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PERIOD-DOUBLINGS AND ORBIT-BIFURCATIONS IN SYMMETRIC SYSTEMS

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We describe a new approach, based on an equivariant Liapunov–Schmidt reduction, to discuss the bifurcation of periodic solutions near some nontrivial periodic solution of symmetric systems depending on a parameter. We describe in detail some of the elementary bifurcations that can occur in such systems; these bifurcations include period-doublings and orbit-pitchforks.

1. Introduction

We consider one-parameter autonomous systems of the form

$$(1.1)_\lambda \qquad \dot{x} = f(x, \lambda)$$

under the following assumptions:

(a) $f\colon \boldsymbol{R}^n \times \boldsymbol{R} \to \boldsymbol{R}^n$ is (sufficiently) smooth;

(b) $f(\gamma x, \lambda) = \gamma f(x, \lambda)$, $\forall \gamma \in \Gamma$, where Γ is a closed subgroup of $O(n) := \{\gamma \in \mathscr{L}(\boldsymbol{R}^n) | \gamma^T \gamma = I\}$.

We are interested in *secondary* bifurcations of periodic solutions of (1.1); by this we mean the bifurcation (as λ is changed) of periodic solutions from a *nonconstant* periodic solution $\tilde{x}_0(t)$ of $(1.1)_{\lambda_0}$, for some $\lambda_0 \in \boldsymbol{R}$. (Hopf bifurcation describes the bifurcation of periodic solutions from a *constant* solution.) Because of the equivariance condition (ii) also $\gamma\tilde{x}_0(t)$ is a periodic solution of $(1.1)_{\lambda_0}$ for each $\gamma \in \Gamma$; we make the following assumption:

(c) $\tilde{\tilde{C}}_0 := \{\gamma\tilde{x}_0(t) | t \in \boldsymbol{R},\ \gamma \in \Gamma\}$ is a one-dimensional submanifold of $\boldsymbol{R}^n$.

This hypothesis implies that $\tilde{\tilde{C}}_0$ consists of a finite number of periodic orbits of $(1.1)_{\lambda_0}$; it excludes the case of a higher-dimensional manifold foliated by periodic orbits, obtained one from the other by the action of the symmetry group Γ. We also restrict our attention to periodic orbits near $\tilde{\tilde{C}}_0$ and having a minimal period near Np_0, for some $N \geqslant 1$ and with $p_0 > 0$ the minimal period of the given solution $\tilde{x}_0$. In case $N = 1$ we will talk about orbit-bifurcations, $N = 2$ corresponds to period-doubling, and $N \geqslant 3$ to subharmonic bifurcation.

In the case of a trivial symmetry (i.e. $\Gamma = \{\mathrm{Id}\}$) there are for one-parameter problems of the form (1.1) only two bifurcations which appear generically, namely the turn (a special case of an orbit-bifurcation) and the simple period-doubling (see further for a more precise description). For nontrivial symmetries we expect a more complicated behaviour, arising from the fact that solutions, and in particular periodic solutions, carry a certain symmetry which can break at bifurcations. The aim of this paper is to describe some of the bifurcations which can arise in such symmetric systems.

As a first step we have to find a way to describe the symmetry of a given periodic solution $\tilde{x}(t)$ of $(1.1)_\lambda$. We can do this by introducing the following subgroups of Γ:

$$K := \{\gamma \in \Gamma \mid \gamma\tilde{x}(t) = \tilde{x}(t),\ \forall t \in \boldsymbol{R}\} \tag{1.2}$$

and

$$H := \{\gamma \in \Gamma \mid \gamma(\tilde{C}) = \tilde{C}\}, \tag{1.3}$$

where $\tilde{C} := \{\tilde{x}(t) \mid t \in \boldsymbol{R}\}$ is the orbit of $\tilde{x}$. By the uniqueness of the solutions of (1.1) we see that a given $\gamma \in \Gamma$ will belong to K if and only if $\gamma\tilde{x}(t_0) = \tilde{x}(t_0)$ for some $t_0 \in \boldsymbol{R}$, and to H if and only if $\gamma\tilde{x}(t_0) = \tilde{x}(t_1)$ for some $t_0, t_1 \in \boldsymbol{R}$. We call K the *spatial symmetry* of $\tilde{x}$, and H the *orbital symmetry*. It is clear that K is a subgroup of H; in fact, denoting by $p > 0$ the minimal period of $\tilde{x}$, there exists a group homomorphism $\tilde{\theta}$: $H \to S^1 := \boldsymbol{R}/\boldsymbol{Z}$ such that

$$\gamma\tilde{x}(t) = \tilde{x}(t + \tilde{\theta}(\gamma)p), \qquad \forall t \in \boldsymbol{R},\ \forall \gamma \in H; \tag{1.4}$$

we have then that $K = \ker\tilde{\theta}$. It follows that K is a normal subgroup of H, and that H/K is isomorphic to either $\boldsymbol{Z}_m := m^{-1}\boldsymbol{Z}/\boldsymbol{Z}$ for some $m \geqslant 1$, or to S^1; in case $H/K \cong S^1$ we say that $\tilde{x}$ is a *rotating wave solution.* The symmetry of the periodic solution $\tilde{x}$ is completely described by the orbital symmetry H and by the homomorphism $\tilde{\theta}$: $H \to S^1$ appearing in (1.4).

Let now $\lambda_0 \in \boldsymbol{R}$ and $\tilde{x}_0(t)$ a nonconstant periodic solution of $(1.1)_{\lambda_0}$ such that (c) holds. Denote by K_0 and H_0 the spatial, respectively orbital symmetry of $\tilde{x}_0$, and by $\tilde{\theta}_0$: $H_0 \to S^1$ the corresponding homomorphism, i.e. we have

$$\gamma\tilde{x}_0(t) = \tilde{x}(t_0 + \tilde{\theta}_0(\gamma)p_0), \qquad \forall t \in \boldsymbol{R},\ \forall \gamma \in H_0. \tag{1.5}$$

The standard approach for the study of bifurcations near the orbit $\tilde{C}_0 := \{\tilde{x}_0(t) \mid t \in \boldsymbol{R}\}$ of $\tilde{x}_0$ is by the introduction of a Poincaré map, as follows. One takes a section S at a point $\tilde{x}_0(t_0) \in \tilde{C}_0$, such that S is transversal to $\tilde{C}_0$ and $S \cap \tilde{C}_0 = \{\tilde{x}_0(t_0)\}$. Then the Poincaré map π: $S \times \boldsymbol{R} \to S$ is defined as the first return map on S under the flow of $(1.1)_\lambda$. Periodic solutions of $(1.1)_\lambda$ near $\tilde{C}_0$ and with minimal period near Np_0 are then obtained from the fixed point equation

$$\pi^N(x, \lambda) = x, \tag{1.6}$$

where π^l: $S \times \boldsymbol{R} \to S$ $(l \geq 1)$ is defined recursively by $\pi^1 = \pi$, $\pi^{l+1}(x, \lambda) = \pi(\pi^l(x, \lambda), \lambda)$. Since we want to keep track of the symmetries it is important to see how the symmetry of $\tilde{x}_0$ is reflected in the Poincaré map π. This is no problem for the spatial symmetry K_0: one can choose the section S to be K_0-invariant, and then π is K_0-equivariant. For the orbital symmetry things are more complicated, since it is clearly impossible to make π equivariant with respect to H_0 (or some group isomorphic to H_0). One way out is by use of "partial" Poincaré maps, as constructed by Fiedler in [2]. In this approach, however, one has to treat the cases $H_0/K_0 \cong \boldsymbol{Z}_m$ and $H_0/K_0 \cong S^1$ separately; also one does not have the nice "equivariant" setting which has now become standard for bifurcation problems with symmetry (see e.g. [4] or [6]).

In an earlier paper [7] we have proposed a different approach which does not have these shortcomings; in section 2 we briefly outline this approach, which leads to a reduced problem whose equivariance reflects the symmetry of $\tilde{x}_0$. In section 3 we make a kind of "generic assumption", outline the form of the corresponding bifurcation equations, and give then a detailed description of some of the secondary bifurcations which one obtains from an analysis of these bifurcation equations. We thereby put the emphasis mainly on the qualitative aspects of the bifurcation, in particular the symmetry of the bifurcating solutions, and much less on the computational aspects of the problem. For those we refer to a forthcoming paper in collaboration with B. Fiedler.

2. Reduction to equivariant bifurcation equations

Let $\tilde{x}_0$ be a periodic solution of $(1.1)_{\lambda_0}$ for some $\lambda_0 \in \boldsymbol{R}$, with minimal period $p_0 > 0$; we assume that the hypotheses (a)–(c) are satisfied. Fixing some $N \geq 1$ we want to describe, for λ near λ_0, all periodic solutions of (1.1) with a period near Np_0 and an orbit near $\tilde{\tilde{C}}_0$.

Let Z (respectively X) be the Banach space of all C^0 (respectively C^1) 1-periodic mappings z: $\boldsymbol{R} \to \boldsymbol{R}^n$, endowed with the appropriate supremum norms. Define $x_0 \in X$ and $\sigma_0 \in \boldsymbol{R}$ by

$$x_0(t) := \tilde{x}_0(Np_0 t), \qquad \sigma_0 := (Np_0)^{-1}; \tag{2.1}$$

then we have $\tilde{M}(x_0, \lambda_0, \sigma_0) = 0$, where $\tilde{M}$: $X \times \boldsymbol{R}^2 \to Z$ is defined by

$$\tilde{M}(x, \lambda, \sigma)(t) := -\sigma\dot{x}(t) + f(x(t), \lambda), \qquad \forall t \in \boldsymbol{R}. \tag{2.2}$$

Now the group $\Gamma \times S^1$ acts on Z by

$$(\gamma, \varphi)\cdot z(t) := \gamma z(t-\varphi), \quad \forall t \in \boldsymbol{R}\ \forall(\gamma, \varphi)\in\Gamma\times S^1, \tag{2.3}$$

and $\tilde{M}$ is equivariant with respect to this action:

$$\tilde{M}((\gamma, \varphi)\cdot x, \lambda, \sigma) = (\gamma, \varphi)\cdot\tilde{M}(x, \lambda, \sigma), \quad \forall(\gamma, \varphi)\in\Gamma\times S^1. \tag{2.4}$$

Therefore the equation

$$\tilde{M}(x, \lambda, \sigma) = 0 \tag{2.5}$$

has not only the solution $(x_0, \lambda_0, \sigma_0)\in X\times\boldsymbol{R}^2$, but a whole orbit of solutions given by $C_0\times\{\lambda_0\}\times\{\sigma_0\}$, where $C_0 := \{(\gamma, \varphi)\cdot x_0 | (\gamma, \varphi)\in\Gamma\times S^1\}$ is the orbit in X generated by the group $\Gamma\times S^1$ acting on x_0. Our problem reduces then to that of finding all solutions (x, λ, σ) of (2.5) in a neighborhood of $C_0 \times\{\lambda_0\}\times\{\sigma_0\}$ in $X\times\boldsymbol{R}^2$; to each such solution (with $\sigma > 0$) there corresponds a $1/\sigma$-periodic solution of $(1.1)_\lambda$, given by $\tilde{x}(t) := x(\sigma t)$; if σ is near σ_0, then the period $1/\sigma$ is near Np_0. Moreover we will see later that the symmetry of $\tilde{x}$ (described by the spatial symmetry K, the orbital symmetry H and the homomorphism $\tilde{\theta}$: $H\to S^1$) can be completely determined from the isotropy subgroup of x, i.e. from

$$\Sigma := \{(\gamma, \varphi)\in\Gamma\times S^1 | (\gamma, \varphi)\cdot x = x\}. \tag{2.6}$$

We will denote the isotropy subgroup of our basic solution x_0 by Σ_0. Because of the condition (c) the tangent space to C_0 in X is one-dimensional and spanned by $\dot{x}_0$: it follows that $\dot{x}_0\in\ker\tilde{L}_0$, where $\tilde{L}_0 := D_x\tilde{M}(x_0, \lambda_0, \sigma_0)$. We can define in Z a projection P on this tangent space by

$$Pz := \langle z, \dot{x}_0\rangle\langle\dot{x}_0, \dot{x}_0\rangle^{-1}\dot{x}_0, \quad \forall z\in Z, \tag{2.7}$$

where the inner product $\langle\cdot,\cdot\rangle$ on Z is given by

$$\langle z_1, z_2\rangle := \int_0^1 (z_1(t), z_2(t))dt, \quad \forall z_1, z_2\in Z. \tag{2.8}$$

We remark that P is Σ_0-equivariant (since $\dot{x}_0$ has isotropy Σ_0), and therefore the subspace $Y := \ker P$ is invariant under the action of Σ_0.

Now we can apply Theorem 8.2.5 of [6] to prove that C_0 has a tubular neighborhood in X of the form

$$\{(\gamma, \varphi)\cdot(x_0+y) | (\gamma, \varphi)\in\Gamma\times S^1, y\in\Omega\}, \tag{2.9}$$

where Ω is a neighborhood of the origin in $X\cap Y$. Therefore we can replace x by $(\gamma, \varphi)\cdot(x_0+y)$ in (2.5); using the $\Gamma\times S^1$-equivariance of $\tilde{M}$ and projecting with P and $(I-P)$ gives the equivalent problem

$$P\tilde{M}(x_0+y, \lambda, \sigma) = 0, \tag{2.10}$$

$$(I-P)\tilde{M}(x_0+y, \lambda, \sigma) = 0. \tag{2.11}$$

Since $D_\sigma \tilde{M}(x_0, \lambda_0, \sigma_0) = -\dot{x}_0$ we can solve (2.10) by the implicit function theorem for $\sigma = \tilde{\sigma}(y, \lambda)$; bringing this solution in (2.11) gives the reduced equation

$$M(y, \lambda) := (I-P)\tilde{M}(x_0+y, \lambda, \tilde{\sigma}(y, \lambda)) = 0. \tag{2.12}$$

The mapping $M: (Y \cap X) \times \mathbf{R} \to Y$ is smooth and Σ_0-equivariant, with $M(0, \lambda_0) = 0$; also $L_0 := D_y M(0, \lambda_0) \in \mathscr{L}(X \cap Y, Y)$ is a Fredholm operator with index zero, and $\operatorname{Im} L_0 = (I-P)\operatorname{Im} \tilde{L}_0$; $\ker L_0$ can be related to $\ker \tilde{L}_0$ as follows: if $\dot{x}_0 \notin \operatorname{Im} \tilde{L}_0$ then $\ker L_0 = (I-P)\ker \tilde{L}_0$; if $\dot{x}_0 \in \operatorname{Im} \tilde{L}_0$ then there exists a unique $y_0 \in Y \cap X$ such that $\tilde{L}_0 y_0 = \dot{x}_0$, and $\ker L_0 = (I-P)\ker \tilde{L}_0 \oplus \operatorname{span}\{y_0\}$. Finally, the elements of $\ker \tilde{L}_0$ can be obtained from the eigenvectors corresponding to the characteristic multipliers of $\tilde{x}_0$ which are Nth roots of unity.

As a last step before the main bifurcation analysis we can now apply an equivariant Liapunov–Schmidt reduction to (2.12) (see [6]); the foregoing shows that all the necessary hypotheses for such reduction are satisfied. As a result we find a finite-dimensional system of bifurcation equations, with dimension equal to $\dim \ker L_0$, and equivariant with respect to the action of Σ_0 on $\ker L_0$. This Σ_0-equivariance reflects the symmetries (both spatial and orbital) of our original solution: this can be seen from the following lemma which relates the isotropy Σ of a solution $(x, \lambda, \sigma) \in X \times \mathbf{R}^2$ of (2.5) to the spatial and orbital symmetries of the corresponding solution $\tilde{x}(t) := x(\sigma t)$ of $(1.1)_\lambda$.

LEMMA 1. *Let* $x \in X$, $\sigma > 0$, *and* $\tilde{x}(t) := x(\sigma t)$, $\forall t \in \mathbf{R}$.
Let

$$\Sigma := \{(\gamma, \varphi) \in \Gamma \times S^1 \mid (\gamma, \varphi)\cdot x = x\}, \tag{2.13}$$

$$T := \{\varphi \in S^1 \mid (\mathrm{Id}, \varphi) \in \Sigma\}, \tag{2.14}$$

$$H := \{\gamma \in \Gamma \mid (\gamma, \varphi) \in \Sigma \text{ for some } \varphi \in S^1\}, \tag{2.15}$$

and

$$K := \{\gamma \in \Gamma \mid (\gamma, 0) \in \Sigma\}. \tag{2.16}$$

Then either

(i) $T = S^1$, $H = K$, *and* x *is constant,*

or

(ii) $T = \mathbf{Z}_M$ *for some* $M \geq 1$ *and* $\tilde{x}$ *has minimal period* $(\sigma M)^{-1}$, *spatial symmetry* K *and orbital symmetry* H; *moreover*

$$\gamma x(t) = \tilde{x}(t + \tilde{\theta}(\gamma)(\sigma M)^{-1}), \quad \forall t \in \mathbf{R},\ \forall \gamma \in H, \tag{2.17}$$

where $\tilde{\theta}: H \to S^1$ *is defined by* $\tilde{\theta}(\gamma) = M\varphi$ *for all* $\gamma \in H$ *and any* $\varphi \in S^1$ *such that* $(\gamma, \varphi) \in \Sigma$.

Now the group $\Gamma \times S^1$ acts on Z by

$$(2.3) \qquad (\gamma, \varphi)\cdot z(t) := \gamma z(t-\varphi), \qquad \forall t \in \boldsymbol{R}\ \forall(\gamma, \varphi) \in \Gamma \times S^1,$$

and $\tilde{M}$ is equivariant with respect to this action:

$$(2.4) \qquad \tilde{M}((\gamma, \varphi)\cdot x, \lambda, \sigma) = (\gamma, \varphi)\cdot \tilde{M}(x, \lambda, \sigma), \qquad \forall(\gamma, \varphi) \in \Gamma \times S^1.$$

Therefore the equation

$$(2.5) \qquad \tilde{M}(x, \lambda, \sigma) = 0$$

has not only the solution $(x_0, \lambda_0, \sigma_0) \in X \times \boldsymbol{R}^2$, but a whole orbit of solutions given by $C_0 \times \{\lambda_0\} \times \{\sigma_0\}$, where $C_0 := \{(\gamma, \varphi)\cdot x_0 | \ (\gamma, \varphi) \in \Gamma \times S^1\}$ is the orbit in X generated by the group $\Gamma \times S^1$ acting on x_0. Our problem reduces then to that of finding all solutions (x, λ, σ) of (2.5) in a neighborhood of $C_0 \times \{\lambda_0\} \times \{\sigma_0\}$ in $X \times \boldsymbol{R}^2$; to each such solution (with $\sigma > 0$) there corresponds a $1/\sigma$-periodic solution of $(1.1)_\lambda$, given by $\tilde{x}(t) := x(\sigma t)$; if σ is near σ_0, then the period $1/\sigma$ is near Np_0. Moreover we will see later that the symmetry of $\tilde{x}$ (described by the spatial symmetry $\tilde{K}$, the orbital symmetry $\tilde{H}$ and the homomorphism $\tilde{\theta}$: $\tilde{H} \to S^1$) can be completely determined from the isotropy subgroup of x, i.e. from

$$(2.6) \qquad \Sigma := \{(\gamma, \varphi) \in \Gamma \times S^1 | \ (\gamma, \varphi)\cdot x = x\}.$$

We will denote the isotropy subgroup of our basic solution x_0 by Σ_0. Because of the condition (c) the tangent space to C_0 in X is one-dimensional and spanned by $\dot{x}_0$: it follows that $\dot{x}_0 \in \ker \tilde{L}_0$, where $\tilde{L}_0 := D_x \tilde{M}(x_0, \lambda_0, \sigma_0)$. We can define in Z a projection P on this tangent space by

$$(2.7) \qquad Pz := \langle z, \dot{x}_0 \rangle \langle \dot{x}_0, \dot{x}_0 \rangle^{-1} \dot{x}_0, \qquad \forall z \in Z,$$

where the inner product $\langle \cdot, \cdot \rangle$ on Z is given by

$$(2.8) \qquad \langle z_1, z_2 \rangle := \int_0^1 (z_1(t), z_2(t))\, dt, \qquad \forall z_1, z_2 \in Z.$$

We remark that P is Σ_0-equivariant (since $\dot{x}_0$ has isotropy Σ_0), and therefore the subspace $Y := \ker P$ is invariant under the action of Σ_0.

Now we can apply Theorem 8.2.5 of [6] to prove that C_0 has a tubular neighborhood in X of the form

$$(2.9) \qquad \{(\gamma, \varphi)\cdot(x_0 + y) | \ (\gamma, \varphi) \in \Gamma \times S^1, \ y \in \Omega\},$$

where Ω is a neighborhood of the origin in $X \cap Y$. Therefore we can replace x by $(\gamma, \varphi)\cdot(x_0 + y)$ in (2.5); using the $\Gamma \times S^1$-equivariance of $\tilde{M}$ and projecting with P and $(I-P)$ gives the equivalent problem

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Since $D_\sigma \tilde{M}(x_0, \lambda_0, \sigma_0) = -\dot{x}_0$ we can solve (2.10) by the implicit function theorem for $\sigma = \tilde{\sigma}(y, \lambda)$; bringing this solution in (2.11) gives the reduced equation

$$M(y, \lambda) := (I-P)\tilde{M}(x_0+y, \lambda, \tilde{\sigma}(y, \lambda)) = 0. \tag{2.12}$$

The mapping $M: (Y \cap X) \times \boldsymbol{R} \to Y$ is smooth and Σ_0-equivariant, with $M(0, \lambda_0) = 0$; also $L_0 := D_y M(0, \lambda_0) \in \mathscr{L}(X \cap Y, Y)$ is a Fredholm operator with index zero, and $\operatorname{Im} L_0 = (I-P)\operatorname{Im} \tilde{L}_0$; $\ker L_0$ can be related to $\ker \tilde{L}_0$ as follows: if $\dot{x}_0 \notin \operatorname{Im} \tilde{L}_0$ then $\ker L_0 = (I-P)\ker \tilde{L}_0$; if $\dot{x}_0 \in \operatorname{Im} \tilde{L}_0$ then there exists a unique $y_0 \in Y \cap X$ such that $\tilde{L}_0 y_0 = \dot{x}_0$, and $\ker L_0 = (I-P)\ker \tilde{L}_0 \oplus \operatorname{span}\{y_0\}$. Finally, the elements of $\ker \tilde{L}_0$ can be obtained from the eigenvectors corresponding to the characteristic multipliers of $\tilde{x}_0$ which are Nth roots of unity.

As a last step before the main bifurcation analysis we can now apply an equivariant Liapunov–Schmidt reduction to (2.12) (see [6]); the foregoing shows that all the necessary hypotheses for such reduction are satisfied. As a result we find a finite-dimensional system of bifurcation equations, with dimension equal to $\dim\ker L_0$, and equivariant with respect to the action of Σ_0 on $\ker L_0$. This Σ_0-equivariance reflects the symmetries (both spatial and orbital) of our original solution: this can be seen from the following lemma which relates the isotropy Σ of a solution $(x, \lambda, \sigma) \in X \times \boldsymbol{R}^2$ of (2.5) to the spatial and orbital symmetries of the corresponding solution $\tilde{x}(t) := x(\sigma t)$ of $(1.1)_\lambda$.

LEMMA 1. *Let $x \in X$, $\sigma > 0$, and $\tilde{x}(t) := x(\sigma t)$, $\forall t \in \boldsymbol{R}$.*
Let

$$\Sigma := \{(\gamma, \varphi) \in \Gamma \times S^1 \mid (\gamma, \varphi)\cdot x = x\}, \tag{2.13}$$

$$T := \{\varphi \in S^1 \mid (\mathrm{Id}, \varphi) \in \Sigma\}, \tag{2.14}$$

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and

$$K := \{\gamma \in \Gamma \mid (\gamma, 0) \in \Sigma\}. \tag{2.16}$$

Then either

(i) $T = S^1$, $H = K$, *and x is constant,*

or

(ii) $T = \boldsymbol{Z}_M$ *for some $M \geq 1$ and $\tilde{x}$ has minimal period $(\sigma M)^{-1}$, spatial symmetry K and orbital symmetry H; moreover*

$$\gamma x(t) = \tilde{x}(t + \tilde{\theta}(\gamma)(\sigma M)^{-1}), \quad \forall t \in \boldsymbol{R},\ \forall \gamma \in H, \tag{2.17}$$

where $\tilde{\theta}: H \to S^1$ is defined by $\tilde{\theta}(\gamma) = M\varphi$ for all $\gamma \in H$ and any $\varphi \in S^1$ such that $(\gamma, \varphi) \in \Sigma$.

Proof. Since T is a closed subgroup of S^1, we have either $T = S^1$ or $T = Z_M$ for some $M \geqslant 1$. In the first case, x is constant and $H = K$. In the second case, x has minimal period M^{-1}, and hence $\tilde{x}$ has minimal period $(\sigma M)^{-1}$. Moreover, we have then for each $(\gamma, \varphi) \in \Sigma$ and each $\psi \in S^1$ that $(\gamma, \varphi + \psi) \in \Sigma$ if and only if $\psi \in T$. Therefore, if we put

$$\theta(\gamma) := \{\varphi \in S^1 \mid (\gamma, \varphi) \in \Sigma\} \tag{2.18}$$

for each $\gamma \in H$, then θ is a group homomorphism from H into S^1/T, and

$$\Sigma = \{(\gamma, \varphi) \mid \gamma \in H, \varphi \in \theta(\gamma)\}. \tag{2.19}$$

(Remember that the elements of S^1/T are cosets of the form $\varphi + T \subset S^1$, for some fixed $\varphi \in S^1$.) Since $T = Z_M$, it follows that the mapping $\tilde{\theta}: H \to S^1$ given in the statement is well defined, and a group homomorphism; also (2.17) follows easily from

$$\gamma x(t) = x(t+\varphi), \quad \forall t \in \boldsymbol{R}, \ \forall (\gamma, \varphi) \in \Sigma. \tag{2.20}$$

From (2.17) we see that H is contained in the orbital symmetry of $\tilde{x}$. To prove equality, suppose that

$$\gamma \tilde{x}(t) = \tilde{x}(t + \psi(\sigma M)^{-1}), \quad \forall t \in \boldsymbol{R}, \tag{2.21}$$

for some $(\gamma, \psi) \in \Gamma \times S^1$; then

$$\gamma x(t) = x(t + \psi M^{-1}), \quad \forall t \in \boldsymbol{R}, \tag{2.22}$$

$(\gamma, M^{-1}\psi) \in \Sigma$, $\gamma \in H$ and $\psi = \tilde{\theta}(\gamma)$. Finally, it follows from (2.16) that $K = \ker \tilde{\theta} = \ker \theta$, and hence K is the spatial symmetry of $\tilde{x}$.

3. Some elementary bifurcations

The bifurcation equations resulting from a Liapunov–Schmidt reduction of (2.12) may take many different forms and show different degrees of complexity, depending on the structure of $\ker L_0$ and the action of Σ_0 on this space. In order to proceed we make now the following assumption:

(d)(i) the operator $L_0 := D_y M(0, \lambda_0)$ has zero as a semi-simple eigenvalue, i.e.

$$Y = \ker L_0 \oplus \operatorname{Im} L_0. \tag{3.1}$$

(ii) the action of Σ_0 on $U := \ker L_0$ is absolutely irreducible; this means that the only linear operators on U which commute with the Σ_0-action are the scalar multiples of the identity.

One can argue that the conditions (d) will be satisfied for generic Σ_0-equivariant one-parameter problems of the form (2.12) (see [4]). However, this does not prove that (d) holds for generic Γ-equivariant one-parameter systems of the form (1.1) although it is plausible that this is actually the case: a formal proof has only been given for the case of trivial symmetry (see e.g. [1] or [5]) and for cyclic Γ (see [3]).

Under the hypothesis (d) the bifurcation equations take the form

$$G(u, \lambda) = 0 \tag{3.2}$$

where $G\colon U \times \boldsymbol{R} \to U$ is smooth, with $G(0, \lambda_0) = 0$, $D_u G(0, \lambda_0) = 0$, and

$$G((\gamma, \varphi), \mu, \lambda) = (\gamma, \varphi)\cdot G(\mu, \lambda), \qquad \forall (\gamma, \varphi) \in \Sigma_0. \tag{3.3}$$

To each solution (μ, λ) near $(0, \lambda_0)$ of (3.2) there corresponds a solution $(\tilde{y}(u, \lambda), \lambda)$ of (2.12), and a solution $(x_0 + \tilde{y}(u, \lambda), \lambda, \tilde{\sigma}(\tilde{y}(u, \lambda), \lambda))$ of (2.5); this, in turn, generates a whole family of solutions of (2.5), by the $\Gamma \times S^1$-action. The symmetry of these solutions is determined by the isotropy subgroup

$$\Sigma_u := \{(\gamma, \varphi) \in \Sigma_0 \mid (\gamma, \varphi)\cdot u = u\}. \tag{3.4}$$

In particular, all bifurcating solutions will at least have the symmetry given by

$$\Sigma := \{(\gamma, \varphi) \in \Sigma_0 \mid (\gamma, \varphi)\cdot u = u, \ \forall u \in U\}. \tag{3.5}$$

Σ is a normal subgroup of Σ_0, and Σ_0/Σ acts absolutely irreducibly on U. One way of classifying the possible bifurcations is according to $\dim U$ and the action of Σ_0/Σ on U.

There is one consequence of our hypothesis (d) which we should clarify before we start looking at particular cases. If we let $T_0 := \{\varphi \in S^1 \mid (\mathrm{Id}, \varphi) \in \Sigma_0\}$ (compare with Lemma 1) then we know from (2.1) that $T_0 = \boldsymbol{Z}_N$. Now $T_0 \cong \{\mathrm{Id}\} \times T_0$ commutes with all elements of Σ_0, and hence our assumption (d)(ii) implies that each element of T_0 acts on U as a scalar multiple of the identity; in particular, there is some $\alpha \in \boldsymbol{R}$ such that $(\mathrm{Id}, 1/N)\cdot u = \alpha u$ for all $u \in U$. Since $(\mathrm{Id}, 1/N)^N$ equals the identity we have the condition $\alpha^N = 1$; this gives us the following possibilities:

(1) $(\mathrm{Id}, 1/N)\cdot u = u$ for all $u \in U$; this implies that all bifurcating solutions of (2.5) will have the same minimal period $1/N$ as x_0. It follows that in the corresponding bifurcation picture for (1.1) all periodic solutions will have a minimal period near $N^{-1}\sigma_0^{-1} = p_0$, and hence these solutions can also be obtained by taking $N = 1$ in (2.1);

(2) $(\mathrm{Id}, 1/N)\cdot u = -u$ for all $u \in U$, and N is even. Then $(\mathrm{Id}, 2/N) \in \Sigma$, and all bifurcating solutions of (2.5) will have a minimal period $2/N$, which is twice the minimal period of x_0. In the corresponding bifurcation picture for (1.1), all periodic solutions will have a minimal period near $2p_0$, and hence it is sufficient to take $N = 2$ in (2.1).

We conclude that under the hypothesis (d) it is sufficient to consider the cases $N = 1$ and $N = 2$, while in this last case we may also assume that $(\mathrm{Id}, 1/2)$ acts as minus the identity on U.

The easiest case to discuss is when $\Sigma = \Sigma_0$; the condition (d)(ii) then implies that $\dim U = 1$, and (3.2) becomes a scalar equation. Assuming that

$$D_\lambda G(0, \lambda_0) \neq 0, \tag{3.6}$$

we can solve for $\lambda = \lambda^*(u)$, with $\lambda^*(0) = \lambda_0$ and $D\lambda^*(0) = 0$; if also $D_u^2 G(0, \lambda_0) \neq 0$ then $D^2\lambda^*(0) \neq 0$ and we have an *orbit-turn*: two periodic orbits with the same symmetry exist for $\lambda < 0$ (or for $\lambda > 0$), coalesce as λ goes to zero, and disappear for $\lambda > 0$ (resp. for $\lambda < 0$). We remark that for such orbit-turn one will in general have that $\dot{x}_0 \in \operatorname{Im} L_0$, i.e. zero will be a non-semisimple eigenvalue of L_0. To see this notice that in case of a turn the equation (2.5) has a solution branch $(x^*(u), \lambda^*(u), \sigma^*(u))$, parametrized by $u \in U$, and with $D\lambda^*(0) = 0$; differentiating the identity $\tilde{M}(x^*(u), \lambda^*(u), \sigma^*(u)) = 0$ one finds

$$L_0 Dx^*(0) = \dot{x}_0 D\sigma^*(0); \tag{3.7}$$

assuming that $D\sigma^*(0) \neq 0$ we see that $\dot{x}_0 \in \operatorname{Im} L_0$. The hypothesis (d) is then equivalent to $\ker L_0 = \operatorname{span}\{\dot{x}_0\}$ and $Dx^*(0) \notin \operatorname{Im} L_0$.

From now on we will assume that Σ is a proper subgroup of Σ_0. The hypothesis d(ii) implies that U is irreducible under the Σ_0-action, i.e. that U has no proper subspaces which are invariant under the action of Σ_0. Indeed, the orthogonal projection on such subspace (using the inner product (2.8)) commutes with the operators from Σ_0, and hence this projection must be either the identity or the zero operator on U. Now

$$U^{\Sigma_0} := \{u \in U \mid (\gamma, \varphi)u = u, \ \forall (\gamma, \varphi) \in \Sigma_0\}$$

is such a subspace; since $U^{\Sigma_0} \neq U$ by our assumption that $\Sigma \neq \Sigma_0$, we conclude that

$$U^{\Sigma_0} = \{0\}. \tag{3.8}$$

Together with (3.3) this implies that

$$G(0, \lambda) = 0, \quad \forall \lambda. \tag{3.9}$$

So (2.5) has a solution curve $\{(\bar{x}(\lambda), \lambda, \bar{\sigma}(\lambda))\}$ parametrized by λ, passing through $(x_0, \lambda_0, \sigma_0)$, and with all solutions on the curve having the same symmetry as x_0. Solutions bifurcating from this "trivial branch" correspond to solutions (u, λ) of (3.2) with $u \neq 0$ and hence with a symmetry Σ_u strictly contained in Σ_0 (by (3.8)).

We consider now in more detail the different possibilities which can arise when $\dim U = 1$. In that case we necessarily have $\Sigma_0/\Sigma \cong \mathbf{Z}_2$ and all bifurcating solutions have the symmetry Σ. The equation (3.2) is then a scalar equation, with

$$G(-u, \lambda) = -G(u, \lambda). \tag{3.10}$$

Writing $G(u, \lambda) = uH(u, \lambda)$ the nonzero solutions of (3.2) have to satisfy

$$H(u, \lambda) = 0. \tag{3.11}$$

We have $H(0, \lambda_0) = 0$ and $H(-u, \lambda) = H(u, \lambda)$. Assuming that

$$D_\lambda H(0, \lambda_0) = D_u D_\lambda G(0, \lambda_0) \neq 0 \tag{3.12}$$

we find a unique solution branch $\{(u, \tilde{\lambda}(u))\}$ for (3.11), with $\tilde{\lambda}(0) = \lambda_0$ and $\tilde{\lambda}(-u) = \tilde{\lambda}(u)$, i.e. we have a pitchfork bifurcation for (3.2) whereby the two nontrivial branches are related to each other by the symmetry operators from $\Sigma_0 \backslash \Sigma$. In order to describe the corresponding bifurcation for (1.1) we have to see how Σ can sit into Σ_0; as already mentioned before, we have only to consider the cases $N = 1$ and $N = 2$.

If $N = 1$ then we see from Lemma 1 that Σ_0 and Σ have the form

$$\Sigma_0 = \{(\gamma, \tilde{\theta}_0(\gamma)) | \ \gamma \in H_0\} \tag{3.13}$$

and

$$\Sigma = \{(\gamma, \tilde{\theta}_0(\gamma)) | \ \gamma \in H\}, \tag{3.14}$$

respectively, with H_0 the orbital symmetry of our original solution $\tilde{x}_0$, $\tilde{\theta}_0$: $H_0 \to S^1$ as in (1.5), and with H a normal subgroup of H_0 such that $H_0/H \cong Z_2$. The spatial symmetry of $\tilde{x}_0$ is given by $K_0 = \ker \tilde{\theta}_0$, while the bifurcating solutions have orbital symmetry H and spatial symmetry $K := H \cap K_0$. Hence the whole bifurcation takes place in the subspace

$$(R^n)^K := \{x \in R^n | \ \gamma x = x, \ \forall \gamma \in K\}$$

on which the group H_0/K acts (remark that K is a normal subgroup of H_0). Restricting equation (1.1) to this subspace we have to replace H_0, K_0 and H in the foregoing by H_0/K, K_0/K and H/K, respectively; we will denote these quotient groups again by H_0, K_0 and H. Thus we may assume that K is trivial, i.e. H in (3.14) is a normal subgroup of H_0 such that

$$H_0/H \cong Z_2 \quad \text{and} \quad H \cap K_0 = H \cap \ker \tilde{\theta}_0 = \{I\}. \tag{3.15}$$

We remind also that H_0/K_0 is isomorphic to im $\tilde{\theta}_0$, so that $H_0/K_0 \cong Z_m$ for some $m \in N$ or $H_0/K_0 \cong S^1$.

There are now two possibilities, depending on whether K_0 is trivial or not. If K_0 is trivial then $H_0 \cong Z_m$ or $H_0 \cong S^1$; such H_0 has a normal subgroup H with $H_0/H \cong Z_2$ only if $H_0 \cong Z_m$ with m even, in which case $H = \tilde{\theta}_0^{-1}(Z_{m/2}) \cong Z_{m/2}$. Suppose next that K_0 is nontrivial. Let $\sigma \in K_0$ and $\gamma \in H$; since $\sigma H = H\sigma$ there exists some $\gamma' \in H$ such that $\sigma\gamma = \gamma'\sigma$. It follows that $\tilde{\theta}_0(\gamma) = \tilde{\theta}_0(\sigma\gamma) = \tilde{\theta}_0(\gamma'\sigma) = \tilde{\theta}_0(\gamma')$; by (3.15) $\tilde{\theta}_0$ is injective on H, so that we may conclude that $\gamma' = \gamma$ and

$$\sigma\gamma = \gamma\sigma, \quad \forall \sigma \in K_0, \ \forall \gamma \in H. \tag{3.16}$$

If σ is a nontrivial element of K_0, then (3.15) also implies that $H_0 = H \cup \sigma H$ and $\sigma^2 H = H$; hence $\sigma^2 \in H \cap K_0 = \{I\}$, i.e. $\sigma^2 = I$. Fixing such σ we see that each nontrivial element of K_0 has the form $\sigma\gamma$ for some $\gamma \in H$; the condition $\sigma\gamma \in K_0$ implies $0 = \tilde{\theta}_0(\sigma\gamma) = \tilde{\theta}_0(\gamma)$, i.e. $\gamma \in H \cap K_0 = \{I\}$. So we conclude that $K_0 = \{I, \sigma\} \cong Z_2$. Also $\operatorname{im}\tilde{\theta}_0 = \tilde{\theta}_0(H_0) = \tilde{\theta}_0(H \cup \sigma H) = \tilde{\theta}_0(H)$, so that H is isomorphic to $\operatorname{im}\tilde{\theta}_0$, i.e. $H \cong H_0/K_0$. Finally (3.16) shows that $H_0 \cong H \times K_0$; such H_0 is itself cyclic only if $H \cong Z_m$ with m odd.

Returning to the original setting (with K not necessarily trivial) we can summarize the case $N = 1$ by saying that we have found two different types of "orbit-pitchforks":

(1) one for which there is breaking of orbital symmetry but conservation of spatial symmetry; this is only possible when $H_0/K_0 \cong Z_m$ with m even, and then $H = \tilde{\theta}_0^{-1}(Z_{m/2})$;

(2) one for which both the orbital and the spatial symmetry break; this is only possible if H_0 has a normal subgroup H, and K_0 has a normal subgroup K, such that $H/K \cong H_0/K_0$, $K_0/K \cong Z_2$ and $(H_0/K) \cong (H/K) \times (K_0/K)$.

We now turn to the case $N = 2$; it follows from Lemma 1 that Σ_0 has the form

$$\Sigma_0 = \{(\gamma, \varphi) \mid \gamma \in H_0,\ \varphi \in \theta_0(\gamma)\},$$

where $\theta_0 \colon H_0 \to S^1/Z_2$ is a group homomorphism connected to the homomorphism $\tilde{\theta}_0 \colon H_0 \to S^1$ appearing in (1.5) by

$$\theta_0(\gamma) = \frac{1}{2}\tilde{\theta}_0(\gamma) + Z_2 = \left\{\frac{1}{2}\tilde{\theta}_0(\gamma), \frac{1}{2}\tilde{\theta}_0(\gamma) + \frac{1}{2}\right\}, \quad \forall \gamma \in H_0. \tag{3.17}$$

We may assume that $(0, 1/2) \notin \Sigma$, since otherwise the problem can be further reduced to the case $N = 1$. Since $\Sigma_0/\Sigma \cong Z_2$ it follows that Σ must have the form

$$\Sigma = \{(\gamma, \theta(\gamma)) \mid \gamma \in H_0\}$$

with $\theta \colon H_0 \to S^1$ a homomorphism such that $\theta(\gamma) \in \theta_0(\gamma)$ for each $\gamma \in H_0$. We conclude that the bifurcating solutions will have the same orbital symmetry as the solutions along the trivial branch. The spatial symmetry of the bifurcating solutions is given by $K := \ker\theta$; K is a normal subgroup of H_0, and since $\theta(\gamma) \in \theta_0(\gamma)$ we have $K_0 = \ker\theta_0 = \theta^{-1}(Z_2)$, so that K is also a normal subgroup of K_0. We have $K = K_0$ or $K_0/K \cong Z_2$ depending on whether $1/2 \notin \operatorname{im}\theta$ or $1/2 \in \operatorname{im}\theta$. Let us consider these two cases separately.

If $K = K_0$ and $1/2 \notin \operatorname{im}\theta$ then we have necessarily $\operatorname{im}\theta = Z_m$ with m odd. Putting $m = 2l + 1$ we have then

$$H_0/K_0 = H_0/K \cong \operatorname{im}\theta = Z_{2l+1}. \tag{3.18}$$

Moreover $\tilde{\theta}_0(\gamma) = 2\theta(\gamma) \pmod 1$ for each $\gamma \in H_0$, and hence also $\operatorname{im}\tilde{\theta}_0 = 2\boldsymbol{Z}_{2l+1} = \boldsymbol{Z}_{2l+1}$. If $\gamma \in H_0$ is such that $\tilde{\theta}_0(\gamma) = 1/(2l+1)$ then

$$\theta_0(\gamma) = \left\{\frac{1}{2(2l+1)}, \frac{l+1}{2l+1}\right\}$$

and

$$\theta(\gamma) = \frac{l+1}{2l+1},$$

since $\theta(\gamma) \in \theta_0(\gamma)$ and $\operatorname{im}\theta = \boldsymbol{Z}_{2l+1}$.

If $K_0/K \cong \boldsymbol{Z}_2$ and $1/2 \in \operatorname{im}\theta$ then we have either $\operatorname{im}\theta = \boldsymbol{Z}_{2m}$ for some $m \in \boldsymbol{N}$ or $\operatorname{im}\theta = S^1$. If $H_0/K \cong \operatorname{im}\theta = \boldsymbol{Z}_{2m}$ then $H_0/K_0 \cong \operatorname{im}\tilde{\theta}_0 = 2\boldsymbol{Z}_{2m} = \boldsymbol{Z}_m$; if $\gamma \in H_0$ is such that $\theta(\gamma) = 1/2m$ then $\tilde{\theta}_0(\gamma) = 1/m$; this implies that γ^m is an orbital symmetry along the primary branch, but corresponds to a shift over half a period for the bifurcating (double-period) solutions. If $H_0/K \cong \operatorname{im}\theta = S^1$ then $H_0/K_0 \cong \operatorname{im}\tilde{\theta}_0 = 2S^1 = S^1$; if $\gamma \in H_0$ is such that $\theta(\gamma) = 1/2$ then γ is an orbital symmetry along the primary branch, and corresponds to a shift over half a period for the bifurcating solutions.

To summarize: we have found two types of period doublings:

(3) one for which both the orbital and the spatial symmetry are conserved; this is only possible when $H_0/K_0 = \boldsymbol{Z}_m$ with m odd;

(4) one for which the orbital symmetry is conserved but the spatial symmetry is broken; this case is only possible if K_0 has a normal subgroup K such that $K_0/K \cong \boldsymbol{Z}_2$, while K is also normal in H_0 and H_0/K is cyclic.

We finish with some remarks. In [2] and [3] Fiedler studies the cases $\Gamma = \boldsymbol{Z}_q$ and $\Gamma = S^1$, which implies that all subgroups which appear are cyclic. He proved that for one-parameter problems only the orbit turn and the four bifurcations (1)–(4) considered above can appear generically. In this terminology case (1) is a flip pitchfork, case (2) a flip-flop pitchfork, case (3) a flip doubling and case (4) a flop doubling; this terminology did arise from considerations concerning the Poincaré map attached to the periodic solution $\tilde{x}_0(t)$.

One can also attempt to discuss (under the hypothesis (d)) the cases with $\dim U \geqslant 2$; then Σ_0/Σ is isomorphic to a group acting absolutely irreducibly on U, and hence cannot be commutative. One can show that for such cases there will always be a breaking of both the orbital and the spatial symmetries. We hope to discuss this in a forthcoming paper.

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