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On a Fučík problem

by VALTER ŠEDA (Bratislava)

Dedicated to the memory of Jacek Szarski

In paper [6] S. Fučík considered the boundary value problem $u''+h(u) = f(t)$, $u(0) = u(\pi) = 0$, where h is a continuous function such that

$$\lim_{u\to\infty}\frac{h(u)}{u} \neq \lim_{u\to-\infty}\frac{h(u)}{u}.$$

He called a function h with this property *jumping*. In the mentioned paper he solved a number of problems with jumping non-linearities using the concept of the Leray–Schauder degree of a mapping. The problems have also been attacked in the book [7]; they can be handled in the framework of the so-called *Fredholm alternative* for non-linear operators [5]; for other results see paper [9] by J. Nečas. In [6] S. Fučík formulated many open problems both for ordinary and partial differential (d.) equations. L. Aguinaldo and K. Schmitt [1] solved one of the listed problems using the famous Mawhin continuation theorem.

In this paper the problem of jumping non-linearities is generalized to the case where h depends not only on u and the existence of limits at ∞ and $-\infty$ is not necessary. The method of solving problems is elementary, but it goes to the core of the problem. Although the statement of the results resembles the Fredholm alternative for non-linear operators, no abstract theory is given.

Let $n > 1$ be an integer, $a > 0$ and $p \in C([0, \pi])$. Write $x^- = \max\{-x, 0\}$ and consider the boundary value problem

$$x''+n^2x = ax^-+p(t) \quad (0 \leqslant t \leqslant \pi), \tag{1}$$

$$x(0) = 0 = x(\pi). \tag{2}$$

First we show that the method of lower and upper solutions of (1), (2), as given in Theorem 4, [10], p. 528, is not applicable. Recall that a function

$f \in C^2([0, \pi])$ is a lower (upper) solution of (1) in the interval $i \subset [0, \pi]$ if $f'' + n^2 f \geqslant af^- + p(t)$ $(f'' + n^2 f \leqslant af^- + p(t))$ in i. It is known that any solution of (1) can be extended to the whole interval $[0, \pi]$ and it is uniquely determined by the initial conditions.

LEMMA 1. *Suppose that x is a lower and y an upper solution of* (1) *on* $[a, b] \subset [0, \pi]$ *such that* $x(t) \leqslant y(t)$ *and* $x(t) \not\equiv y(t)$ *in* $[a, b]$. *Then* $b - a \leqslant \pi/n$ *and, in the case of* $b - a = \pi/n$, x *and* y *satisfy* (1) *in* $[a, b]$, $x(a) = y(a)$, $x(b) = y(b)$ *and* $x(t) \geqslant 0$, $y(t) \geqslant 0$ *in* $[a, b]$.

Proof. x (resp. y) fulfils the d. equation $x'' + n^2 x = ax^- + p_1(t)$ (resp. $y'' + n^2 y = ay^- + p_2(t)$) in $[a, b]$ with $p_1(t) \geqslant p(t) \geqslant p_2(t)$ in $[a, b]$. Thus $u(t) = y(t) - x(t)$ satisfies $u'' + n^2 u = a\big(y^-(t) - x^-(t)\big) + p_2(t) - p_1(t)$ in $[a, b]$ and the variation of parameters formula yields

$$u(t) = v(t) + \int_a^t \frac{\sin n(t-s)}{n} \big[a\big(y^-(s) - x^-(s)\big) + p_2(s) - p_1(s)\big]\, ds \tag{3}$$

$$(a \leqslant t \leqslant b),$$

where v is the solution of $v'' + n^2 v = 0$ determined by $v(a) = y(a) - x(a) \geqslant 0$ and $v'(a) = y'(a) - x'(a)$. Suppose that $b - a \geqslant \pi/n$. As $y^-(t) - x^-(t) \leqslant 0$ in $[a, b]$, (3) implies that

$$0 \leqslant u(t) \leqslant v(t) \quad \text{in } [a, a + \pi/n]. \tag{4}$$

Two cases may occur. If $v(a) > 0$, then there is a t_1, $a < t_1 < a + \pi/n$, such that $v(t) > 0$ in $[a, t_1)$, $v(t_1) = 0$ and $v(t) < 0$ in $(t_1, t_1 + \varepsilon)$ for $\varepsilon > 0$ sufficiently small. This contradicts (4) and hence $b - a < \pi/n$.

If $v(a) = 0$, then $v(t) = A \sin n(t - a)$ for an $A \geqslant 0$, $v(a + \pi/n) = 0$, and by (4) $u'(a) \geqslant 0$, $u'(a + \pi/n) \leqslant 0$. (3) yields

$$0 \leqslant u(a + \pi/n)$$
$$= \int_a^{a+\pi/n} \frac{\sin n(a + \pi/n - s)}{n} \big[a\big(y^-(s) - x^-(s)\big) + p_2(s) - p_1(s)\big]\, ds \leqslant 0$$

and thus $p_1(t) = p(t) = p_2(t)$ in $[a, a + \pi/n]$, which shows that x and y are two solutions of (1) in $[a, a + \pi/n]$. At the same time $u(a + \pi/n) = 0$. If $u'(a + \pi/n) = 0$ and $b - a > \pi/n$, then we obtain (4) on $[a + \pi/n, c]$ with $c = \min(a + 2\pi/n, b)$, but the initial conditions for v at $a + \pi/n$ imply $v(t) \equiv 0$ in $[a + \pi/n, c]$ and thus $u(t) \equiv 0$. Proceeding in this way we arrive at the conclusion that x and y are identical on $[a, b]$, which contradicts the assumption. Hence $b - a \leqslant \pi/n$. If $u'(a + \pi/n) < 0$, we get $b - a = \pi/n$.

By inspecting the proof we see that $b - a = \pi/n$ implies that x and y are two different solutions of (1) in $[a, b]$ and $x(a) = y(a)$, $x(b) = y(b)$, as well as $x(t) \geqslant 0$, $y(t) \geqslant 0$ in $[a, b]$.

Now consider two different solutions x, y of (1) in $[0, \pi]$. By Lemma 1 their graphs have at least one point of intersection in each interval $[a, b] \subset [0, \pi]$ with $b-a \geqslant \pi/n$. On account of the uniqueness of initial value problems for (1) the total number of points of intersection is finite. Call the t-coordinate of any point of intersection a *knot* of the pair of solutions x, y. We can order the set of all knots of the pair x, y into a finite increasing sequence $0 \leqslant t_0 < t_1 < \ldots < t_m \leqslant \pi$, where $m \geqslant n$.

The estimation from below of the distance of two consecutive knots is given by the following lemma.

LEMMA 2. *Let x and y be two solutions of* (1) *on $[a, b] \subset [0, \pi]$ such that $x(a) = y(a)$, $x(b) = y(b)$, $x(t) < y(t)$ in (a, b). Then $b-a \geqslant \pi/\sqrt{n^2+a}$ and if $b-a = \pi/\sqrt{n^2+a}$, then $x(t) \leqslant 0$, $y(t) \leqslant 0$ in $[a, b]$.*

Proof. Write $c^+ = \max\{c, 0\}$. Then $c = c^+ - c^-$ and the difference $u(t) = y(t)-x(t)$ $(a \leqslant t \leqslant b)$ satisfies $u''+(n^2+a)u = a\big(y^+(t)-x^+(t)\big)$. Thus

$$(5) \qquad u(t) = w(t) + \int_a^t \frac{\sin\sqrt{n^2+a}(t-s)}{\sqrt{n^2+a}}\, a[y^+(s)-x^+(s)]ds \qquad (a \leqslant t \leqslant b)$$

where w is the solution of $w''+(n^2+a)w = 0$ determined by $w(a) = y(a) - x(a) = 0$ and $w'(a) = y'(a)-x'(a) > 0$. Hence $w(t) = A\sin\sqrt{n^2+a}(t-a)$ with $A > 0$. Suppose that $b-a < \pi/\sqrt{n^2+a}$. Then (5) implies that $u(t) \geqslant A\sin\sqrt{n^2+a}(t-a)$ $(a \leqslant t \leqslant b)$, which contradicts the assumption $u(b) = 0$. Therefore $b-a \geqslant \pi/\sqrt{n^2+a}$. If $b-a = \pi/\sqrt{n^2+a}$, then $y^+(t) - x^+(t) \equiv 0$ in $[a, b]$, which in view of $x(t) < y(t)$ in (a, b) is equivalent to $y(t) \leqslant 0$, $x(t) \leqslant 0$ in $[a, b]$.

The next lemma gives a sufficient condition for three solutions of (1) to have the same consecutive knots.

LEMMA 3. *Let x, y and z be three solutions of* (1) *in $[a, b] \subset [0, \pi]$ such that $x(a) = y(a)$, $x(b) = y(b)$, $z(t) < x(t) < y(t)$ in (a, b). Then $z(a) = x(a) = y(a)$, $z(b) = x(b) = y(b)$.*

Proof. For the differences $y-x$, $y-z$ we have in $[a, b]$ the equalities

$$\big(y(t)-x(t)\big)''+n^2\big(y(t)-x(t)\big) = a[y^-(t)-x^-(t)],$$
$$\big(y(t)-z(t)\big)''+n^2\big(y(t)-z(t)\big) = a[y^-(t)-z^-(t)].$$

Multiplying the first equality in $[a, b]$ by $y(t)-z(t)$ and the second one by $-\big(y(t)-x(t)\big)$ and adding the resulting equalities, we get

$$\big(y(t)-x(t)\big)''\big(y(t)-z(t)\big)-\big(y(t)-x(t)\big)\big(y(t)-z(t)\big)'' = a\Big[\big(y^-(t)-x^-(t)\big)\big(y(t)-z(t)\big)-\big(y^-(t)-z^-(t)\big)\big(y(t)-x(t)\big)\Big].$$

In view of $y(b) = x(b)$, $y(a) = x(a)$, integration over $[a, b]$ yields

$$(6)\qquad \big(y'(b)-x'(b)\big)\big(y(b)-z(b)\big) - \big(y'(a)-x'(a)\big)\big(y(a)-z(a)\big)$$
$$= a\int\limits_a^b \Big[\big(y^-(t)-x^-(t)\big)\big(y(t)-z(t)\big) - \big(y^-(t)-z^-(t)\big)\big(y(t)-x(t)\big)\Big]\,dt.$$

Since the integrand on the right-hand side of (6) is non-negative and the left-hand side is non-positive, the two sides must be equal to 0 and thus $y(b)-z(b) = 0$, $y(a)-z(a) = 0$, which was to be proved.

Now consider the homogeneous equation

$$(7)\qquad x''+n^2x = ax^-.$$

To this equation there corresponds a unique natural k such that

$$(8)\qquad 0 \leqslant \beta = \pi - k\left(\frac{\pi}{n}+\frac{\pi}{\sqrt{n^2+a}}\right) < \frac{\pi}{n}+\frac{\pi}{\sqrt{n^2+a}}.$$

If x, y are two different solutions of the problem (7),

$$(9)\qquad x(0) = 0,$$

then we get the following relations for their knots:

$$t_{2j} = j\left(\frac{\pi}{n}+\frac{\pi}{\sqrt{n^2+a}}\right) \quad (j = 0, 1, \ldots, k),$$

while the odd knots satisfy the inequalities

$$(10)\qquad t_{2j}+\frac{\pi}{\sqrt{n^2+a}} \leqslant t_{2j+1} \leqslant t_{2j}+\frac{\pi}{n} \quad (j = 0, 1, \ldots, k-1 \text{ or } k).$$

t_{2k} is the last knot iff $0 \leqslant \beta < \pi/\sqrt{n^2+a}$. Otherwise t_{2k+1} is the last one. Especially, if x is a non-trivial and z a trivial solution of the problem (7), (9), then instead of (10) we have

$$t_{2j+1} = t_{2j}+\frac{\pi}{n} \quad \left(t_{2j+1} = t_{2j}+\frac{\pi}{\sqrt{n^2+a}}\right) \quad (j = 0, 1, \ldots, k-1 \text{ or } k)$$

according to whether $x'(0) > 0$ or $x'(0) < 0$. Therefore the problem (7), (2) has only the trivial solution iff one of the following three cases occurs:

$$(11)\qquad 0 < \beta < \frac{\pi}{\sqrt{n^2+a}},$$

$$(12)\qquad \frac{\pi}{\sqrt{n^2+a}} < \beta < \frac{\pi}{n},$$

$$(13)\qquad \frac{\pi}{n} < \beta < \frac{\pi}{n}+\frac{\pi}{\sqrt{n^2+a}}.$$

Define a mapping $F: R \to R$ by: $F(c) = d$ iff the solution x of (7), (9) (in $[0, \pi]$) with $x'(0) = c$ is such that $x(\pi) = d$. The d. equation (7) has the property that kx is its solution for any $k \geqslant 0$ if x is. This implies that $F(k) = kF(1)$ and $F(-k) = kF(-1)$ for any $k \geqslant 0$. A short computation shows that if β satisfies (11) then $F(1) > 0$ and $F(-1) < 0$, if β satisfies (12) then both $F(1)$, $F(-1)$ are positive and, finally, for β satisfying (13) we obtain $F(1) < 0$ and $F(-1) > 0$.

According to (8), β can also satisfy one of the relations:

$$\beta = 0, \tag{14}$$

$$\beta = \frac{\pi}{\sqrt{n^2+a}}, \tag{15}$$

and

$$\beta = \frac{\pi}{n}. \tag{16}$$

If (14) holds, then $F(1) = F(-1) = 0$. In the case (15) we have $F(1) > 0$ and $F(-1) = 0$, and in the case (16) we have $F(-1) > 0$ and $F(1) = 0$. The behaviour of F gives existence statements for the boundary value problem (7),

$$x(0) = 0, \quad x(\pi) = d. \tag{17}$$

Consider the problem (1), (17). For equation (1) the function F_p can be defined just as F was defined for equation (7), F_p is continuous on R and by Theorem 2.1, [3], pp. 8, 19, which is also valid for vector d. equations, there exists an $M > 0$ depending only on p, n, a such that $|F_p(c) - F(c)| \leqslant M$ $(c \in R)$. The properties of F imply the following lemma.

LEMMA 4. *The problem* (7), (2) *has only the trivial solution iff β satisfies one of the relations* (11)–(13). *If β satisfies* (11) *or* (13), *then the boundary value problem* (1), (17) *has a solution for each $d \in R$. In the case* (12) *there exists a $d_0 \in R$ such that* (1), (17) *has a solution for each $d \geqslant d_0$ and no solution for $d < d_0$. If β fulfils* (14), *then two cases may occur: Either the problem* (1), (17) *has a solution exactly for one d or there are $d_1 < d_2$ in R such that* (1), (17) *has a solution for each d, $d_1 < d < d_2$, and no solution for $d < d_1$ and $d > d_2$. If β satisfies* (15) *or* (16), *there is a $d_0 \in R$ such that* (1), (17) *has a solution for each $d > d_0$ and no solution for $d < d_0$.*

Remarks. 1. Lemma 4 shows that if β satisfies (12), then the problem (1), (2) need not have a solution, although the corresponding homogeneous problem (7), (2) has only the trivial solution. In the case (11) and (13) the problem does have a solution.

2. The uniqueness of solution to (1), (2) may fail in the case where there exists a non-negative solution of (1), (2), as the following example shows.

EXAMPLE. The problem (1), (2) with $p(t) = (n^2-1)\sin t$ $(0 \leqslant t \leqslant \pi)$ has as solutions a non-negative function $x(t) = \sin t$ and infinitely many perturbations of that solution $y(t) = \sin t + k \sin nt$ (for all $k > 0$ sufficiently small) which are, of course, also non-negative.

3. We know that (n being fixed) for each $a > 0$ there exists a unique natural $k = k(a)$ such that (8) holds. The same is true for $a = 0$. The function k defined in this way is non-decreasing, piecewise constant and continuous from the right in $[0, \infty)$. When $n = 2m$, then $k(0) = m$ and $0 < a < 8m+4$ is a sufficient condition for (11) to be true. In the case of $n = 2m+1$, we again have $k(0) = m$ and the condition $0 < a < 8m+12+6/m+1/m^2$ implies (13).

Lemma 4 guarantees the existence of a solution to (1), (2). Under the assumption of the lemma an estimate for solutions of that problem can be given; this is the contents of the next lemma. To formulate it we denote the sup-norm in $C([0, \pi])$ by $\|\cdot\|$ and we define the norm in $C_1([0, \pi])$ by $\|x\|_1 = \|x\| + \|x'\|$ $(x \in C_1([0, \pi]))$.

LEMMA 5. *Let* (11) *or* (12) *or* (13) *be true. Then there exists a* $K = K(n, a) > 0$ *such that for any solution* x *of the problem* (1), (2) *we have*

$$\|x\|_1 \leqslant K \|p\|. \tag{18}$$

Proof. Denote by y the solution of the initial value problem $y'' + n^2 y = a y^-$, $y(0) = x(0) = 0$, $y'(0) = x'(0) = c$. By Theorem 2.1 [3], pp. 8, 19, we have the estimate

$$|x(t)-y(t)| + |x'(t)-y'(t)| \leqslant \frac{\|p\|}{n^2+a}\left(e^{(n^2+a)t}-1\right) \quad (0 \leqslant t \leqslant \pi), \tag{19}$$

which implies that

$$|y(\pi)| \leqslant \frac{\|p\|}{n^2+a}\left(e^{(n^2+a)\pi}-1\right). \tag{20}$$

Let F have the same meaning as before; then (20) gives an estimate for $|cF(1)|$ or for $|cF(-1)|$, according to whether $c \geqslant 0$ or $c < 0$. Hence, denoting $\min(|F(1)|, |F(-1)|) = m$ and

$$\frac{1}{m} \cdot \frac{1}{n^2+a}\left(e^{(n^2+a)\pi}-1\right) = k,$$

we get

$$|x'(0)| = |c| \leqslant k\|p\|. \tag{21}$$

Clearly, k depends only on n, a.

Now we multiply the equality for x resulting from equation (1) by $2x'$ and integrate in $[0, t] \subset [0, \pi]$. We obtain

$$x'^2(t)+n^2x^2(t) = x'^2(0)+a\int_0^t 2x^-(s)x'(s)\,ds + \int_0^t 2p(s)x'(s)\,ds \qquad (0 \leqslant t \leqslant \pi)$$

and thus

$$x'^2(t)+n^2x^2(t) \leqslant \Big(x'^2(0)+\int_0^\pi p^2(s)\,ds\Big)+a\int_0^t x^2(s)\,ds+(a+1)\int_0^t x'^2(s)\,ds$$
$$\leqslant \Big(x'^2(0)+\int_0^t p^2(s)\,ds\Big)+(a+1)\int_0^t [x'^2(s)+n^2x^2(s)]\,ds.$$

The Gronwall lemma then gives the estimate

$$x'^2(t)+n^2x^2(t) \leqslant \Big(x'^2(0)+\int_0^\pi p^2(s)\,ds\Big)e^{(a+1)\pi} \qquad (0 \leqslant t \leqslant \pi). \tag{22}$$

(22) and (21) imply (18) with

$$K = \frac{n+1}{n}\exp\big(\tfrac{1}{2}(a+1)\pi\big)(k^2+\pi)^{1/2}.$$

The next lemma follows from a modification of the Kneser theorem (see Lemma 2.1.1 in [2], p. 95).

LEMMA 6. *Let f: $[0, \pi]\times R^2 \to R$ be continuous and assume that all solutions of*

$$x''+n^2x = ax^-+f(t, x, x') \tag{23}$$

can be extended to $[0, \pi]$. Let $[a, b]$ be a compact interval. Then the set $G = \{x(\pi)$: x is a solution of (23) which satisfies

$$x(0) = 0, \quad x'(0) = c \tag{24}$$

and $c \in [a, b]\}$ is a compact interval or a one-point set.

Now we generalize Lemma 4 as follows:

THEOREM 1. *Let f: $[0, \pi]\times R^2 \to R$ be continuous and such that*

$$\lim_{|x|+|y|\to\infty} \frac{|f(t, x, y)|}{|x|+|y|} = 0 \tag{25}$$

uniformly in $t \in [0, \pi]$. Let β be determined by (8). Then the following statements are true:

If β satisfies (11) *or* (13), *then the boundary value problem* (23), (17) *has a solution for every $d \in R$. If β satisfies* (12), *then there exists a $d_0 \in R$ such that* (23), (17) *has a solution for all $d \geqslant d_0$ and no solution for $d < d_0$. If β fulfils* (14), *then two cases may occur: Either the problem* (23), (17) *has a solution exactly for one $d \in R$ or there are d_1, d_2 $(-\infty \leqslant d_1 \leqslant d_2 \leqslant \infty)$ such that* (23), (17) *has a solution for each d, $d_1 < d < d_2$ and no solution for $d < d_1$ (if $-\infty < d_1$) and $d > d_2$ (if $\infty > d_2$). In the cases* (15) *and* (16) *there is a d_3, $d_3 \geqslant -\infty$, such that* (23), (17) *has a solution for all $d > d_3$ and no solution for $d < d_3$ (if $-\infty < d_3$).*

Proof. Write

$$\max_{\substack{0 \leqslant t \leqslant \pi \\ |x|+|y| \leqslant r}} |f(t, x, y)| = \gamma(r) \qquad (0 \leqslant r < \infty).$$

By (25), to any $\varepsilon > 0$ there corresponds an $r_0 = r_0(\varepsilon) > 0$ such that for $r > r_0$ we have $\gamma(r) \leqslant \gamma(r_0) + \varepsilon r$, and this implies

$$\lim_{r \to \infty} \frac{\gamma(r)}{r} = 0. \tag{26}$$

Let $c \in R$ be given. Consider a solution x of the initial value problem (23), (24) and the solution y of the problem (7), (24). As

$$y(t) = \frac{c \sin nt}{n} + \int_0^t \frac{\sin n(t-s)}{n} a y^-(s) ds \qquad (0 \leqslant t \leqslant \pi),$$

we get the estimates

$$|y(t)| \leqslant |c| M, \qquad |y'(t)| \leqslant |c|(1 + aM\pi) \qquad (0 \leqslant t \leqslant \pi) \tag{27}$$

with $M = \frac{1}{n} \exp\left(\frac{a}{n}\pi\right)$. By (19) it follows that there exists a constant $L > 0$ (independent of c) such that

$$|x(t) - y(t)| + |x'(t) - y'(t)| \leqslant L \max_{0 \leqslant t \leqslant \pi} |f[t, x(t), x'(t)]| \qquad (0 \leqslant t \leqslant \pi). \tag{28}$$

Inequalities (27) and (28) yield

$$|x(t)| + |x'(t)| \leqslant |c|(1 + M + aM\pi) + + L \max_{0 \leqslant t \leqslant \pi} |f[t, x(t), x'(t)]| \qquad (0 \leqslant t \leqslant \pi). \tag{29}$$

Let $\max_{0 \leqslant t \leqslant \pi} \big(|x(t)| + |x'(t)|\big) = r$. Then $|c| \leqslant r$; by means of (29) we obtain $|c| \leqslant r \leqslant |c|(1 + M + aM\pi) + L\gamma(r)$ and in view of (26), for all sufficiently great $|c|$ we have

$$|c| \leqslant r \leqslant |c| P \tag{30}$$

with a $P > 0$. (28) implies $|x(\pi)-y(\pi)| \leqslant L\gamma(r)$; on account of (30) and (26) we get

$$\lim_{|c|\to\infty} \frac{|x(\pi)-y(\pi)|}{|c|} = 0. \tag{31}$$

If F is the function defined as before then $y(\pi) = cF(1)$ ($c \geqslant 0$) and $y(\pi) = |c|F(-1)$ ($c < 0$). Thus (31) implies that

$$\lim_{c\to\infty} \frac{x(\pi)}{c} = F(1), \qquad \lim_{c\to-\infty} \frac{x(\pi)}{|c|} = F(-1).$$

Hence, if β satisfies (11), then $\lim\limits_{c\to\infty} x(\pi) = \infty$, $\lim\limits_{c\to-\infty} x(\pi) = -\infty$ and Lemma 6 completes the proof. The use of this lemma is justified by the fact that under condition (25) all solutions of (23) can be extended to the whole interval $[0, \pi]$. The other cases can be dealt with in a similar way.

Remark. Theorem 1 gives an affirmative answer to the question of the existence of a solution to (23), (2) when β satisfies (11) or (13). Now we consider the case (12). The main tool in the proof of the existence of a solution to (23), (2) will be the Leray–Schauder theorem ([4], p. 189).

First we notice that problem (23), (2) is equivalent to the integro-differential equation

$$x(t) = \int_0^\pi G(t, s)\big[-n^2x(s) + ax^-(s) + f\big(s, x(s), x'(s)\big)\big]ds \qquad (0 \leqslant t \leqslant \pi) \tag{32}$$

where G is the Green function of the problem $x'' = 0$, (2). We shall show that under condition (25) equation (32) has a solution $x \in C_1([0, \pi])$. To that aim consider the system of operators $T(\cdot, k)$ ($0 \leqslant k \leqslant 1$) defined by

$$T(x, k)(t) = \int_0^\pi G(t, s)\big[-n^2x(s) + ax^-(s) + kf\big(s, x(s), x'(s)\big)\big]ds$$
$$(0 \leqslant t \leqslant \pi,\ x \in C_1([0, \pi])).$$

This system has the following properties:

a. *$T(\cdot, k)$ maps $C_1([0, \pi])$ into itself for every $k \in [0, 1]$.*

Write

$$c_0 = \max_{0\leqslant t\leqslant\pi} \int_0^\pi |G(t, s)|\,ds, \qquad c_1 = \max_{0\leqslant t\leqslant\pi} \int_0^\pi \left|\frac{\partial G(t, s)}{\partial t}\right| ds.$$

Let $x \in C_1([0, \pi])$. By the uniform continuity of f on compact subsets of $[0, \pi] \times R^2$, to an arbitrary $\varepsilon > 0$ there corresponds a $\delta > 0$, $\delta \leqslant \varepsilon$, such that for any $y \in C_1([0, \pi])$ with $\|x-y\| < \delta$, $\|x'-y'\| < \delta$, we have

$$|f[t, y(t), y'(t)] - f[t, x(t), x'(t)]| < \varepsilon \qquad (0 \leqslant t \leqslant \pi)$$

and thus

$$\|T(y, k) - T(x, k)\|_1 = \|T(y, k) - T(x, k)\| + \|T(y, k)' - T(x, k)'\|$$
$$\leqslant (c_0 + c_1)[(n^2 + a)\delta + k\varepsilon] \leqslant (c_0 + c_1)(n^2 + a + k)\varepsilon;$$

this, in turn, implies the property

b. *$T(\cdot, k)$ is continuous in $C_1([0, \pi])$ for every k, $0 \leqslant k \leqslant 1$.*

Further we have

c. *For x in bounded sets of $C_1([0, \pi])$, $T(x, \cdot)$ is uniformly continuous in $[0, 1]$.*

In fact, for any bounded set $S \subset C_1([0, \pi])$ the boundedness of f on compact subsets of $[0, \pi] \times R^2$ implies

$$|f[t, x(t), x'(t)]| \leqslant M \qquad (x \in S) \tag{33}$$

for an $M > 0$. Then $\|T(x, k_1) - T(x, k_2)\|_1 \leqslant (c_0 + c_1)M|k_1 - k_2|$, which implies the statement c.

d. *$T(\cdot, k)$ is compact for any $k \in [0, 1]$.*

To prove this, consider a bounded set $S \subset C_1([0, \pi])$. By (33) we get $\|T(x, k)\|_1 \leqslant (c_0 + c_1)[(n^2 + a)\|x\| + kM]$ $(x \in S)$, and hence $T(S)$ is bounded, too. This proves the uniform boundedness of the functions $T(x, k)$, $T(x, k)'$ as well as the equicontinuity of $T(x, k)$. The relation

$$|T(x, k)'(t_1) - T(x, k)'(t_2)|$$
$$\leqslant \int_0^\pi \left| \frac{\partial G(t_1, s)}{\partial t} - \frac{\partial G(t_2, s)}{\partial t} \right| ds \cdot [(n^2 + a)\|x\| + kM]$$

implies the equicontinuity of $T(x, k)'$. The Ascoli lemma then yields the statement d.

e. *There exists an $N > 0$ such that every possible solution x of $x = T(x, k)$ $(x \in C_1([0, \pi]), k \in [0, 1])$, satisfies $\|x\|_1 < N$.*

In fact, any such solution satisfies $x'' + n^2 x = ax^- + kf(t, x, x')$ and boundary conditions (2). Let $\max_{t \in [0, \pi]} (|x(t)| + |x'(t)|) = r$. Then by Lemma 5 we get

$$r \leqslant \|x\|_1 \leqslant Kk\|f[t, x(t), x'(t)]\| = Kk\gamma(r).$$

Here $\gamma(r)$ has the same meaning as in the proof of Theorem 1. Thus in the case of $r > 0$ we obtain $\gamma(r)/r \geqslant 1/Kk \geqslant 1/K$. In view of (26) this implies that there exists an $r_0 > 0$ such that $r < r_0$. Then $\|x\|_1 < N = 2r_0$, which was to be proved.

f. *Finally, the equation* $x = T(x, 0)$, *i.e. the problem* (7), (2) *has a unique solution in* $C_1([0, \pi])$ *(namely, the trivial one).*

By the Leray–Schauder theorem there exists a solution x of $x = T(x, 1)$. Hence the following theorem is true.

THEOREM 2. *If the assumptions of Theorem* 1 *are fulfilled and* β *satisfies* (12), *then the boundary value problem* (23), (2) *has at least one solution and thus* d_0 *from Theorem* 1 *satisfies* $d_0 \leqslant 0$.

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