infty (D_j \times V_j)$$

is a neighborhood of $D \times F$. Given $(z, w) \in V$ we define $\tilde{f}(z, w)$ by $\tilde{f}(z, w) = f_j(z, w)$, where j is an arbitrary integer such that $(z, w) \in D_j \times V_j$. To be sure that this definition is correct we have to show that $f_p(z, w) = f_q(z, w)$ if $(z, w) \in (D_p \times V_p) \cap (D_q \times V_q)$ $(p \neq q)$. We may assume that $p < q$. Then the functions f_p and f_q are both analytic in $D_p \times V_q$ and $f_p = f_q$ in $D_p \times F$. Hence by Remark 1.1 $f_p = f_q$ in $D_p \times V_q$. By the same reasoning if $\tilde{f}_1$ and $\tilde{f}_2$ are analytic functions in V and $\tilde{f}_1 = \tilde{f}_2$ in $D \times F$, then $\tilde{f}_1 \equiv \tilde{f}_2$. Hence the continuation $\tilde{f}$ of f is unique.

§ 6. Locally bounded separately analytic functions. We shall start with the following

LEMMA 6.1. *Let D be a plane domain satisfying (r_0) and let E be a compact subset of D such that $\partial\hat{E} \in (L)$. Let G be a domain in C^n, let F be a compact subset of G and let $(G, F) \in (A_0)$. Finally, let $f(z, w) = f(z, w_1, \ldots, w_n)$ be a separately analytic function in $X = (D \times F) \cup (E \times G)$ such that*

$$\text{(i)} \qquad |f| \leqslant M \quad \text{in } X.$$

Then 1° *f is continuable to an analytic function $\tilde{f}$ in*

$$\Omega = \{(z, w) \in D \times G\colon h_D(z, E) + h_G(w, F) < 1\},$$

2° $|\tilde{f}| \leqslant M$ *in Ω.*

Proof. Ad 1°. By the Approximation Lemma there exists a sequence $\{Q_\nu\}$ of analytic functions in $D \times G$ such that

$$f(z, w) = \sum Q_\nu(z, w) \quad \text{in } D \times F$$

and

$$|Q_\nu| \leqslant Mc(\sigma, \tau)(k\nu+1)e^{-k\lambda\nu[\sigma-\varepsilon-\tau-(\sigma-\varepsilon)h_G(w,F)]}, \tag{6.1}$$

$$z \in D_\tau, \quad w \in G, \quad \nu \geqslant \nu_0,$$

where $\varepsilon > 0$, $\sigma_0 < \tau < \sigma < 1$. Since $\partial E \in (L)$, we have $\sigma_0 = 0$. Take an arbitrary τ $(0 < \tau < 1)$ and θ $(0 < \theta < 1)$ such that $\tau + \theta < 1$. Then there exist $\varepsilon > 0$ and σ $(\tau < \sigma < 1)$ such that $\sigma - \varepsilon - \tau - (\sigma - \varepsilon)\theta > 0$. This implies that the series $\sum Q_\nu$ is uniformly convergent in $D_\tau \times G_\theta$, where $D_\tau = \{z \in D: h_D(z, E) < \tau\}$ and $G_\theta = \{w \in G: h_G(w, F) < \theta\}$. By the arbitrariness of τ and θ the function $\tilde{f} = \sum Q_\nu$ gives the analytical continuation of f into Ω.

Ad 2°. Suppose inequality 2° is not true. Then there exists $(a, b) \in \Omega$ such that $M_0 = |\tilde{f}(a, b)| > M$. So the function $g(z, w) = 1/[f(z, w) - \tilde{f}(a, b)]$ satisfies all the assumptions of Lemma 6.1 with (i) replaced by $|g| \leqslant 1/(M_0 - M)$ in X. By 1° the function g is continuable to an analytic function $\tilde{g}$ in Ω. The point (a, b) does not belong to X, in particular, $a \notin E$. There is τ $(0 < \tau < 1)$ such that $g(z, b)$ is analytic for $z \in D_\tau$ and $g(z, b) = 1/[f(z, b) - \tilde{f}(a, b)]$ for $z \in E$. By the principle of analytical continuation $g(z, b) = 1/[\tilde{f}(z, b) - \tilde{f}(a, b)]$ for $z \in D_\tau$. This equation cannot hold for $z = a$, because the right-hand side function is not analytic for $z = a$. This contradiction ends the proof of 2°.

THEOREM 6.1. *Let D be an arbitrary domain in the complex z-plane and let E be a compact subset of D such that $\partial E \in (L)$. Let G be a domain in the space C^n of n complex variables $w = (w_1, \ldots, w_n)$. Let F be a compact subset of G and let $(G, F) \in (A)$. Let $f(z, w)$ be defined, locally bounded and separately analytic in $X = (D \times F) \cup (E \times G)$.*

Then f is continuable to an analytic function $\tilde{f}$ in

$$\Omega = \{(z, w) \in D \times G: H_D(z, E) + H_G(w, F) < 1\}.$$

The domain Ω is the envelope of holomorphy of X.

Proof. Let $\{D_j\}$ be a sequence of relatively compact subdomains of D such that $E \subset D_j \subset D_{j+1}$, $D = \bigcup_1^\infty D_j$ and D_j satisfies (r_0). Let $\{G_j\}$ be a sequence of relatively compact subdomains of G such that $F \subset G_j \subset G_{j+1}$, $G = \bigcup_1^\infty G_j$ and $(G_j, F) \in (A_0)$. By Lemma 6.1 the function f is continuable to an analytic function f_j in

$$\Omega_j = \{(z, w) \in D_j \times G_j: h_{D_j}(z, E) + h_{G_j}(w, F) < 1\}, \quad j = 1, 2, \ldots$$

Since the function $V_j(z, w) = h_{D_j}(z, E) + h_{G_j}(w, F)$ is plsh. in Ω_j, $V_j < 1$ in Ω_j and $\lim\limits_{(z,w)\to(a,b)} V_j(z, w) = 1$ for $(a, b) \in \partial\Omega_j$, so Ω_j is a domain of holomorphy (see [8]). But $\Omega = \lim \Omega_j$, so by Behnke-Stein Theorem Ω is also a domain of holomorphy. The function $\tilde{f}(z, w) = \lim f_j(z, w)$, $(z, w) \in \Omega$, gives the required continuation.

COROLLARY 6.1. *Let the assumptions of Theorem 6.1 be satisfied. If $H_D(z, E) = 0$ in D or $H_G(w, F) = 0$ in G, then f is continuable to an analytic function $\tilde{f}$ in $D \times G$ and $D \times G$ is the envelope of holomorphy of X.*

§ 7. The assumption of the local boundedness is superfluous. To begin with we shall prove the following

LEMMA 7.1. *Let D be a domain in the z-plane. Let E be a compact subset of D with $d(E) > 0$. Let F be a compact subset of the $w = (w_1, \ldots, w_n)$-space with $F \in (L)$. Let U be an open neighborhood of F. Let f be a separately analytic function in $X = (D \times F) \cup (E \times U)$.*

Then f is continuable to an analytic function in a neighborhood of $D \times F$. In particular, f is locally bounded on $D \times F$.

Proof. Let G be an open neighborhood of F relatively compact in U such that each component of G intersects F. For every $j = 1, 2, \ldots$ define E_j by

$$E_j = \{z \in E\colon \sup_{w \in G} |f(z, w)| \leqslant j\}.$$

Then $E_j \subset E_{j+1}$, $E = \bigcup E_j$. We claim that E_j is closed. Indeed, let $z_k \in E_j$ $(k = 1, 2, \ldots)$, $z_0 = \lim_{k\to\infty} z_k$ and let ω be a fixed component of G. Then $|f(z_k, w)| \leqslant j$, $w \in \omega$, $k = 1, 2, \ldots$ By the Montel theorem on normal families of analytic functions there exists a subsequence $f(z_{k_s}, w)$, $s = 1, 2, \ldots$ converging uniformly on every compact subset of ω to an analytic function $g(w)$. The function $f_1(w) = g(w) - f(z_0, w)$ is analytic in ω and $f_1(w) = 0$ in $F \cap \omega$. By Remark 1.1 $f_1(w) = 0$ in ω. By the arbitrariness of ω we have $f_1(w) = 0$ in G, i.e. $\lim_{k\to\infty} f(z_k, w) = f(z_0, w)$ in G. Hence $z_0 \in E_j$.

We have proved that E_j $(j = 1, 2, \ldots)$ is closed. Since $E = \bigcup_1^\infty E_j$ and $d(E) > 0$, there exists j_0 such that $d(E_{j_0}) > 0$. Put $E_0 = E_{j_0}$. We have

$$|f(z, w)| \leqslant M = j_0, \quad z \in E_0, \ w \in G. \tag{7.1}$$

So, by Corollary 5.1, the function f is continuable to an analytic function in an open neighborhood of $D \times F$.

COROLLARY 7.1. *Let D, E and U be the same as in Lemma 7.1. Let $f(z, w)$ be defined in $D \times U$ and let*

(i) *for every fixed $a \in E$ the function $f(a, w)$ be analytic in G,*

(ii) *for every fixed $b \in U$ the function $f(z, b)$ be analytic in D.*

Then f is analytic in $D \times U$.

Indeed, given any fixed point $b \in G$ let F denote a closed polycilinder $F = \{w \in C^n\colon |w_k - b_k| \leqslant r_k,\ k = 1, \dots, n\}$ contained in U. By Lemma 7.1 the function f is analytic in $D \times F$. Hence, by the arbitrariness of $b \in U$ the function f is analytic in $D \times U$.

This corollary is equivalent to Proposition 1.1 of [28]. On the other hand, it follows from [28] that in Lemma 7.1 one cannot drop the assumption that $d(E) > 0$.

THEOREM 7.1. *Let D_k be a domain in the complex z_k-plane ($k = 1, \dots, n$). Let E_k be a compact subset of D_k such that $\partial \hat{E}_k \in (L)$. Let f be defined in*

$$(\ast) \qquad X = (D_1 \times E_2 \times \dots \times E_n) \cup \dots \cup (E_1 \times \dots \times E_{n-1} \times D_n)$$

and seperately analytic in X, i.e. for each fixed $(a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n) \in (E_1 \times \dots \times E_{k-1} \times E_{k+1} \times \dots \times E_n)$ the function $f(a_1, \dots, a_{k-1}, z_k, a_{k+1}, \dots, a_n)$ is analytic in D_k ($k = 1, \dots, n$).

Then

1° *f is continuable to an analytic function $\tilde{f}$ in*

$$(7.2) \qquad \Omega = \{z \in D_1 \times \dots \times D_n\colon h_{D_1}(z_1, E_1) + \dots + h_{D_n}(z_n, E_n) < 1\},$$

2° *Ω is the envelope of holomorphy of X.*

Proof. (Induction with respect to n). If $n = 1$, Theorem 7.1 is obviously true. Suppose it is true in the case of n variables. We shall prove that it is also true for $n+1$ variables. Indeed, by the induction assumption there exists a function f_1 defined in

$$X_1 = (E_1 \times G) \cup (D_1 \times F),$$

where $G = \{w \in D_2 \times \dots \times D_{n+1}\colon h_{D_2}(w_2, E_2) + \dots + h_{D_{n+1}}(w_{n+1}, E_{n+1}) < 1\}$ and $F = E_2 \times \dots \times E_{n+1}$, such that $f_1 = f$ in $D_1 \times F$ and f_1 is separately analytic in X_1. In view of Theorem 1.1 we have $F \in (L)$. Hence, by Lemma 7.1 the function f_1 is locally bounded in $D_1 \times F$ and, consequently, the function f is bounded on every compact subset of $D_1 \times F = D_1 \times E_2 \times \dots \times E_{n+1}$. Changing the numeration of the variables we are derived to the conclusion that f is locally bounded in X.

We claim that f_1 is locally bounded in X_1. We have only to prove that f_1 is locally bounded in $E_1 \times G$. Let D_{0k} ($k = 2, \dots, n$) be a relatively compact subdomain of D_k satisfying (r_0) and containing E_k in its interior. Put

$$X_0 = (D_{02} \times E_3 \times \dots \times E_{n+1}) \times \dots \times (E_2 \times \dots \times E_n \times D_{0,n+1}).$$

Then

$$|f| \leqslant M = \text{const} \quad \text{in } E_1 \times X_0,$$

M depending on D_{0k} $(k = 2, \dots, n+1)$. For every $a_1 \in E_1$ the function $f_1(a_1, w)$ is an analytic continuation of $f(a_1, w)$ into

$$G_0 = \{w \in D_{02} \times \dots \times D_{0,n+1}\colon\ h_{D_{02}}(w_2, E_2) + \dots + h_{D_{0,n+1}}(w_{n+1}, E_{n+1}) < 1\}\,.$$

By the standard reasoning used to prove 2° of Lemma 6.1 we conclude that $|f_1(a_1, w)| < M$ in G_0. By the arbitrariness of D_{0k} $(k = 2, \dots, n+1)$ and of $a_1 \in E_1$ we get the local boundedness of f_1 in $E_1 \times G$. Hence, by virtue of Example 3.2 and of Theorem 6.1, the function f_1 is continuable to an analytic function $\tilde{f}_1$ in Ω. But $\tilde{f}_1 = f_1 = f$ in $E_1 \times \dots \times E_{n+1}$, so the function $\tilde{f}_1$ represents the analytic continuation of f into Ω.

By 1° and by Remark 3.1 the domain Ω is the envelope of holomorphy of X. Theorem 7.1 is proved.

Observe that 2° of the last theorem extends a result contained in [2] concerning the analytical extension of a union of two confocal elliptical polycilinders.

In [27] Theorem 7.1 has been proved under the assumption that E_k is a line segment in D_k and D_k is symmetric with respect to the line in which E_k is contained. In [27] we used a theorem on expansion of functions analytic in a line interval into a series of Chebyshev polynomials instead of the Approximation Lemma.

COROLLARY 7.2. *Let the assumptions of Theorem 7.1 be satisfied. Moreover, let $h_{D_k}(z_k, E_k) = 0$ (e.g. $D_k = C$) $(k = 2, \dots, n)$. Then the function f may be continued to an analytic function in $D_1 \times \dots \times D_n$. In particular, $D_1 \times \dots \times D_n$ is the envelope of holomorphy of X.*

We shall now prove that the assumptions of Corollary 7.2 may be weakened. We shall first prove three lemmas.

LEMMA 7.2. *Let $G = G_1 \times \dots \times G_n$, where G_j is a domain in the w_j-plane. Let $F = F_1 \times \dots \times F_n$, where F_j is a compact subset of G_j with $d(F_j) > 0$. Put $h(w) = h_{G_1}(w_1, F_1) + \dots + h_{G_n}(w_n, F_n)$, $w \in G$.*

If $U(w)$ is a plsh. function in G such that $U \leqslant m$ on F and $U \leqslant M$ in G, then

$$U(w) \leqslant m + (M - m)h(w)\,, \qquad w \in G\,.$$

This lemma may be easily proved by a reasoning quite analogous to the reasoning used in the proof of (3.5).

LEMMA 7.3. *Assume that:*

(a) *D is a domain in the z-plane;*

(b) *E is a compact subset of D and $d(E) > 0$;*

(c) *$G = G_1 \times \dots \times G_n$, where G_j is a domain in the w_j-plane;*

(d) *$F = F_1 \times \dots \times F_n$, where F_j is a compact subset of G_j and $d(F_j) > 0$;*

(e) *$f(z, w)$ is defined and separately analytic in $X = (D \times F) \cup (E \times G)$.*

Then for every relatively compact subdomain D_0 of D there are compact subsets E_0 of E and F_{j0} of F_j $(j = 1, \dots, n)$ such that 1° $d(E_0) > 0$ *and* $d(F_{j0}) > 0$ $(j = 1, \dots, n)$, 2° *for every relatively compact subdomain ω of G there is a constant M such that*

$$|f| \leqslant M \quad in\ (D_0 \times F_0) \cup (E_0 \times \omega) \qquad (F_0 = F_{10} \times F_{20} \times \dots \times F_{n0}).$$

Proof. Let $\{G^s\}$ be a sequence of relatively compact subdomains of G such that $F \subset G^s \subset G^{s+1}$, $G = \bigcup_1^\infty G^s$. We shall first prove that there exists a sequence $\{E_s\}$ of compact subsets of E such that

1° $E = E_1 \supset E_2 \supset \dots$, $d(E_j) > d(E_{j-1}) \exp(2^{-j+1})$, $j = 2, 3, \dots$,

2° $|f(z, w)| \leqslant M_s = \text{const}$, $(z, w) \in E_s \times G^s$, $s = 2, 3, \dots$

Put $E_1 = E$. Suppose $E_1, \dots, E_{k-1}$ are closed subsets of E already defined in such a way that $E_1 \supset E_2 \supset \dots \supset E_{k-1}$, $d(E_j) > d(E_{j-1}) \exp(-2^{-j+1})$ and $|f(z, w)| \leqslant M_j = \text{const}$ for $(z, w) \in E_j \times G^j$ $(j = 2, \dots, k-1)$. Put

$$E_{kr} = \{z \in E_{k-1}\colon \sup_{w \in G^k} |f(z, w)| \leqslant r\}, \qquad r = 1, 2, \dots$$

By Montel's theorem and in view of Remark 1.2, E_{kr} is closed and $\bigcup_{r=1}^\infty E_{kr} = E_{k-1}$. Therefore, by Lemma 1.1 $\lim_{r \to \infty} d(E_{kr}) = d(E_{k-1})$. Take $r = r(k)$ so large that $d(E_{kr}) > d(E_{k-1}) \exp 2^{-k+1}$. Then $E_1, \dots, E_k = E_{kr}$ are the first k members of the required sequence. By the induction we get the sequence satisfying 1° and 2°. Put $E_0 = \bigcap_1^\infty E_k$. Then $d(E_0) \geqslant d(E) e^{-1} > 0$ and

$$|f(z, w)| \leqslant M_s = \text{const} \qquad \text{for } (z, w) \in E_0 \times G^s, \ s = 2, 3, \dots$$

Hence, f is bounded on every compact subset of $E_0 \times G$.

Let D_0 be a relatively compact subdomain of D. Without loss of generality we may assume that for every $a \in F_j$ $(j = 1, \dots, n)$ and for every $r > 0$ the set $F_j \cap \Delta(a, r)$, where $\Delta(a, r) = \{z\colon |w_j - a| \leqslant r\}$, has the positive transfinite diameter. Put

$$F^s = \{w \in F\colon \sup_{z \in D_0} |f(z, w)| \leqslant s\}, \qquad s = 1, 2, \dots$$

The set F^s is closed, $F^s \subset F^{s+1}$ and $F = \bigcup_1^\infty F^s$. By the Baire theorem one can find s and a polycilinder $P = \{w \in C^n\colon |w_j - a_j| \leqslant r_j, j = 1, \dots, n\}$ with the center $a = (a_1, \dots, a_n) \in F$ and with radii $r_j > 0$ such that $(P \cap F^s) \subset F$. Put $F_0 = P \cap F^s$, $F_{j0} = \{w_j \in F_j\colon |w_j - a_j| \leqslant r_j\}$ $(j = 1, \dots, n)$. Then $d(F_{j0}) > 0$ $(j = 1, \dots, n)$ and

$$|f(z, w)| \leqslant s \qquad \text{for } (z, w) \in D_0 \times F_0.$$

The sets E_0 and F_{j0} $(j = 1, \ldots, n)$ have the required properties.

LEMMA 7.4. *Assume that conditions* (a), (b), (c), (d) *and* (e) *of Lemma* 7.3 *are satisfied. Moreover, assume that*

(f) $h_{G_j}(w_j, F_j) = 0$ *in* G_j $(j = 1, \ldots, n)$;

(g) *for every relatively compact subdomain* ω *of* G *there is a constant* $M = M(f, \omega)$ *such that*

$$|f| \leqslant M \quad \text{in } (D \times F) \cup (E \times \omega) .$$

Then there exists a (unique) function $\tilde{f}$ *analytic in* $D \times G$ *such that* $\tilde{f} = f$ *in* X.

Proof. Let $\{G_{js}\}_{s=1,2,\ldots}$ be a sequence of compact subdomains of G_j such that $F_j \subset G_{js} \subset G_{j,s+1}$, $G_j = \bigcup_{s=1}^{\infty} G_{js}$, G_{js} is regular with respect to the classical Dirichlet problem. The function

$$h_s(w) = h_{G_{1s}}(w_1, F_1) + \ldots + h_{G_{ns}}(w_n, F_n) \quad (s = 1, 2, \ldots)$$

is plsh. in

$$G^s = G_{1s} \times \ldots \times G_{ns} \quad (s = 1, 2, \ldots)$$

and $\lim h_s(w) = 0$ for $w \in G$, the convergence being uniform on every compact subset of G.

By the proof of the Approximation Lemma there is a sequence $\{Q_\nu(z, w)\}$ of analytic functions in $D \times G$ such that

(i) the series $\sum Q_\nu$ converges to f in $D \times F$,

(ii) $|Q_\nu| \leqslant 2M_s(k\nu+1)e^{k\lambda\tau\nu}$, $z \in D_\tau$, $w \in G^s$ $(s = 1, 2, \ldots)$, $\nu = 1, 2, \ldots$,

(iii) $|Q_\nu| \leqslant M_s c(\sigma, \tau) \exp[-k\lambda\nu(\sigma - \varepsilon - \tau)]$, $z \in D_\tau$, $w \in F$, $\nu \geqslant \nu_0 = \nu_0(\varepsilon, \sigma)$, $s \geqslant 1$, where $\varepsilon > 0$, $\sigma_0 < \tau < \sigma < 1$, $0 \leqslant \sigma_0 = \sigma_0(D, E)$ and $M_s = \sup\{|f(z, w)|: z \in D, w \in G^s\}$.

Given any fixed $z \in D_\tau$ and $\nu \geqslant \nu_0$ the function $V(w) = \mathrm{Log}[|Q_\nu| e^{-k\lambda\tau\nu}/(k\nu+1)]$ is plsh. in G and satisfies the inequalities

$$V(w) \leqslant m = \mathrm{Log}(M_s c) - k\lambda\nu(\sigma - \varepsilon), \quad w \in F,$$

$$V(w) \leqslant M = \mathrm{Log}(2M_s), \quad w \in G^s .$$

Hence in view of Lemma 7.2 we get

$$V(w) \leqslant m + (M - m) h_s(w) \quad \text{in } G^s,$$

and after obvious transformations,

$$|Q_\nu| \leqslant (k\nu+1) c(\sigma, \tau) M_s \exp\big(-k\lambda\nu[\sigma - \varepsilon - \tau - (\sigma - \varepsilon) h_s(w)]\big),$$
$$z \in D_\tau, \ w \in G^s, \ \nu \geqslant \nu_0, \ s \geqslant 1 .$$

Given τ ($\sigma_0 < \tau < 1$) and a compact subset ω of G we may find σ ($\tau < \sigma < 1$), $\varepsilon > 0$, and s such that

$$\sigma - \varepsilon - \tau - (\sigma - \varepsilon) h_s(w) > 0 , \qquad w \in \omega .$$

Therefore the series $\sum Q_\nu$ is uniformly convergent in $D_\tau \times \omega$. By the arbitrariness of ω and of τ ($\sigma_0 < \tau < 1$) the series converges uniformly on every compact subset of $D \times G$. Its sum $\tilde{f} = \sum Q_\nu$ gives the required continuation of f.

From the last two lemmas one easily gets the following

THEOREM 7.2. *Assume that:* (1) D_k *is a domain in the* z_k*-plane* ($k = 1, \ldots, n$), (2) E_k *is a compact subset of* D_k *and* $d(E_k) > 0$ ($k = 1, \ldots, n$), (3) $h_{D_k}(z_k) = 0$ *in* D_k ($k = 2, \ldots, n$) *and* (4) f *is separately analytic in* X *given by* ($\ast$) (*see Theorem* 7.1).

Then f *may be continued to an analytic function* $\tilde{f}$ *in the product* $D_1 \times \ldots \times D_n$. *In particular, this product is the envelope of holomorphy of* X.

COROLLARY 7.3. *If* $E_k \subset D_k$, $d(E_k) > 0$ *and* D_k *is identical with the whole* z_k*-plane* ($k = 1, \ldots, n$), *then every function* f *defined in* X *and entire with respect to each variable* z_k *separately is continuable to a function analytic in* C^n.

References

[1] V. Avanisian, *Sur l'harmonicité des fonctions separément harmoniques*, Séminaire de probabilité, Dept. de Math., Strasbourg, Février 1967.

[2] S. Bochner and W. T. Martin, *Several complex variables*, Princeton 1948.

[3] M. Brelot, *Éléments de la théorie classique du potentiel*, Paris 1961.

[4] F. E. Browder, *Real analytic functions on product spaces and separate analyticity*, Canad. J. Math. 13 (1961), pp. 650-656.

[5] R. H. Cameron and D. A. Storvick, *Analytic continuation for functions of several variables*, Trans. Amer. Math. Soc. 125 (1966), pp. 7-12.

[6] G. Choquet, *Capacibilité en potentiel logarithmique*, Bull. classe Sci., Bruxelles 44 (1958), pp. 321-326.

[7] R. M. Dudley and B. Randol, *Implications of pointwise bounds on polynomials*, Duke Math. J. 29 (3) (1962), pp. 455-458.

[8] B. A. Fuks, *Special chapters of the theory of analytic functions of several complex variables*, Moscow 1963 (in Russian).

[9] J. Górski, *Sur certains fonctions harmoniques jouissant des propriétés extrémales par rapport à un ensemble*, Ann. Soc. Polon. Math. 23 (1950), pp. 259-271.

[10] F. Hartogs. *Zur Theorie der analytischen Funktionen mehrerer Veränderlichen*, Math. Ann. 62 (1906), pp. 1-88.

[11] M. Hukuhara, *L'extensions du théorème d'Osgood et de Hartogs* (en Japonais) Kansu-hoteisiki oyobi Oyo-kaiseki (1930), p. 48.

[12] K. Koseki, *Neuer Beweis des Hartogsschen Satzes*, Math. J. Okoyama Univ. 12 (1966), pp. 63-70.

[13] N. S. Landkof, *Foundations of modern potential theory*, Moscow 1966 (in Russian).

[14] F. Leja, *Sur les suites des polynômes bornés presque partout sur la frontière d'un domaine*, Math. Ann. 108 (1933), pp. 517-524.

[15] — *Sur une propriété des suites des fonctions bornées sur une courbe*, C. R. Ac. Sci. Paris 196 (1933), p. 321.

[16] — *Une nouvelle démonstration d'un théorème sur les séries de fonctions analytiques*, Actas de la Ac. de Lima, 13 (1950), pp. 3-7.

[17] — *Une condition de regularité et d'irrégularité des points frontières dans le problème de Dirichlet*, Ann. Soc. Polon. Math. 20 (1947), pp. 223-228.

[18] — *Teoria funkcji analitycznych*, Warszawa 1957.

[19] P. Lelong, *Fonctions plurisousharmoniques et fonctions analytiques de variables reélles*, Ann. Inst. Fourier 11 (1961), pp. 515-562.

[20] W. F. Osgood, *Note über analytische Funktionen mehrerer Veränderlichen*, Math. Ann. 52 (1899), pp. 462-464.

[21] — *Zweite Note über analytische Funktionen mehrerer Veränderlichen*, ibidem 53 (1900), pp. 461-464.

[22] H. Renggli, *An inequality for logarithmic capacities*, Pacific J. Math. 11 (1961), pp. 313-314.

[23] W. Rothstein, *Ein neuer Beweis des Hartogsscen Satzes und seine Ausdehnung auf meromorphe Funktionen*, Math. Z. 53 (1950), pp. 84-95.

[24] I. Shimoda, *Notes on the functions of two complex variables*, J. Gakugei Tokushima Univ. 8 (1957), pp. 1-3.

[25] J. Siciak, *Some applications of the method of extremal points*, Coll. Math. 11 (1964), pp. 209-250.

[26] — *Asymptotic behaviour for harmonic polynomials bounded on a compact set*, Ann. Polon. Math. 20 (1968), pp. 267-278.

[27] — *Analyticity and separate analyticity of functions defined on lowerdimensional subsets of C^n*, Zeszyty Nauk. UJ 13 (1969), pp. 53-70.

[28] T. Terada, *Sur une certaine condition sous laquelle une fonction de plusieurs variables est holomorphe* (Diminution de la condition dans le théorème de Hartogs), Publ. Research Inst. for Math. Sci., Ser. A (Kyoto), 2 (1967), pp. 383-396.

[29] J. L. Walsh, *Interpolation and approximation*, Third edition. Boston 1960.

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Reçu par la Rédaction le 11. 4. 1968