On the Gross property*

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Abstract. If \( f(z) \) is a function meromorphic and non-constant in the unit disk and if \( f(z) \) has an asymptotic value \( a \), finite or infinite, along a boundary path \( L \) whose end is the unit circumference, then we prove that the inverse function \( z = \varphi(w) \) of \( f(z) \) has the property that each regular functional element of \( z = \varphi(w) \), with center at \( w_0 \neq \infty \), can be continued analytically using only regular elements along each ray \( \arg(w-w_0) = \theta \) up to the point \( w = \infty \), except for at most a set of values \( \theta \) of measure zero.

1. Introduction. Let \( z = \varphi(w) \) be an analytic function whose domain is its Riemann surface \( \Phi \). If, for each \( Q(w; w_0) \), \( w_0 \neq \infty \), a regular functional element of \( z = \varphi(w) \), \( Q(w; w_0) \) can be continued analytically using only regular elements along each ray \( \arg(w-w_0) = \theta \) up to the point \( w = \infty \), except for at most a set of values \( \theta \) of measure zero, then \( z = \varphi(w) \) will be said to have the Gross property. Gross [2] proved that if \( w = f(z) \) is a non-rational meromorphic function in \( |z| < +\infty \) and \( z = \varphi(w) \) is its inverse, then \( z = \varphi(w) \) has the Gross property. More recently, Stebbins [7] proved that if \( f(z) \) is non-constant and meromorphic in \( |z| < 1 \) and \( f(z) \) has \( \infty \) as an asymptotic value along a spiral, then the inverse \( z = \varphi(w) \) of \( f(z) \) has the Gross property. In this paper we prove that if \( f(z) \) is a function of class \( (P^*) \), a larger class of functions than that considered by Stebbins, then the inverse \( z = \varphi(w) \) of \( f(z) \) has the Gross property.

2. Definitions and a lemma. Denote by \( D \) the unit disk in the complex plane and by \( C \) the unit circumference. By a boundary path of \( D \) we shall mean a simple continuous curve \( S: z = s(t), \ 0 \leq t < 1, \) in \( D \) such that \( |s(t)| \rightarrow 1 \) as \( t \rightarrow 1 \). In particular, if \( \arg s(t) \rightarrow +\infty \) or \( \arg s(t) \rightarrow -\infty \) as \( t \rightarrow 1 \), the boundary path \( S \) will be called a spiral in \( D \). The end of a boundary path \( S \), denoted by \( E(S) \), will be the set of limit points of \( S \) on \( C \).

Suppose that the function \( f(z) \) is meromorphic in \( D \) and that \( S \) is a boundary path. We say that \( f(z) \) tends to a complex value \( a \), finite or

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infinite, along $S$, if
\[ \lim f(z) = a \quad \text{for} \quad |z| \to 1, \quad z \in S, \]
and $a$ will be said to be an asymptotic value of $f(z)$ along $S$.

**Definition 1.** Let $(P^*)$ be the class of functions non-constant and meromorphic in $D$ which have an asymptotic value $a$, finite or infinite, along a boundary path $S$ whose end $E(S)$ is $C$.

**Example.** The example we give here is due to Bagemihl, Erdős, and Seidel [1].

Let $\{n_k\}$ be a sequence of increasing positive integers such that
\[ \lim_{k \to +\infty} n_k/n_{k-1} = +\infty, \quad n_1 > 1. \]
Then, let
\[ g(z) = \prod_{j=1}^{\infty} \left( \left( \frac{z}{1-1/n_j} \right)^{n_j} - 1 \right). \]
The function $g(z)$ is holomorphic in $D$ and it possesses $n_j$ simple zeros on the circle $|z| = 1 - 1/n_j$ for $j = 1, 2, \ldots$. Let us, now, denote by $\gamma_j$ the $n_j$ circles of radius $1/j^2 n_j$ with center at the zeros of $g(z)$ on $|z| = 1 - 1/n_j$, and let $\Gamma_j$ denote $\gamma_j$ with its interior. Then there exists $j_0 > 0$ such that all the $\Gamma_j, j > j_0$, are disjoint. If $\Lambda = D - \bigcup_{j=j_0}^{\infty} \Gamma_j$, then $g(z)$ has the property that it converges uniformly to $\infty$ in $\Lambda$ as $|z| \to 1$ [1], p. 137. If we consider $f(z) = 1/g(z)$, then it is clear that (i) $f(z) \in (P^*)$, and, hence, $(P^*) \neq \emptyset$, and (ii) $w = 0$ is the only asymptotic value for $f(z)$, and, hence, $f(z)$ is not in the class of functions considered by Stebbins [7].

In the sequel $w = f(z)$ will be a function of class $(P^*)$ and $z = \varphi(w)$ will be its inverse function with domain the Riemann surface $\Phi$. The functional element $Q(w; \omega_0)$, which may be rational or algebraic, shall serve a double duty, not only representing a functional element of $z = \varphi(w)$, but also representing a point of the Riemann surface $\Phi$. The projection of $Q(w; \omega_0)$ onto the $w$-plane is $w = \omega_0$.

The proof of the main result of this paper depends somewhat on the theory of transcendental singularities. Because of this, we give the following background.

**Definition 2.** Let
\[ w = Q(w; w(t)), \quad 0 \leq t < 1, \]
with \( \lim_{t \to 1} w(t) = \omega \), be a curve on the Riemann surface $\Phi$ of $z = \varphi(w)$.
Then the curve $\Lambda$ defines a transcendental singularity $\Omega$ of $z = \varphi(w)$ on $\Phi$, with projection $w = \omega$, if

(i) for any positive number $\delta, \delta < 1$, the system of functional elements $Q(w; w(t)), 0 \leq t \leq \delta$, defines an analytic continuation (possibly, of algebraic character), but
(ii) for any functional element \( Q(w; w_0) \), rational or algebraic, with center at \( w = \omega \), the system \( Q(w; w(t)) \), \( 0 \leq t < 1 \), where \( w(1) = \omega \), never defines an analytic continuation.

**Definition 3.** Let \( r > 0 \). Suppose that
\[
\lambda : w = Q(w; w(t)), \quad 0 \leq t < 1,
\]
with \( \lim_{t \to \infty} w(t) = \omega \), defines a transcendental singularity \( \Omega \) on \( \phi \). Let \( t_r \) be the last value for \( t \), \( 0 \leq t < 1 \), such that \( |w(t_r) - \omega| = r \), counting from \( t = 0 \). Then by an \( r \)-neighborhood of \( \Omega \), denoted by \( U_r(\Omega) \), we mean all points \( Q(w; c) \) of \( \phi \) such that \( |c - \omega| < r \) and \( Q(w; c) \) is an analytic continuation (possibly, of algebraic character) of \( Q(w; w(t_r)) \) along a curve lying inside the disk \( |w - \omega| < r \). If the transcendental singularity lies above the point \( w = \infty \), then the circle we use to define \( U_r(\Omega) \) is \( |w| = r \) and the disk used is \( |w| > r \).

**Lemma.** Let \( \lambda : w = Q(w; w(t)) \), \( 0 < t < 1 \), with \( \lim_{t \to 1} w(t) = \omega \), define a transcendental singularity \( \Omega \) on \( \phi \). Then
\[
\bigcap_{r > 0} U_r(\Omega) = \emptyset.
\]

**Proof.** Suppose that there exists a functional element \( Q(w; w_0) \) such that \( Q(w; w_0) \in \bigcap_{r > 0} U_r(\Omega) \). Let \( \{r_n\} \) be a sequence of positive numbers which decrease to zero as \( n \to +\infty \). Then,
\[
Q(w; w_0) \in \bigcap_{r > 0} U_r(\Omega) \subset \bigcap_{n=1}^{\infty} U_{r_n}(\Omega).
\]
Thus, \( Q(w; w_0) \in U_{r_n}(\Omega) \) for every \( n \). This implies that \( |w_0 - \omega| < r_n \) for every \( n \). Thus, \( w_0 = \omega \), and \( Q(w; w_0) = Q(w; \omega) \).

Suppose that \( p \) is the radius of convergence of \( Q(w; \omega) \). There exists \( t_0, 0 < t_0 < 1 \), such that \( |w(t) - \omega| < p \) for all \( t, t_0 < t < 1 \). Thus, by definition 3, \( Q(w; w(t)) \in U_p(\Omega) \) for all \( t, t_0 < t < 1 \). Also, \( Q(w; \omega) \in U_p(\Omega) \). Hence, for each \( t, t_0 < t < 1 \), \( Q(w; w(t)) \) is an analytic continuation of \( Q(w; \omega) \) along a path lying in the circle of convergence of \( Q(w; \omega) \). It follows, then, that \( Q(w; w(t)) \), for \( t_0 < t < 1 \), is a direct analytic continuation of \( Q(w; \omega) \). Thus, we may adjoin \( Q(w; \omega) \) to \( \lambda \) to complete the analytic continuation of \( \lambda \) for \( 0 \leq t \leq 1 \). This is a contradiction. Thus,
\[
\bigcap_{r > 0} U_r(\Omega) = \emptyset.
\]

The importance of transcendental singularities is that there exists a one-to-one correspondence between the transcendental singularities of \( z = \varphi(w) \) and the asymptotic boundary paths of \( w = f(z) \), where \( z = \varphi(w) \) is the inverse of \( w = f(z) \). This result was first proved for \( f(z) \)
an entire function by Iversen [3]. Later, Noshiro [5] proved this result for \( f(z) \) analytic on a Riemann surface \( F \).

3. The main result.

**Theorem.** Let \( w = f(z) \in (P^*) \) and let \( z = \varphi(w) \) be its inverse. Then \( z = \varphi(w) \) has the Gross property.

**Proof.** We remark that part of our proof is patterned after the proof of the Gross Star theorem given in Nevanlinna [4], p. 292–294.

We continue the regular functional element \( Q(w; w_\theta) \) of \( z = \varphi(w) \) analytically with regular elements along the ray \( \arg(w - w_\theta) = \theta \) until either a singular point (algebraic or transcendental singularity) or the point \( w = \infty \) is reached. Let \( S_\theta \) be the resulting (simply-connected) star-shaped region in the \( w \)-plane made up of such segments.

Let \( M \) be the set of values \( \theta \) in \( 0 \leq \theta \leq 2\pi \) such that the ray \( \arg(w - w_\theta) = \theta \) of the star-shaped region \( S_\theta \) terminates in a finite transcendental singularity at \( w = w_\theta \) and this is the first singularity encountered on \( \arg(w - w_\theta) = \theta \) as one continues \( Q(w; w_\theta) \) analytically on \( \arg(w - w_\theta) = \theta \) from \( w = w_\theta \). Since all the zeros of \( f'(z) \) are isolated points, we have that the set of algebraic singularities is countable. Thus, to prove our theorem it will suffice to show that \( m^* (M) = 0 \) \( (m^* (M) \) denotes the outer Lebesgue measure of the set \( M \).

Let \( R > 0 \). Let \( M_R = \{ \theta : \theta \in M \text{ and } |w_\theta - w_0| < R \} \). Since \( M_{R_1} \subseteq M_{R_2} \) for \( R_1 < R_2 \) and \( M = \bigcup_{n=1}^{\infty} M_n, m^* (M) = \lim_{n \to \infty} m^* (M_n) \). Thus, it suffices to show that \( m^* (M_n) = 0 \) for an arbitrary integer \( n \).

Since \( f(z) \in (P^*), f(z) \to \alpha \) on a boundary path \( S: z = s(t), 0 \leq t < 1 \), with \( E(S) = C \). In order to prove our theorem we must consider the following three cases: (i) \( \alpha \) is finite and \( w_\theta \neq \alpha \), (ii) \( \alpha \) is finite and \( w_\theta = \alpha \), and (iii) \( \alpha \) is infinite.

Let \( \alpha \) be finite and, further, let \( \alpha \neq w_\theta \). Let \( p \) be the radius of convergence of the regular element \( Q(w; w_\theta) \). Choose an integer \( n \) sufficiently large so that \( |\alpha - w_\theta| < n \). Let \( p_0 = \frac{1}{2} \min(|\alpha - w_\theta|, p) \). For each integer \( m > 1 \) we define the set \( W_m \) as follows:

\[
W_m = \{ w : |\arg(w - w_\theta) - \arg(\alpha - w_\theta)| \leq \pi/m \text{ and } |w_\theta| \leq |w - w_\theta| \leq n \}. 
\]

Let \( M_{n,m} = \{ \theta : \theta \in M_n \text{ and } w_\theta \in W_m \} \). Since \( M_{n,m+1} \supseteq M_{n,m} \) and \( M_n - \{ \arg(\alpha - w_\theta) \} = \bigcup_{m=2}^{\infty} M_{n,m} \), we have \( m^* (M_n) \leq \lim_{m \to \infty} m^* (M_{n,m}) \). Thus, it suffices to show that \( m^* (M_{n,m}) = 0 \) for an arbitrary integer \( m \).

Let us, now, assume that \( m \) is an arbitrary, but fixed, integer. Suppose, also, that \( M_{n,m} \neq \emptyset \). Since \( f(z) \to \alpha \) on \( S \), there exists \( t_0, 0 < t_0 < 1 \), such that \( f(\alpha) \in W_m \) for all \( t, t_0 < t < 1 \). Let \( S^1 : z = s(t), t_0 < t < 1 \). Clearly, \( E(S^1) = E(S) = C \). We map the simply-connected region \( D - S^1 \) in a one-
to-one conformal manner by $\zeta = \zeta(z)$ onto the disk $|\zeta| < 1$ in such a way that the unique prime end $P$ of $D - S^1$ whose impression is $C$ corresponds to $\zeta = 1$ under $\zeta = \zeta(z)$.

Let $S^* = [S_0 \cap \{|w - w_0| < n\}] - W_m$. We consider the function $F(w) = \zeta(\phi(w))$ on $S^*$. Then, $F(w)$ maps $S^*$ in a one-to-one conformal manner onto a simply-connected subregion $B$ of $|\zeta| < 1$. Each ray of $S^*$ which terminates at a transcendental singularity of $S^*$ is mapped by $\phi(w)$ onto a boundary path $L$ in $D$ disjoint from $S^1$, and $L$ is mapped by $\zeta = \zeta(z)$ onto a path in $B$ which terminates at $\zeta = 1$ (since $E(S^1) = E(S) = C$).

At this point, we remark that the remainder of the proof of case (i), which we are about to give, will apply also to case (ii) and (iii).

Consider the function

$$t(w) = \frac{F(w) - 1}{F(w) - F(w_0)}$$

on $S^*$. The function $t(w)$ is a one-to-one conformal map of $S^*$ onto a schlicht region $S_t$. Since $w_0$ is an interior point of $S^*$, $t = \infty$ is an interior point of $S_t$. The point $t = 0$ is an exterior point or a boundary point of $S_t$. Since

$$|F(w) - F(w_0)| \leq |F(w)| + |F(w_0)| \leq 2,$

$t(w) \to 0$ if and only if $F(w) \to 1$. Since we have assumed that $M_{n,m} \neq \emptyset$ there exists a ray $\arg(w - w_0) = \theta$ of $S^*$ which terminates at a transcendental singularity at $w = w_0$. Along this ray $F(w)$ tends to 1 as $w$ tends to $w_0$. Thus, $t(w)$ tends to 0 as $w$ tends to $w_0$ along $\arg(w - w_0) = \theta$. Hence, $t = 0$ is a boundary point of $S_t$.

Let $r > 0$. Let $A_t(r)$ be the finite or countable collection of component arcs of $|t| = r$ which fall into the region $S_t$. These arcs separate the boundary point $t = 0$ from the interior point $t = \infty$ of $S_t$.

Let $A_w(r)$ be the image arcs of the arcs of $A_t(r)$ under the map $w(t)$ which is the inverse of $t(w)$. The arcs of $A_w(r)$ are crosscuts of $S^*$ which separate the transcendental singularities of $S^*$ from $w = w_0$. Therefore, $m^*(M_{n,m})$ is less than or equal to the sum of the variations of $\arg(w - w_0)$ on the crosscuts $A_w(r)$. But, the sum of the variations of $\arg(w - w_0)$ on the crosscuts $A_w(r)$ is less than or equal to $s(r)/d_r$, where $s(r)$ is the sum of the lengths of the crosscuts $A_w(r)$ and $d_r$ is the shortest distance from $w = w_0$ to the crosscuts $A_w(r)$.

We now fix $r_o, \ r_o > 0$. Since $w_0 \in S^*$ corresponds to $\infty \in S_t$ under $t(w)$ and since $r_o$ is fixed and finite, $d_r > 0$. Then, for $0 < r < r_o$, $d_r > d_r$. Hence,

$$m^*(M_{n,m}) \leq s(r)/d_r \leq s(r)/d_{r_o}.$$

Thus, it suffices to show that $s(r)$ can be made arbitrarily small by a suitable selection of $r$. 
Let \( z = z(\zeta) \) be the inverse of \( \zeta = \xi(x) \). Let \( t = re^{i\theta} \). Let

\[
G(t) = f \left( \frac{t F(w_0) - 1}{t - 1} \right).
\]

Then \( G(t) \) maps \( S_1 \) onto \( S^\ast \). By the Schwarz inequality

\[
(s(r))^2 = \left( \int_{a}^{b} |G'(t)|^2 \, dt \right)^2 \leq \left( \int_{a}^{b} |G'(t)| \, dt \right)^2 \leq 2\pi r \int_{a}^{b} |G'(t)|^2 r \, d\theta.
\]

But

\[
\int_{a}^{b} |G'(t)|^2 r \, d\theta \leq \frac{dA}{dr},
\]

where \( A(r) \) denotes the area of that subregion of \( S^\ast \) which contains the point \( w = w_0 \) and which is bounded by the crosscuts \( A_w(r) \). Integrating between \( r \) and \( r_0 \) \((r_0 > r)\), we get

\[
(1) \quad \int_{r}^{r_0} \frac{(s(r))^2}{r} \, dr \leq - \int_{r}^{r_0} \frac{2\pi r}{r} \frac{dA}{dr} \, dr = -2\pi (A(r_0) - A(r)) \leq 2\pi^2 n^2.
\]

This inequality holds for all \( r, 0 < r < r_0 \).

Let \( \delta > 0 \). Suppose \( s(r) \geq \delta \) for all \( r, 0 < r < r_0 \). Then

\[
\int_{r}^{r_0} \frac{(s(r))^2}{r} \, dr \geq \delta^2 \log \frac{r_0}{r} \to +\infty
\]

as \( r \to 0 \). This contradicts inequality (1) above. Thus, there exists a number \( r_1, 0 < r < r_0 \), such that \( s(r) < \delta \). Therefore, \( m^\ast(M_{n,m}) < \delta/d_{a_0} \). Since \( \delta \) is an arbitrary positive number, \( m^\ast(M_{n,m}) = 0 \). This completes the proof for case (i).

We consider the case with \( a \) finite and \( w_0 = a \). By the lemma above, there exists \( p_0 > 0 \) such that the functional element \( Q(w; w_0) = Q(w; a) \) \( \notin U_{p_0}(\Omega) \), where \( \Omega \) is the transcendental singularity of \( z = \varphi(w) \) which corresponds to the asymptotic boundary path \( S \). Let \( p^\ast = \min(p_0, p) \), where \( p \) is the radius of convergence of \( Q(w, a) \). There exists \( t_0, 0 < t_0 < 1 \), such that \( |f(s(t)) - a| < p^\ast \) for all \( t, t_0 \leq t < 1 \). Let \( S^1: z = s(t), t_0 \leq t < 1 \). We map the simply-connected region \( D - S^1 \) onto \( |\zeta| < 1 \) in the same manner as before by the one-to-one conformal map \( \zeta = \xi(x) \).

Let \( n \) be any fixed integer with \( n > p_0 \). Let \( S^\ast = S_0 \cap \{|w - a| < n\} \). The function \( F(w) = \zeta(\varphi(w)) \) maps \( S^\ast \) in a one-to-one conformal manner onto a simply-connected subregion \( B \) of \( |\zeta| < 1 \). The image of \( S^1 \) under \( w = f(z) \) lies in \( U_{p^\ast}(\Omega) \). Each ray of \( S^\ast \) which terminates at a transcendental singularity is mapped by \( z = \varphi(w) \) onto a boundary path \( L \) in \( D \). The path \( L \) is disjoint from \( S^1 \) in \( D \). Indeed, if not, then any point common
to $S^1$ and $L$ is mapped by $f(z)$ onto a point $Q(w; b)$ with $|b - a| < p^*$. Then $Q(w; b) \in U_{p^*}(\Omega)$ and $Q(w; b)$ is a direct analytic continuation of $Q(w; a)$. Hence, $Q(w; a) \in U_{p^*}(\Omega)$. But this is impossible. Thus, $L$ is disjoint from $S^1$ in $D$ and, hence, $L$ is mapped by $\zeta = \zeta(z)$ onto a path in $B$ which terminates at $\zeta = 1$.

We now consider the function $t(w)$ on $S^*$ of case (i) and we use the analysis of case (i) to show $m^*(M_n) = 0$.

For our last case we begin by assuming $a = \infty$. Then, for a fixed integer $n$, the exists $t_0$, $0 < t_0 < 1$, such that $|f(s(t))| > |w_0| + n$ for all $t$, $t_0 < t < 1$. Let $S^1: z = s(t)$, $t_0 < t < 1$. Let $\zeta = \zeta(z)$ be the map of case (i) which maps $D - S^1$ onto $|\zeta| < 1$. Let $S^* = S_0 \cap \{ w - w_0 | < n\}$. We consider the one-to-one conformal map $F(w) = \zeta(\varphi(w))$ on $S^*$ which maps $S^*$ onto a subregion $B$ of $|\zeta| < 1$. Each ray of $S^*$ which terminates at a transcendental singularity is mapped by $\varphi(w)$ onto a boundary path $L$ in $D$ disjoint from $S^1$, and $L$ is mapped by $\zeta = \zeta(z)$ onto a path in $B$ which terminates at $\zeta = 1$.

We can now form the function $t(w)$ on $S^*$ as in case (i) and once again, apply the analysis of case (i) to prove $m^*(M_n) = 0$. This completes the proof of our theorem.

Remark. Although the Gross property is a necessary condition for the inverse function of a function of class $(P^*)$, it is not a sufficient condition. The inverse of the elliptic modular function $w = \wp(z)$ for the unit disk has the Gross property, but $\wp(z) \in (P^*)$.

Corollary 1. Let $w = f(z) \in (P^*)$ and let $z = \varphi(w)$ be its inverse function with Riemann surface $\Phi$. Let $Q(w; w_0)$ be an arbitrary functional element (regular or algebraic) of $z = \varphi(w)$ with center $w_0$ lying in $|w - a| < r$. Then a continuous path $L$ can be found lying inside $|w - a| < r$, starting at $w = w_0$ and terminating at $w = a$, such that there exists an analytic continuation of $Q(w; w_0)$ of algebraic character on $\Phi$ above $L$ except perhaps at the endpoint $w = a$ of $L$ (Iversen's property).

Corollary 2. Let $f(z) \in (P^*)$. If $a$ is a complex value, finite or infinite, which is taken by $f(z)$ only finitely many times in $D$, then $a$ is an asymptotic value of $f(z)$ along some boundary path $L$ of $D$.

Proof. The proof is the same as that given in Noshiro [6], p. 4–5.

References


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