COLLOQUIUM MATHEMATICUM

VOL. LIII 1987 FASC. 1

ON A MODIFIED GAME OF SIERPIŃSKI

BY

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Introduction. In 1924, W. Sierpiński [13] proved that every uncountable Borel set contains a perfect subset. In the proof he made use of some multivariate function of sets, by means of which R. Telgársky [15] defined a topological game, where the function becomes a winning strategy of Player II. That game is an infinite positional two-person game with perfect information. We shall call it the *game of Sierpiński* and denote by $S(X, Y)$.

Let X be a subset of a topological space Y. The players construct a decreasing sequence of uncountable sets $A_1 \supset B_1 \supset A_2 \supset B_2 \supset \ldots$, where A_n (B_n) is chosen by Player I (resp., Player II). Player II wins the play $(A_1, B_1, A_2, B_2, \ldots)$ of the Sierpiński game $S(X, Y)$ iff $\bigcap_n \bar{B}_n \subset X$.

The game $S(X, Y)$ points out some common features with the famous Banach–Mazur game (see [10]) and with its generalizations studied by J. C. Morgan II [8].

In this paper we shall study a modified game of Sierpiński in which the players choose countable and dense in themselves sets, and we shall establish some relationships of this game to the Choquet game (see [1], [11], [16], [17], [18], [19]).

Definitions and notation. A subset A of Y is said to be a *Q-set* if $A \approx Q$, i.e. A is homeomorphic to the set Q of all rational numbers. It is well known that each countable and dense in itself subset A of a metrizable space is a Q-set ([12]; [7]).

We define the *modified game of Sierpiński* $S_0(X, Y)$ as follows. Let X be a subset of a metrizable space Y. Player I chooses a Q-set (or the empty set) $A_1 \subset X$. After that Player II chooses a Q-set (or the empty set) $B_1 \subset A_1$. Assume inductively that $A_1 \supset B_1 \supset \ldots \supset A_n \supset B_n$ have been chosen. Then Player I chooses a Q-set (or the empty set) $A_{n+1} \subset B_n$. After that Player II chooses a Q-set (or the empty set) $B_{n+1} \subset A_{n+1}$. The player who first chooses the empty set loses the play $(A_1, B_1, A_2, B_2, \ldots)$. If all sets A_i and B_i

are Q-sets, then Player II wins the play $(A_1, B_1, A_2, B_2, \ldots)$ of the game $S_0(X, Y)$ if $\bigcap_n \bar{B}_n \subset X$; otherwise Player I wins.

A strategy of Player I is a function s defined for all finite (including empty) decreasing sequences $(A_1, B_1, \ldots, A_n, B_n)$ of subsets of X so that $s(\emptyset) = A_1 \subset X$, and $s(A_1, B_1, \ldots, A_n, B_n) = A_{n+1} \subset B_n$ for each $n \in N$. A stationary strategy of Player I is a function t defined for all subsets of X and for the empty set such that $t(\emptyset) = A_1 \subset X$ and $t(B) = A \subset B$. Of course, if t is a stationary strategy of Player I, then the function s defined as $s(\emptyset) = t(\emptyset)$, $s(A_1, B_1, \ldots, A_n, B_n) = t(B_n)$ is a strategy of Player I. A strategy and a stationary strategy of Player II can be defined similarly.

In this paper we shall use the following notations. $\mathrm{I} \uparrow S_0(X, Y)$ (resp. $\mathrm{II} \uparrow S_0(X, Y)$) means that Player I (resp. Player II) has a winning strategy in the game $S_0(X, Y)$. $\mathrm{I} \Uparrow S_0(X, Y)$ ($\mathrm{II} \Uparrow S_0(X, Y)$) means that Player I (resp., Player II) has a stationary winning strategy in the game $S_0(X, Y)$. The game $S_0(X, Y)$ is said to be *determined* (*s-determined*) if either Player I or Player II has a winning strategy (resp., a stationary winning strategy) in $S_0(X, Y)$.

It follows directly from the definition of the game $S_0(X, Y)$ that Player II wins the game provided that X is a scattered set (i.e., if X does not contain a dense in itself subset).

G. Choquet in his lectures on analysis ([1], p. 116) considered the following game, denoted here by $G(X, Y)$. Let Y be a topological space and $X \subset Y$. Player I chooses (x_1, U_1), where $x_1 \in X$ and U_1 is an open neighborhood of x_1 in Y. After that Player II chooses an open set V_1 in Y such that $x_1 \in V_1 \subset U_1$. Assume inductively that $(x_1, U_1), V_1, \ldots, (x_n, U_n), V_n$ have been chosen. Then Player I chooses (x_{n+1}, U_{n+1}), where $x_{n+1} \in X \cap V_n$, $U_{n+1} \subset V_n$ and U_{n+1} is an open neighborhood of x_{n+1} in Y. After that Player II chooses an open set V_{n+1} in Y such that $x_{n+1} \in V_{n+1} \subset U_{n+1}$. Player II wins the play $((x_1, U_1), V_1, (x_2, U_2), V_2, \ldots)$ of the game $G(X, Y)$ if $\emptyset \neq \bigcap_n V_n \subset X$; otherwise Player I wins.

The game $G(X, Y)$ has been further studied by E. Porada [11] and R. Telgársky [16], [17], [18], [19].

Definition 1. A metrizable space X is said to be a *strongly Baire space* (in the notation: $X \in \mathrm{sB}$) if it contains no closed subset F, which is of the first category in itself.

THEOREM 2. *Let Y be a complete metric space and $X \subset Y$. Then the following conditions are equivalent*:

(0) $X \notin \mathrm{sB}$.

(1) *There exists a Q-set $F \subset X$ with $\bar{F} \cap X = F$.*

(2) *There exists a Q-set $F \subset X$ with $F \in G_\delta(X)$.*

(3) $\mathrm{I} \uparrow G(X, Y)$.

(4) $\mathrm{I} \Uparrow G(X, Y)$.

(5) $\mathrm{I} \uparrow S_0(X, Y)$.
(6) $\mathrm{I} \Uparrow S_0(X, Y)$.

Proof. (0) $\Leftrightarrow$ (1). This is a theorem of W. Hurewicz ([5], p. 90).

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). For these equivalences we refer to Theorem 1.3 of R. Telgársky [19].

(1) $\Rightarrow$ (6). Let $F \subset X$, $\bar{F} \cap X = F$ and $F \approx Q$. We shall define a stationary strategy s of Player I as follows: $s(\emptyset) = A_1 = F$. Let $F = \{a_n\}_{n=1}^{\infty}$. If B is a set chosen by Player II, then we define $s(B) = A$ such that $A \subset B$, $A \approx Q$, $\operatorname{diam} A \leqslant \min(1, \frac{1}{2}\operatorname{diam} B)$ and $a_{n_0} \notin \bar{A}$, where $n_0 = \min\{n \in N: a_n \in B\}$. Then s just defined is a stationary winning strategy of Player I in $S_0(X, Y)$. For, if $(A_1, B_1, A_2, B_2, \ldots)$ is a play such that $A_1 = s(\emptyset)$ and $A_{n+1} = s(B_n)$ for $n \in N$, then $\bigcap_n \bar{B}_n = \bigcap_n \bar{A}_n = \{p\}$ and $p \neq a_n$ for each $n \in N$. Hence $p \notin F$ and, moreover, $p \notin X$, because $p \in \bar{F}$ and $\bar{F} \cap X = F$. Thus $\bigcap_n \bar{B}_n \cap (Y-X) \neq \emptyset$.

(6) $\Rightarrow$ (5). This is obvious.

(5) $\Rightarrow$ (3). Let s be a winning strategy of Player I in $S_0(X, Y)$. We shall define a strategy t for Player I in the Choquet game $G(X, Y)$ as follows. In the game $S_0(X, Y)$ Player I chooses $s(\emptyset) = A_1$, $A_1 \subset X$, $A_1 \approx Q$. In the game $G(X, Y)$ we set $t(\emptyset) = (x_1, U_1)$ such that $x_1 \in A_1$, $U_1 \ni x_1$ and $\operatorname{diam} U_1 \leqslant 1$. Let V_1 be an open set chosen by Player II, where $x_1 \in V_1 \subset U_1$. Now let Player II in $S_0(X, Y)$ choose $B_1 = A_1 \cap V_1$ (note that $B_1 \subset A_1$ and $B_1 \approx Q$), and let Player I choose in reply $s(A_1, B_1) = A_2$. We have $A_2 \subset B_1$ and $A_2 \approx Q$. In $G(X, Y)$ we set $t((x_1, U_1), V_1) = (x_2, U_2)$ such that $x_2 \in A_2 \cap U_2$, $\bar{U}_2 \subset V_1$ and $\operatorname{diam} U_2 \leqslant \frac{1}{2}$. Let V_2 be an open set chosen by Player II, where $x_2 \in V_2 \subset U_2$. Now, let in $S_0(X, Y)$ Player II choose $B_2 = A_2 \cap V_2$ (again, we have $B_2 \subset A_2$ and $B_2 \approx Q$), and let Player I choose in reply $s(A_1, B_1, A_2, B_2) = A_3$. Then $A_3 \subset B_2$ and $A_3 \approx Q$. In $G(X, Y)$ we set $t((x_1, U_1),\ V_1,\ (x_2, U_2),\ V_2) = (x_3, U_3)$ such that $x_3 \in A_3 \cap U_3$, $\bar{U}_3 \subset V_2$, $\operatorname{diam} U_3 \leqslant \frac{1}{3}, \ldots$, and so on.

Since s is a winning strategy of Player I in the game $S_0(X, Y)$, we have $\bigcap_n \bar{B}_n = \{p\} \subset Y-X$. But $\{q\} = \bigcap_n \bar{U}_n = \bigcap_n V_n \supset \bigcap_n \bar{B}_n = \{p\}$; hence $q = p$ and $q \in Y-X$. Thus t is a winning strategy for Player I in the Choquet game $G(X, Y)$. $\square$

By Theorem 2 the investigation of the game $S_0(X, Y)$ can be reduced to the case where X is a strongly Baire space.

Now we shall show the following

THEOREM 3 (AC). *If* $\mathrm{II} \uparrow S_0(X, Y)$, *then* $\mathrm{II} \Uparrow S_0(X, Y)$.

To prove the theorem we shall use an unpublished construction[1] of

[1] The construction of F. Galvin was published in [20].

F. Galvin [2] which deals with a fairly general class of games; here his proof is slightly modified and adopted for the game $S_0(X, Y)$.

Proof. Let t be a winning strategy of Player II in $S_0(X, Y)$. Without loss of generality we may assume that X is not scattered. Let T be the set of all decreasing sequences $(A_1, B_1, \ldots, A_n)$ of Q-sets so that $B_k = t(A_1, B_1, \ldots, A_k)$ for each $k < n$. Let $<$ be a fixed well ordering of the family of all Q-sets contained in X. We define linear ordering $\prec$ in T as follows: $(A_1, B_1, \ldots, A_m) \prec (A'_1, B'_1, \ldots, A'_n)$ if either $m >$ and $(A_1, B_1, \ldots, A_n) = (A'_1, B'_1, \ldots, A'_n)$ or there is a $k < \min\{m, n\}$ such that $(A_1, B_1, \ldots, A_{k-1}) = (A'_1, B'_1, \ldots, A'_{k-1})$ and $A_k < A'_k$.

Let $T(A)$ be the set of all $(A_1, B_1, \ldots, A_n)$ in T with $A_n = A$. Now we shall show that if A is a Q-set in X such that $\bar{A} - X \neq \emptyset$, then $T(A)$ is well ordered by $\prec$. For, suppose $T(A)$ is not well ordered. Then there is a sequence

$$\{(A_1^{(k)}, B_1^{(k)}, \ldots, A_{n(k)}^{(k)}): k \in N\} \subset T(A)$$

so that

$$(A_1^{(k+1)}, B_1^{(k+1)}, \ldots, A_{n(k+1)}^{(k+1)}) \prec (A_1^{(k)}, B_1^{(k)}, \ldots, A_{n(k)}^{(k)})$$

for each $k \in N$. Now, there is a sequence $k_1 < k_2 < \ldots$ such that

$$A_i^{(k_i)} = A_i^{(k_i+1)} = A_i^{(k_i+2)} = \ldots \quad \text{and} \quad B_i^{(k_i)} = B_i^{(k_i+1)} = B_i^{(k_i+2)} = \ldots$$

for each $i \in N$. Notice that

$$B_i^{(k_i)} = t(A_1^{(k_i)}, B_1^{(k_i)}, \ldots, A_i^{(k_i)})$$

for each $i \in N$. Let us put $A_i = A_i^{(k_i)}$ and $B_i = B_i^{(k_i)}$. Then $(A_1, B_1, A_2, B_2, \ldots)$ is a play of $S_0(X, Y)$ consistent with t. Since $A_i = A_i^{(k_i)} \supset A_{n(k_i)}^{(k_i)} = A$, it follows that $\bigcap_{i=1}^{\infty} \bar{A}_i \supset \bar{A}$ and therefore $\bigcap_{i=1}^{\infty} \bar{A}_i \cap (Y - X) \neq \emptyset$. This is a contradiction.

Now we define a stationary strategy s by setting $s(A) = A$ if $\bar{A} \subset X$ and $s(A) = t(A_1, B_1, \ldots, A_n)$, where $(A_1, B_1, A_2, B_2, \ldots) = \min T(A)$, if $\bar{A} - X \neq \emptyset$. Finally, let $(A_1, B_1, A_2, B_2, \ldots)$ be a play of $S_0(X, Y)$ such that $s(A_k) = B_k$ for each $k \in N$. Without loss of generality we may assume that $\bar{A}_k - X \neq \emptyset$ for each $k \in N$. Let us fix $k \in N$. Then $T(A_k)$ is well ordered and $s(A_k) = t(A_1^{(k)}, B_1^{(k)}, \ldots, A_{n(k)}^{(k)})$. Since

$$B_k = s(A_k) = t(A_1^{(k)}, B_1^{(k)}, \ldots, A_{n(k)}^{(k)})$$

and $A_{k+1} \subset B_k$, it follows that $(A_1^{(k)}, B_1^{(k)}, \ldots, A_{n(k)}^{(k)}, B_k, A_{k+1}) \in T(A_{k+1})$. Hence

$$(A_1^{(k+1)}, B_1^{(k+1)}, \ldots, A_{n(k+1)}^{(k+1)}) \preceq (A_1^{(k)}, B_1^{(k)}, \ldots, A_{n(k)}^{(k)}, B_k, A_{k+1}).$$

Since

$$(A_1^{(k)}, B_1^{(k)}, \ldots, A_{n(k)}^{(k)}, B_k, A_{k+1}) \prec (A_1^{(k)}, B_1^{(k)}, \ldots, A_{n(k)}^{(k)})$$

we have

$$(A_1^{(k+1)}, B_1^{(k+1)}, \ldots, A_{n(k+1)}^{(k+1)}) \prec (A_1^{(k)}, B_1^{(k)}, \ldots, A_{n(k)}^{(k)}).$$

Hence we infer, as above, the existence of a sequence $k_1 < k_2 < \ldots$ such that $A_i^{(k_i)} = A_i^{(k_i+1)} = A_i^{(k_i+2)} = \ldots$ and $B_i^{(k_i)} = B_i^{(k_i+1)} = B_i^{(k_i+2)} = \ldots$ Let us put $A'_i = A_i^{(k_i)}$ and $B'_i = B_i^{(k_i)}$. Since

$$(A'_1, B'_1, \ldots, A'_n) = (A_1^{(k_1)}, B_1^{(k_1)}, \ldots, A_n^{(k_n)}) = (A_1^{(k_n)}, B_1^{(k_n)}, \ldots, A_n^{(k_n)})$$

for each $n \in N$, the play $(A'_1, B'_1, A'_2, B'_2, \ldots)$ is consistent with t. Since $A'_i = A_i^{(k_i)} \supset A_{n(k_i)}^{(k_i)} = A_{k_i}$ for each $i \in N$, we have $\bigcap_{i=1}^{\infty} \bar{A}_i \subset \bigcap_{i=1}^{\infty} \bar{A}'_i \subset X$. Thus s is a stationary winning strategy of Player II. □

By Theorems 2 and 3 we get

COROLLARY 4 (AC). $\mathrm{I} \uparrow S_0(X, Y) \Leftrightarrow \mathrm{I} \Uparrow S_0(X, Y)$ *and* $\mathrm{II} \uparrow S_0(X, Y) \Leftrightarrow \mathrm{II} \Uparrow S_0(X, Y)$.

It follows that under the axiom of choice it is the same to investigate the existence of a winning strategy or a stationary winning strategy; also, the problems of the determinacy and the s-determinacy of the game $S_0(X, Y)$ are therefore equivalent.

Recall that a subset X of a seperable metric space Y is said to be *analytic* if either $X = \emptyset$ or there is a continuous map from the space N^N (of irrational numbers) onto X. A subset X of a topological space Y is said to be a Souslin set in Y if there is an indexed family $\{F(k_1, \ldots, k_n): (k_1, \ldots, k_n) \in N^n, n \in N\}$ of closed subsets of Y such that

$$X = \bigcup \{\bigcap \{F(k_1, \ldots, k_n): n \in N\}: (k_1, k_2, \ldots) \in N^N\}.$$

THEOREM 5. *If X is a Souslin set in a metrizable space Y so that X is a strongly Baire space, then* $\mathrm{II} \uparrow S_0(X, Y)$.

Proof. Let $X = \bigcup \{\bigcap \{F(k_1, \ldots, k_n): n \in N\}: (k_1, k_2, \ldots) \in N^N\}$, where $\{F(k_1, \ldots, k_n): (k_1, \ldots, k_n) \in N^n, n \in N\}$ is a regular system of closed sets in Y.

Let $A_1 \subset X$ be a Q-set chosen by Player I. Then

$$\bar{A}_1 \cap X = \bigcup \{\bigcap \{F(k_1, \ldots, k_n) \cap \bar{A}_1 : n \in N\}: (k_1, k_2, \ldots) \in N^N\},$$

so $\bar{A}_1 \cap X \subset \bigcup_{k \in N} F(k) \cap \bar{A}_1$. Now we claim that there is a $k_1 \in N$ such that $\mathrm{Int}_{\bar{A}_1 \cap X}(F(k_1) \cap \bar{A}_1 \cap X) \neq \emptyset$. For, if for each $k \in N$ we had

$$\mathrm{Int}_{\bar{A}_1 \cap X}(F(k) \cap \bar{A}_1 \cap X) = \emptyset,$$

then $\bigcup_{k \in N} F(k) \cap \bar{A}_1 \cap X = \bar{A}_1 \cap X$ would be a set of the first category in itself. Since $\bar{A}_1 \cap X$ is a closed subset of X, this contradicts $X \in \mathrm{sB}$. So we have

$$A_1 \cap \mathrm{Int}_{\bar{A}_1 \cap X}(F(k_1) \cap \bar{A}_1 \cap X) \neq \emptyset$$

and therefore

$$B_1 = A_1 \cap \mathrm{Int}_{\bar{A}_1 \cap X}(F(k_1) \cap \bar{A}_1 \cap X)$$

is a Q-set contained in A_1. Let us put $s(A_1) = B_1$. Let $A_2 \subset B_1$ be a Q-set chosen by Player I. Then $\bar{A}_2 \cap F(k_1) \cap X \subset \bigcup_{k \in N} F(k_1, k) \cap \bar{A}_2$. Similarly as before we can show that there is a $k_2 \in N$ such that

$$\mathrm{Int}_{\bar{A}_2 \cap F(k_1) \cap X}(F(k_1, k_2) \cap \bar{A}_2 \cap X) \neq \emptyset.$$

We have

$$A_2 \cap \mathrm{Int}_{\bar{A}_2 \cap F(k_1) \cap X}(F(k_1, k_2) \cap \bar{A}_2 \cap X) \neq \emptyset.$$

Hence

$$B_2 = A_2 \cap \mathrm{Int}_{\bar{A}_2 \cap F(k_1) \cap X}(F(k_1, k_2) \cap \bar{A}_2 \cap X)$$

is a Q-set contained in A_2. Let us put $s(A_1, B_1, A_2) = B_2$.

Continuing in this manner we define the strategy s which is a winning strategy for Player II, because

$$\bigcap_n \bar{B}_n \subset \bigcap_n F(k_1, \ldots, k_n) \subset X. \ \Box$$

Since Souslin subsets and analytic subsets of Polish spaces coincide ([7], p. 482) we get from Theorem 2 and Theorem 5 the following.

COROLLARY 6a. *If X is an analytic set in a Polish space Y, then*

(a) $X \notin \mathrm{sB} \Leftrightarrow \mathrm{I} \uparrow S_0(X, Y) \Leftrightarrow \mathrm{I} \Uparrow S_0(X, Y)$;

(b) $X \in \mathrm{sB} \Leftrightarrow \mathrm{II} \uparrow S_0(X, Y) \underset{\mathrm{AC}}{\Leftrightarrow} \mathrm{II} \Uparrow S_0(X, Y)$

and thus the game $S_0(X, Y)$ is determined, and, under the axiom of choice, it is even s-determined.

Remark 7. Let $Y = [0, 1]$. It was announced by K. Gödel [4] and proved by D. S. Novikov [9] that the axiom of constructibility implies the existence of an analytic non-Borel set X such that $Y - X$ is a totally imperfect set. Then $X \in \mathrm{sB}$ (see [19]). Hence, by Theorem 5, we have the following corollary. If we assume the axiom of constructibility, then there is an analytic non-Borel set $X \subset Y = [0, 1]$ such that $\mathrm{II} \Uparrow S_0(X, Y)$.

Remark 8. By Theorem 5 it follows that if X is a G_δ-set in a metrizable space, then $\mathrm{II} \uparrow S_0(X, Y)$. However for X being a G_δ-set we shall prove that Player II can achieve the win on his first move, and therefore $\mathrm{II} \Uparrow S_0(X, Y)$. This is an easy consequence of the following lemma.

LEMMA 9. *If X is a G_δ-set in a metrizable space Y and $A \subset X$ is a Q-set, then there is a Q-set $B \subset A$ such that $\bar{B} \subset X$.*

Proof. Let $X = \bigcap_n G_n$, where G_n are open sets in Y and $G_n \supset G_{n+1}$ for each $n \in N$. Then there are points $b(-1)$, $b(0)$, $b(1) \in A$ and open sets $V(-1)$, $V(0)$, $V(1)$ such that:

(i) $b(n_1) \in V(n_1)$, $n_1 = -1, 0, 1$;

(ii) $\overline{V(n_1)} \cap \overline{V(n_1')} = \emptyset$, $n_1, n_1' = -1, 0, 1$, $n_1 \neq n_1'$;

(iii) $\overline{V(n_1)} \subset G_1$, $n_1 = -1, 0, 1$;

(iv) $\operatorname{diam} V(n_1) \leqslant 1$, $n_1 = -1, 0, 1$.

Analogically, for every $n_1 \in \{-1, 0, 1\}$ there are points $b(n_1, -1)$, $b(n_1, 0)$, $b(n_1, 1) \in A$ and open sets $V(n_1, -1)$, $V(n_1, 0)$, $V(n_1, 1)$ such that:

(i) $b(n_1, n_2) \in V(n_1, n_2)$, $n_2 = -1, 0, 1$;

(ii) $\overline{V(n_1, n_2)} \cap \overline{V(n_1, n_2')} = \emptyset$, $n_2, n_2' = -1, 0, 1$, $n_2 \neq n_2'$;

(iii) $\overline{V(n_1, n_2)} \subset G_2$, $n_2 = -1, 0, 1$;

(iv) $\operatorname{diam} V(n_1, n_2) \leqslant \frac{1}{2}$, $\overline{V(n_1, n_2)} \subset V(n_1)$, $n_2 = -1, 0, 1$;

(v) $b(n_1, 0) = b(n_1)$.

When continuing this construction, we define the points $b(n_1, \ldots, n_k)$ and the open sets $V(n_1, \ldots, n_k)$ for $(n_1, \ldots, n_k) \in \{-1, 0, 1\}^k$ and $k \in N$. Then we set

$$B = \{b(n_1, \ldots, n_k) \colon (n_1, \ldots, n_k) \in \{-1, 0, 1\}^k,\ k \in N\}.$$

Clearly, B is a Q-set contained in A. We shall show that $\bar{B} \subset X$. Indeed, for every $k \in N$,

$$B \subset \bigcup \{V(n_1, \ldots, n_k) \colon (n_1, \ldots, n_k) \in \{-1, 0, 1\}^k\},$$

hence

$$\bar{B} \subset \bigcup \{\overline{V(n_1, \ldots, n_k)} \colon (n_1, \ldots, n_k) \in \{-1, 0, 1\}^k\}.$$

Finally,

$$\bar{B} \subset \bigcap \Big\{\bigcup \{\overline{V(n_1, \ldots, n_k)} \colon (n_1, \ldots, n_k) \in \{-1, 0, 1\}^k\} \colon k \in N\Big\}$$
$$\subset \bigcap \{G_k \colon k \in N\} = X. \ \square$$

Let Y be a Polish space. From a theorem of W. Hurewicz ([5], p. 97) it follows that if $X \subset Y$ is a coanalytic set and $X \notin G_\delta(Y)$, then $X \notin$ sB. Hence, by Theorem 2 and Remark 8, we obtain the following.

COROLLARY 6b. *If X is a coanalytic set in a Polish space Y, then*

(a) $X \in G_\delta(Y) \Leftrightarrow \mathrm{II} \Uparrow S_0(X, Y)$;

(b) $X \notin G_\delta(Y) \Leftrightarrow \mathrm{I} \Uparrow S_0(X, Y)$,

and thus the game $S_0(X, Y)$ is s-determined.

By Corollaries 6a and 6b we get

COROLLARY 10. *If X is an analytic or coanalytic set in a Polish space Y, then the game $S_0(X, Y)$ is determined and under the axiom of choice s-determined.*

Remark 11. It is easy to point out that if $X_1 \subset [0, 1]$ is analytic and $X_2 \subset [1, 2]$ is coanalytic, then the game $S_0(X_1 \cup X_2, [0, 2])$ is determined (and under AC it is also s-determined).

Question (**P 1320**). Let X belong to the σ-algebra generated by analytic subsets of a Polish space Y. Is then $S_0(X, Y)$ determined?

Remark 12. G. Choquet [1] proved that if Y is a metrizable space, then II $\Uparrow G(X, Y)$ iff X is an absolute G_δ-set.

Hence and by Theorem 2 and Remark 8 we have the following.

COROLLARY 13. *If Y is a complete metrizable space, then*

$$\mathrm{I} \Uparrow G(X, Y) \Leftrightarrow \mathrm{I} \Uparrow S_0(X, Y),$$

$$\mathrm{II} \Uparrow G(X, Y) \Rightarrow \mathrm{II} \Uparrow S_0(X, Y).$$

By Theorem 5 and Remark 7, under the axiom of constructibility, the above implication cannot be reversed. We shall show below that under CH there is a non-analytic set which also disproves the converse implication.

LEMMA 14. *Let X be a subset of a metrizable space Y and let $A \subset X$ be a Q-set. Then there is a Q-set $B \subset A$ such that $\bar{B}$ is nowhere dense in Y.*

Proof. Since $Q \approx Q^2$ we have $A \approx A^2$. Let h be a homeomorphism from A onto A^2. Let us pick any point $p \in A$ and let us set $B = h^{-1}(\{p\} \times A)$. Clearly, $B \subset A$ and $B \approx Q$. Since B is relatively closed and nowhere dense in A, and A is dense in itself, it follows that B is nowhere dense in $\bar{A}$, and therefore B (and also $\bar{B}$) is nowhere dense in Y. □

Recall that a subset A of a Polish space Y is said to be a *Lusin set* if A is uncountable and $|A \cap F| \leqslant \aleph_0$ for every nowhere dense set $F \subset Y$. Let us note that under the continuum hypothesis each uncountable Polish space contains Lusin set ([7], p. 525).

THEOREM 15. *If X is a subset of a Polish space Y, where $Y - X$ is a Lusin set, then* II $\Uparrow S_0(X, Y)$.

Proof. Let A be a Q-set in X. We define a stationary strategy s for Player II as follows. If $\bar{A} \subset X$, we set $s(A) = A$. If $\bar{A} \cap (Y - X)$ is countable and nonvoid, then $\bar{A} \cap X$ is a G_δ-set in Y contained in X. Hence by Lemma 6b there is a Q-set $B \subset A$ such that $\bar{B} \subset \bar{A} \cap X$. Then we set $s(A) = B$. If $\bar{A} \cap (Y - X)$ is uncountable, then by Lemma 14 there is a Q-set $B \subset A$ such that $\bar{B}$ is nowhere dense in Y. Since $Y - X$ is a Lusin set in Y, it follows that $\bar{B} \cap (Y - X)$ is countable. Again by Lemma 6b we find a Q-set $C \subset B$ such that

$\bar{C} \subset \bar{B} \cap X$, and we set $s(A) = C$. Clearly, s is a stationary winning strategy for Player II in $S_0(X, Y)$. ⌐

Remark 16. One can show similarly as above that if $Y - X$ is an n-chain concentrated about a countable set (cf. [14]) or, if $Y - X$ is concentrated of type n (cf. [3]), then II $\Uparrow S_0(X, Y)$.

By Theorem 15 and Remark 12 we have the following.

COROLLARY 17 (CH). *The games $G(X, Y)$ and $S_0(X, Y)$ are not equivalent.*

COROLLARY 18. *Let Y be an uncountable Polish space. If $X \subset Y$ is a Lusin set, then* I $\Uparrow S_0(X, Y)$.

Indeed, since the Lusin set X is uncountable, it is not scattered. Hence X contains a Q-set A. By Lemma 14 there is a Q-set $B \subset A$ so that $\bar{B}$ is nowhere dense in Y. Since X is a Lusin set, $\bar{B} \cap X$ is countable and, moreover, it is a Q-set. If we put $F = \bar{B} \cap X$, then the condition (1) of Theorem 2 is fulfilled, so I $\Uparrow S_0(X, Y)$.

Remark 19. It is easy to see that a stronger version of Corollary 18 is true: If $E \subset X \subset Y$, where Y is a Polish space and $\bar{E} \cap X$ is a Lusin set in $\bar{E}$, then I $\Uparrow S_0(X, Y)$.

THEOREM 20. *Let X be a subset of a complete metric space Y.*

(a) *If* I $\uparrow S_0(X, Y)$, *then $Y - X$ contains a copy of the Cantor discontinuum.*

(b) *If* II $\uparrow S_0(X, Y)$, *then X is scattered or X contains a copy of the Cantor discontinuum.*

The proof of part (b) is analogical to the proof of the Sierpiński theorem (see [13]) and the proof of part (a) is similar to the proof of (b), so they are omitted.

A subset X of an uncountable Polish space Y is said to be a Bernstein set if neither X nor $Y - X$ contains a copy of the Cantor discontinuum. Assuming the axiom of choice each uncountable Polish space contains a Bernstein set ([7], p. 514). It is easy to check that each Bernstein set is a strongly Baire space.

Since Bernstein sets in Polish spaces are not scattered, we have from Theorem 20 the following.

COROLLARY 21 (AC). *If X is a Bernstein set in a Polish space Y, then the game $S_0(X, Y)$ is undetermined.*

Remark 22. Let Y be a Polish space. The assumption that each strongly Baire space of the class PCA is an absolute G_δ-set is consistent with ZFC (see [6], p. 1063). Therefore, the determinacy and s-determinacy of $S_0(X, Y)$ for each PCA-set $X \subset Y$ is consistent with ZFC. Furthermore, the assumption that each strongly Baire space is an absolute G_δ-set is consistent with ZF + DC (see [6], p. 1063). Hence it follows that the determinacy and s-

determinacy of the game $S_0(X, Y)$ for each subset X of a Polish space Y are consistent with ZF+DC. From Corollary 21 it follows that the axiom of dependent choice (DC) cannot be replaced by the axiom of choice.

Finally, we shall give several examples which testify to the fact that the class of sets for which the game $S_0(X, Y)$ is determined is not closed for unions, intersection or complements.

Example 1 (AC). Let I be the unit interval [0, 1]. We shall show that there are sets $X_1, X_2 \subset I$ such that $\mathrm{I}\uparrow S_0(X_1, I)$, $\mathrm{I}\uparrow S_0(X_2, I)$ but the game $S_0(X, I)$, where $X = X_1 \cup X_2$, is undetermined.

Let B be a Bernstein set in I. Let us set

$$X_1 = ([0, \tfrac{1}{3}] \cap Q) \cup ([\tfrac{1}{3}, \tfrac{2}{3}] \cap P) \cup ([\tfrac{2}{3}, 1] \cap B),$$

$$X_2 = ([0, \tfrac{1}{3}] \cap P) \cup ([\tfrac{1}{3}, \tfrac{2}{3}] \cap Q) \cup ([\tfrac{2}{3}, 1] \cap B),$$

where P is the set of irrational numbers. Then $X_1, X_2 \notin \mathrm{sB}$, so $\mathrm{I}\uparrow S_0(X_1, I)$ and $\mathrm{I}\uparrow S_0(X_2, I)$. We have

$$X = X_1 \cup X_2 = [0, \tfrac{2}{3}] \cup ([\tfrac{2}{3}, 1] \cap B).$$

Since $X \in \mathrm{sB}$, then $\neg(\mathrm{I}\uparrow S_0(X, I))$. If $\mathrm{II}\uparrow S_0(X, I)$ and A_1 is a Q-set in $[\tfrac{2}{3}, 1]$ chosen by Player I, then similarly as in the proof of Theorem 20, one can show that $[\tfrac{2}{3}, 1] \cap B$ contains a copy of the Cantor discontinuum. This is, however, a contradiction with the assumption that B is a Bernstein set. Hence $\neg(\mathrm{II}\uparrow S_0(X, I))$.

Example 2 (AC). There exist sets $X_1, X_2 \subset I$ such that $\mathrm{I}\uparrow S_0(X_1, I)$ and $\mathrm{I}\uparrow S_0(X_2, I)$ but the game $S_0(X, I)$, where $X = X_1 \cap X_2$, is undetermined.

Let

$$X_1 = ([0, \tfrac{1}{3}] \cap Q) \cup ([\tfrac{1}{3}, \tfrac{2}{3}] \cap B),$$

$$X_2 = ([\tfrac{1}{3}, \tfrac{2}{3}] \cap B) \cup ([\tfrac{2}{3}, 1] \cap Q),$$

where B is a Bernstein set in I. Then $X_1, X_2 \notin \mathrm{sB}$, so $\mathrm{I}\uparrow S_0(X_1, I)$ and $\mathrm{I}\uparrow S_0(X_2, I)$. We have $X = X_1 \cap X_2 = [\tfrac{1}{3}, \tfrac{2}{3}] \cap B$ and by Corollary 21 it follows that the game $S_0(X, I)$ is undetermined.

Example 3 (AC). There is a subset X of I such that $\mathrm{I}\uparrow S_0(X, I)$ and the game $S_0(I-X, I)$ is undetermined.

Let B be a Bernstein set in I. Let us set

$$X = ([0, \tfrac{1}{2}] \cap Q) \cup ([\tfrac{1}{2}, 1] \cap B).$$

Then $X \notin \mathrm{sB}$, so $\mathrm{I}\uparrow S_0(X, I)$. But $I - X = ([0, \tfrac{1}{2}] \cap P) \cap ([\tfrac{1}{2}, 1] \cap B^c)$ and $I - X \in \mathrm{sB}$, so $\neg(\mathrm{I}\uparrow S_0(I-X, I))$. Similarly as in Example 1 one can show that $\neg(\mathrm{II}\uparrow S_0(I-X, I))$. Hence $S_0(I-X, I)$ is undetermined.

Remark 23. One can prove similar results for the following modification of the game $S_0(X, Y)$: Player II wins the play $(A_1, B_1, A_2, B_2, \ldots)$ if $\emptyset \neq \bigcap_n \bar{B}_n \subset X$ or $A_n = \emptyset$ for some $n \in N$; otherwise Player I wins.

Note that if Y is a complete space, then the games are equivalent.

I would like to express my deep gratitude to Professor Rastislav Telgársky for his attention and invaluable help during the preparation of this paper.

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Reçu par la Rédaction le 15. 11. 1982