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The translation equation and algebraic objects

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Introduction. A. Zajtz introduced in the note [7] the notion of the algebraic object. This notion is very closly related to the translation equation. The determination of algebraic objects can be reduced to the determination of solutions of the translation equation on a suitable algebraic structure (on a semigroupoid) satisfying the identity condition.

In this note we are going to give the general solution of the translation equation in a groupoid; next, the general solution of this equation with the identity condition.

Some proofs of theorems in this note are similar to proofs of respective theorems in the note [5]. In this note we regard the translation equation a little more generally. Some proofs are given for better readibility, although that proofs are similar to the proofs in [5].

Particularly, we are going to show that every solution of the translation equation in a Brandt groupoid, satisfying the identity condition, is defined on some Cartesian product. It means that every algebraic object over a Brandt groupoid is non-singular.

Next, we prove that every solution F of the translation equation in a groupoid can be extended to a solution $\bar{F}$ defined on a Cartesian product in such a manner that if F satisfies the identity condition, then $\bar{F}$ satisfies the identity condition. It follows that every algebraic object over a groupoid can be extended to a non-singular algebraic object at the same fibre. Author does not know if it is possible for a semigroupoid.

Preliminary.

DEFINITION 1 (cf. [6]). A non-empty set C with one binary interior operation "$\cdot$" defined for some pairs $(\alpha, \beta) \in C \times C$ will be called a *multiplicative system.*

DEFINITION 2 (cf. [5]). Two multiplicative systems C_1, C_2 will be called *isomorphic* if there exists a one-to-one function f transforming C_1 onto C_2 and satisfying the following condition: the product $f(\alpha)\cdot f(\beta)$ is defined if and only if the product $\alpha\cdot\beta$ is defined, and if it then holds:

$$f(\alpha\beta) = f(\alpha)f(\beta).$$

DEFINITION 3 (cf. [1]). A multiplicative system C will be called a *semigroupoid* if there are satisfied the following axioms:

(a) If in the equation

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma$$

one of its sides or both of the products $\beta\gamma$ and $\alpha\beta$ are defined, then both sides of the equation are defined and the equality holds,

(b) To every element α of C there exists exactly one left unit ε_α and exactly one right unit δ_α such that:

$$\varepsilon_\alpha \alpha = \alpha\delta_\alpha = \alpha.$$

(c) If the product $\alpha\beta$ is defined, then $\delta_\alpha = \varepsilon_\beta$.

DEFINITION 4 (cf. [1], [6]). A semigroupoid will be called a *groupoid* (1) if, in addition to axioms (a)-(c), also the following condition holds:

(d) To every element α there exists exactly one element α^{-1} (inverse to α) such that:

$$\alpha\alpha^{-1} = \varepsilon_\alpha, \quad \alpha^{-1}\alpha = \delta_\alpha.$$

DEFINITION 5 (cf. [4], [6]). A groupoid will be called a *Brandt groupoid* if the following condition holds:

(e) To every two elements α, γ there exists such an element β that both the products $\alpha\beta$ and $\beta\gamma$ are defined.

Let A be an arbitrary non-empty set and G an arbitrary group. In the set $A\times A\times G$ we define the operation "$\cdot$" as follows:

The product $(a, b, \alpha)\cdot(c, d, \beta)$ is defined if and only if $b = c$, and then

$$(a, b, \alpha)\cdot(c, d, \beta) = (a, d, \alpha\beta).$$

It is easy to verify that the set $A\times A\times G$, with such an operation "$\cdot$", is a Brandt groupoid. This groupoid will be called a *product groupoid* and it will be also denoted by $A\times A\times G$. The term "product groupoid" was proposed by A. Zajtz.

A. Nijenhuis has proved the theorem which can be formulated in the following way (cf. [4], p. 11):

THEOREM 1. *Every Brandt groupoid is isomorphic to some product groupoid.*

We quote two further theorems; the first of them was proved in [6], the other in [5].

THEOREM 2. *A multiplicative system C is a groupoid if and only if there exists such a decomposition $\{C_i\}_{i\in I}$ of the set C that every set C_i with the operation "$\cdot$" is a Brandt groupoid. If C is a groupoid, then such a decomposition is unique.*

(1) We were using the term "Ehresmann groupoid" instead of the term „groupoid" in note [5].

PROPOSITION 1. *If the product of two elements of a groupoid is defined, then they belong to the same Brandt groupoid.*

We now present the following three definitions given by A. Zajtz (cf. [7], p. 68, 69, 71).

DEFINITION 6. The pair $\Omega = (X, G)$, where X is a non-empty set and G is a semigroupoid, will be called a *left algebraic object over G at the fibre X* (or shortly an *algebraic object*) if the exterior product $(a, x) \to aX$ defined for some pairs $(a, x) \in G \times X$ satisfies the following axioms:

(A) To every element $x \in X$ there exists an element $a \in G$ such that the exterior product ax is defined.

(B) The exterior product is associative in the following sense: if the products βx and $a\beta$ are defined then both sides of the equality

$$a(\beta x) = (a\beta)x$$

are defined and this equality holds.

(C) Any unit is a neutral operator, i.e. if ax is defined, then $\varepsilon_a x$ and $\delta_a x$ are defined and the following equalities hold:

$$\varepsilon_a x = \delta_a x = x.$$

DEFINITION 7. The algebraic object will be called *non-singular* if the exterior product is defined for all pairs $(a, x) \in G \times X$; in the opposite case it will be called *singular*.

DEFINITION 8. Let (X, G) be an algebraic object and let $\bar{X} \subset X$. If a pair $(\bar{X}, G)$ with the restriction of the exterior product in the set X to the set $\bar{X}$ is an algebraic object, then the object $(\bar{X}, G)$ will be called a *subobject* of the object (X, G).

2. The general solution of the translation equation in a groupoid and algebraic objects. We shall denote by $(X \times G; X)$ the family of all functions defined in subsets of the set $X \times G$ and having their values in the set X.

We assume the following.

DEFINITION 9. Let X be an arbitrary set and G an arbitrary multiplicative system. We shall say that a function $F \in (X \times G; X)$ satisfies the translation equation if it satisfies the following condition: If $F(x, \beta)$ and $a\beta$ are defined then both sides of the equation

$$F[F(x, \beta), a] = F(x, a\beta) \tag{1}$$

are defined and this equality holds.

The functional equation (1) will be called the *translation equation.*

DEFINITION 10. We shall say that a function $F \in (X \times G; X)$ satisfies the *identity condition* if there are satisfied the following conditions:

(a) If $F(x, a)$ is defined and $\varepsilon_a a = a$, then $F(x, \varepsilon_a)$ is defined and $F(x, \varepsilon_a) = x$.

(b) If $F(x, a)$ is defined and $a\delta_a = a$, then $F(x, \delta_a)$ is defined and $F(x, \delta_a) = x$.

Let us assume that the multiplicative system G in Definitions 9 and 10 is a semigroupoid.

Let us put:

$$ax = F(x, a).$$

It is easy to see that every non-empty solution of the translation equation satisfying the identity condition determines some algebraic object and every algebraic object determines some non-empty solution of the translation equation satisfying the identity condition (see conditions (B) and (C) of Definition 6).

An object determined by a non-empty function $F \epsilon (X \times G; X)$ satisfying the translation equation and the identity condition is non-singular if and only if the domain of the function F is a set of the form

$$\bar{X} \times G \quad (\emptyset \neq \bar{X} \subset X).$$

Now we shall characterize the structure of the general solution of the translation equation in a Brandt groupoid. In view of Theorem 1 we may consider product groupoids only. We shall prove the following

THEOREM 3. *The general solution of the translation equation in a product groupoid $A \times A \times G$, for a basic set X, is a family of functions F which can be obtained in the following manner:*

(a) *To every $a \epsilon A$ we choose arbitrarily a set X_a and $\bar{X}_a$ such that $\bar{X}_a \subset X_a \subset X$ and the sets $\bar{X}_a$ have the same power (for $a \epsilon A$).*

(b) *To every $a \epsilon A$ we construct a function f_a transforming X_a onto $\bar{X}_a$ such that $f_a(x) = x$ for $x \epsilon \bar{X}_a$.*

(c) *Let a_0 be any fixed element of the set A. To every $a \epsilon A$ we construct a one-to-one function h_a transforming $\bar{X}_a$ onto $\bar{X}_{a_0}$.*

(d) *We choose an arbitrary function H defined on $\bar{X}_{a_0} \times G$ and satisfying the translation equation*(²).

(e) *We put:*

$$F(x, a, b, a) = h_a^{-1} H\big(h_b f_b(x), a\big) \quad \text{for } x \epsilon X_b. \tag{2}$$

Proof. At first we shall prove that the function F defined by construction (a)-(e) satisfies the translation equation. Let us consider two elements such that their product is defined, i.e. elements of the form:

$$(a, b, \beta), (b, c, a) \epsilon A \times A \times G.$$

(²) The form of these functions was given in the notes [2] and [3].

From the definition of the product groupoid we have evidently:

$$(a, b, \beta)\cdot(b, c, \alpha) = (a, c, \beta\alpha).$$

Let us, moreover, assume that $F(x, b, c, \alpha)$ is defined. From (2) and from the form of functions h_a it follows that $F(x, b, c, \alpha)\in\bar{X}_b$. Thus $f_b(F(x, b, c, \alpha))$ is defined. This implies that $F[F(x, b, c, \alpha), a, b, \beta]$ is defined and

$$F[F(x, b, c, \alpha), a, b, \beta] = h_a^{-1}H\big(h_b f_b h_a^{-1}H(h_c f_c(x), \alpha), \beta\big).$$

Since $h_b^{-1}H(h_c f_c(x), \alpha)\in\bar{X}_b$, we have:

$$f_b h_b^{-1}H(h_c f_c(x), \alpha) = h_b^{-1}H(h_c f_c(x), \alpha).$$

Hence and from the fact that the function H satisfies the translation equation we obtain:

$$F[F(x, b, c, \alpha), a, b, \beta] = h_a^{-1}H\big(H(h_c f_c(x), \alpha), \beta\big)$$
$$= h_a^{-1}H(h_c f_c(x), \beta\alpha) = F(x, a, c, \beta\alpha).$$

This completes the first part of the proof of Theorem 3.

Now, let us assume that the function $F\in(X\times(A\times A\times G); X)$ satisfies the translation equation.

We are going to show that the function F can be obtained by construction (a)-(e).

We divide the proof into a few parts.

We shall denote by 1 the unity of the group G.

I. Let x be a fixed element of the set X. We shall prove that $F(x, a, b, \alpha)$ is defined if and only if $F(x, b, b, 1)$ is defined. Assume that $F(x, b, b, 1)$ is defined. Since the product $(a, b, \alpha)(b, b, 1)$ is defined and F satisfies the translation equation thus, $F[F(x, b, b, 1), a, b, \alpha]$ is defined and consequently $F(x, a, b, \alpha)$ is defined. Analogously, since the product $(b, a, \alpha^{-1})(a, b, \alpha)$ is defined thus if $F(x, a, b, \alpha)$ is defined, then $F(x, b, b, 1) = F[F(x, a, b, \alpha), b, a, \alpha^{-1}]$ is defined. We define the function f_a by the following condition:

$$f_a(x) = F(x, a, a, 1) \tag{3}$$

and we define the set X_a as the domain of the function f_a and $\bar{X}_a$ as the set of values of this function.

Let $f_a(x)$ be defined. Then it follows from (3) that $f_a f_a(x)$ is defined and $f_a f_a(x) = f_a(x)$. This proves that $\bar{X}_a\subset X_a$ and $f_a(x) = x$ for $x\in\bar{X}_a$.

II. Let a_0 be an arbitrarily fixed element of the set A. Let us consider an arbitrary element $y\in\bar{X}_a$. From the definition of the set $\bar{X}_a$ it follows that there exists an element $x\in X$ such that $y = F(x, a, a, 1)$.

We put:

$$h_a(y) = F(x, a_0, a, 1). \tag{4}$$

Since $F(x, a, a, 1)$ is defined thus it follows from I that $F(x, a_0, a, 1)$ is defined.

We shall show that the relation $y \to h_a(y)$ is independent of the choice of the element x, thus this is a function with the domain X_a.

Suppose that $F(x_0, a, a, 1) = F(x_1, a, a, 1)$. Since F satisfies the translation equation and $F(x_0, a_0, a, 1)$ is defined (see I), we have the following equalities:

$$F(x_0, a_0, a, 1) = F[F(x_0, a, a, 1), a_0, a, 1]$$
$$= F[F(x_0, a, a, 1), a_0, a, 1] = F(x_1, a_0, a, 1).$$

It proves that the function h_a is well defined.

Similary we also obtain: if $F(x_0, a_0, a, 1) = F(x_1, a_0, a, 1)$, then $F(x_0, a, a, 1) = F(x_1, a, a, 1)$.

It shows that the function h_a is an injection.

We shall prove that the function h_a transforms the set $\bar{X}_a$ onto the set $\bar{X}_{a_0}$. First we show that if $F(x, a, b, a)$ is defined, then $F(x, a, b, a) \epsilon \bar{X}_a$. Suppose that $F(x, a, b, a)$ is defined. Then $F[F(x, a, b, a), a, a, 1]$ is defined and we have:

$$F(x, a, b, a) = F[F(x, a, b, a), a, a, 1] = f_a\big(F(x, a, b, a)\big).$$

Thus $F(x, a, b, a) \epsilon X_a$.

In particular, it shows that the values of the function h_a belong to $\bar{X}_{a_0}$. Now let us consider an arbitrary element y of the set $\bar{X}_{a_0}$. It has to have the form:

$$y = F(x_0, a_0, a_0, 1).$$

Then $F(x_0, a, a_0, 1)$ is defined and $F(x_0, a, a_0, 1) \epsilon \bar{X}_a$. It implies that there exists $x_1 \epsilon X$ such that

$$F(x_0, a, a_0, 1) = f_a(x_1) = F(x_1, a, a, 1).$$

Since F satisfies the translation equation, we have:

$$F(x_0, a_0, a_0, 1) = F[F(x_0, a, a_0, 1), a_0, a, 1]$$
$$= F[F(x_1, a, a, 1), a_0, a, 1] = F(x_1, a_0, a, 1).$$

In such a manner we have obtained:

$$h_a\big(F(x_1, a, a, 1)\big) = F(x_1, a_0, a, 1) = F(x_0, a_0, a_0, 1) = y.$$

Thus the function h_a transfroms $\bar{X}_a$ onto $\bar{X}_{a_0}$. Since, as we have proved, h_a is a one-to-one function transforming $\bar{X}_a$ onto $\bar{X}_{a_0}$, thus (for every $a \epsilon A$) the sets $\bar{X}_a$ have the same power.

Now we are going to prove that if $F(x, a_0, a_0, 1)$ is defined then, for every $a \epsilon A$, the equality

$$F(x, a, a_0, 1) = h_a^{-1}\big(F(x, a_0, a_0, 1)\big) \tag{5}$$

holds.

Suppose that $F(x, a_0, a_0, 1)$ is defined. It follows that $F(x, a, a_0, 1)$ is defined and $F(x, a, a_0, 1) \epsilon \overline{X}_a$.

Thus there exists $\bar{x} \epsilon X$ such that:

$$F(\bar{x}, a, a, 1) = F(x, a, a_0, 1). \tag{6}$$

Hence, from the translation equation, we have

$$\begin{aligned} F(\bar{x}, a_0, a, 1) &= F[F(\bar{x}, a, a, 1), a_0, a, 1] \\ &= F[F(x, a, a_0, 1), a_0, a, 1] = F(x, a_0, a_0, 1). \end{aligned} \tag{7}$$

We obtain from (4), (6) and (7):

$$h_a\big(F(x, a, a_0, 1)\big) = h_a\big(F(\bar{x}, a, a, 1)\big) = F(\bar{x}, a_0, a, 1) = F(x, a_0, a_0, 1).$$

This equality implies condition (5).

III. Now we define the function H.

We put:

$$H(x, \alpha) = F(x, a_0, a_0, \alpha) \quad \text{for } x \epsilon \overline{X}_{a_0}, \alpha \epsilon G. \tag{8}$$

We shall prove that the function H is defined on the set $\overline{X}_{a_0} \times G$.

Let x be an arbitrary element of the set $\overline{X}_{a_0}$. It follows from the definition of the set $\overline{X}_{a_0}$ that there exists $\bar{x} \epsilon X_{a_0}$ such that $x = F(\bar{x}, a_0, a_0, 1)$. Since $F(\bar{x}, a_0, a_0, 1)$ is defined, thus for every $\alpha \epsilon G, F(x, a_0, a_0, \alpha)$ is defined (see I), i.e. $H(x, \alpha)$ is defined.

We proved in II that if $F(x, a, b, \alpha)$ is defined, then $F(x, a, b, \alpha) \epsilon \overline{X}_a$. It follows that the values of the function H belong to $\overline{X}_{a_0}$. Since F satisfies the translation equation, we have for every $x \epsilon \overline{X}_{a_0}$ and every $\alpha \epsilon G$:

$$\begin{aligned} H\big(H(x, \alpha), \beta\big) &= F[F(x, a_0, a_0, \alpha), a_0, a_0, \beta] \\ &= F(x, a_0, a_0, \beta\alpha) = H(x, \beta\alpha). \end{aligned}$$

It proves that the function H defined by (8) satisfies the translation equation on the set $\overline{X}_{a_0} \times G$.

Remark 1. The function H defined by (8) satisfies the identity condition, too.

If $x \epsilon \overline{X}_{a_0}$, i.e. x has a form:

$$x = F(\bar{x}, a_0, a_0, 1),$$

then we get:

$$H(x, 1) = F(x, a_0, a_0, 1) = F[F(\bar{x}, a_0, a_0, 1), a_0, a_0, 1] = F(\bar{x}, a_0, a_0, 1) = x.$$

IV. We shall prove that the function F can be written in the form (2). Assume that $F(x, a, b, a)$ is defined. From the assumption that F satisfies the translation equation and from (3), (4), (5) and (8) we obtain:

$$F(x, a, b, a) = F\{F(x, a_0, b, a), a, a_0, 1\} = F\{F[F(x, a_0, b, 1), a_0, a_0, a],$$
$$a, a_0, 1\} = h_a^{-1}\Big(F\{F\big[h_b\big(F(x, b, b, 1), a_0, a_0, a\big], a_0, a_0, 1\}\Big)$$
$$= h_a^{-1}\Big(F\{h_b\big(F(x, b, b, 1)\big), a_0, a_0, a\}\Big) = h_a^{-1}\Big(H\big(h_b f_b(x), a\big)\Big).$$

The definiteness of suitable expressions in the above equalities was proved in part I and II of this proof. It completes the proof of Theorem 3.

PROPOSITION 2. *Let us assume that the function $F \in (X \times B; X)$, where B is a Brandt groupoid, satisfies the translation equation and the identity condition. Let us denote by $\bar{X}$ the set of elements $x \in X$ such that there exists an element $a \in B$ such that $F(x, a)$ is defined. Then the function F is defined on the set $\bar{X} \times B$.*

Proof. In virtue of Theorem 1 we may restrict attention to product groupoids only.

Assume that $x \in \bar{X}$. Thus there exists an element $(a_1, b_1, a_1) \in A \times A \times G$ such that $F(x, a_1, b_1, a_1)$ is defined.

From part I of the proof of Theorem 3 we obtain: for every $a \in A$ and for every $a \in G$, $F(x, a, b_1, a)$ is defined. From this and from Definition 10 (condition (a)) it follows that $F(x, a, a, 1)$ is defined. It implies that, for every $(b, a, a) \in A \times A \times G$, $F(x, b, a, a)$ is defined. It completes the proof.

Remark 2. It is evident that the assumptions of the above proposition can be reduced. In the proof of this proposition we have used condition (a) of Definition 10 only. But Proposition 2 is not true if, instead of the identity condition, we assume condition (b) of Definition 10 only.

It follows immediately from Proposition 2 the following

COROLLARY 1. *Every algebraic object over a Brandt groupoid is non-singular.*

Using Proposition 2 we shall prove the following

COROLLARY 2. *The function F of the form (2) (where H satisfies the identity condition) satisfies the identity condition if and only if every function f_a is the identity function on the set $\bar{X}$ ($\bar{X}$ was defined in Proposition 2).*

Proof. From (2) it follows that $F(x, a, a, 1)$ is defined if and only if $f_a(x)$ is defined. As for every $x \in \bar{X}$, $F(x, a, a, 1)$ is defined (see Proposition 2) and H satisfies the identity condition, we have for every $x \in \bar{X}$:

$$F(x, a, a, 1) = h_a^{-1}\big(H(h_a f_a(x), 1)\big) = h_a^{-1} h_a f_a(x) = f_a(x).$$

It implies the assertion.

Now we shall consider a possibility of a suitable extension of solutions of the translation equation in a Brandt groupoid.

First we prove the following

PROPOSITION 3. *Let $F\epsilon(X\times B; X)$, where B is a Brandt groupoid, satisfy the translation equation. Then F can be extended to a function $\bar{F}$: $X\times B\rightarrow\bar{X}$ satisfying the translation equation.*

The set $\bar{X}$ was defined in Proposition 2.

Proof. It suffices to prove the assertion for a product groupoid. Let the function F satisfy the translation equation. Then F has the form (2). We shall extend functions f_a only. For every $a\epsilon A$ we define the function $\bar{f}_a$ as follows:

$$\bar{f}_a(x) = \begin{cases} f_a(x) & \text{for } x\epsilon X_a, \\ x_a & \text{for } x\epsilon\bar{X}\setminus X_a, \end{cases}$$

where x_a is an arbitrary fixed element of the set $\bar{X}_a$. Put:

$$\bar{F}(x, a, b, \alpha) = h_a^{-1}\big(H(h_b\bar{f}_b(x), \alpha)\big) \quad \text{for } x\epsilon\bar{X},\ (a, b, \alpha)\epsilon A\times A\times G.$$

The function F has the form (2), thus it satisfies the translation equation and it is defined on the set $\bar{X}\times(A\times A\times G)$. It is also clear that $\bar{F}$ is an extension of the function F.

Now we shall prove the following

PROPOSITION 4. *Let B be a Brandt groupoid and let $\bar{\bar{X}}$ be an arbitrary set such that $\bar{X}\subset\bar{\bar{X}}$ and let $F\epsilon(X\times B; X)$ satisfy the translation equation. Then F can be extended to a function $\bar{\bar{F}}\epsilon\bar{\bar{X}}^{\bar{\bar{X}}\times B}$ satisfying the translation equation in such a manner that if F satisfies the identity condition, then F satisfies the identity condition.*

Proof. From Propositions 2 and 3 it follows that it is possible to extend the function F to a function $\bar{F}\epsilon\bar{X}^{\bar{X}\times B}$.

Let us put:

$$\bar{\bar{F}}(x, \alpha) = \begin{cases} \bar{F}(x, \alpha) & \text{for } x\epsilon\bar{X},\ \alpha\epsilon B, \\ x & \text{for } x\epsilon\bar{\bar{X}}\setminus\bar{X},\ \alpha\epsilon B. \end{cases} \tag{9}$$

If F satisfies the identity condition, then $\bar{F}$ satisfies the identity condition. In this case also $\bar{\bar{F}}$ satisfies the identity condition. Moreover, $\bar{\bar{F}}$ satisfies the translation equation. It is sufficient to verify that this is the case for $x\epsilon\bar{\bar{X}}\setminus\bar{X}$. From (9) we obtain for $x\epsilon\bar{\bar{X}}\setminus\bar{X}$:

$$\bar{\bar{F}}[\bar{\bar{F}}(x, \alpha), \beta] = \bar{\bar{F}}[x, \beta] = x = \bar{\bar{F}}[x, \beta\alpha].$$

It completes the proof.

Now we shall determine the general solution of the translation equation in a groupoid. We shall prove the following

THEOREM 4. *The general solution of the translation equation in an arbitrary groupoid E (for the basic set X) is a family of functions F which can be obtained in the following manner:*

Let

$$E = \bigcup_{s \in S} E_s,$$

where every E_s is a Brandt groupoid and $E_{s_1} \cap E_{s_2} = \emptyset$ for $s_1 \neq s_2$ (see Theorem 2).

(a) *For every $s \in S$ we choose an arbitrary function $F_s \in (X \times E_s; X)$ satisfying the translation equation.*

(b) *We put: $F(x, a) = F_s(x, a)$ for $a \in E_s$ and all $x \in X$ such that $F_s(x, a)$ is defined, i.e.*

$$F = \bigcup_{s \in S} F_s.$$

Proof. From Proposition 1 it follows evidently that the function F, defined in such a manner, satisfies the translation equation. Conversely, one can see that, if F satisfies the translation equation and we put:

$$F_s(x, a) = F(x, a) \quad \text{for } a \in E_s \quad \text{and all } x \in X$$

such that $F(x, a)$ is defined, then the function F_s satisfies the translation equation. This shows that the function F can be obtained by construction (a)-(b).

From Theorem 4 we get immediately the following

COROLLARY 3. *The function F satisfying the translation equation in a groupoid satisfies the identity condition if and only if every function F_s does.*

From Theorem 4 and Proposition 4 we obtain immediately also the following

THEOREM 5. *Let E be a groupoid and let $\overline{\overline{X}}$ be an arbitrary set such that $\overline{X} \subset \overline{\overline{X}}$ and let $F \in (X \times E; X)$ satisfy the translation equation. Then F can be extended to a function $\overline{F} \in \overline{\overline{X}}^{\overline{X} \times E}$ satisfying the translation equation in such a manner that if F satisfies the identity condition, then $\overline{F}$ satisfies the identity condition.*

The part of Theorem 5 may be also formulated as follows:

THEOREM 6. *Every algebraic object over a groupoid can be extended to a non-singular algebraic object at the same fibre over this groupoid.*

Theorem 6 can be also given the following equivalent formulation:

THEOREM 6′. *Every algebraic object over a groupoid is a subobject of some non-singular algebraic object at the same fibre over this groupoid.*

The question whether every algebraic object can be extended to a non-singular algebraic object (at the same fibre or not) over a semi-groupoid is open.

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