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THE EXISTENCE OF AN a.c.i.p.m. FOR AN EXPANDING MAP OF THE INTERVAL; THE STUDY OF A COUNTEREXAMPLE

B. SCHMITT

Université de Bourgogne, Laboratoire de Topologie
Dijon Cédex, France

0. Introduction

In this paper, we consider mappings f of a compact interval into itself which are piecewise differentiable and expanding; we consider also f-invariant probability measures.

Such measures always exist; but if we wish to study the statistical behaviour of almost every f-orbit, with respect to the Lebesgue measure on $[0, 1]$ (the compact interval under consideration) one type of them is of particular interest: probability measures μ which satisfy

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} g(f^i(x)) = \int_0^1 g(y)\,d\mu(y) \tag{1}$$

for x belonging to an open set in $[0, 1]$.

These measures, called S-R-B (Sinai–Ruelle–Bowen), when they exist, provide some visualisation of the attractors of the dynamics of f.

We are also interested in another type of invariant measures: absolutely continuous invariant probability measures, abbreviated a.c.i.p.m. The existence of an a.c.i.p.m. tells us that the dynamics of f is "chaotic". Moreover, if an a.c.i.p.m. μ is ergodic – i.e. if the μ-measure of each f-invariant set is zero or one – then it is an S-R-B measure.

We are going to discuss the existence of an a.c.i.p.m. for piecewise differentiable expanding maps of the interval.

This problem was first studied in the case of piecewise linear transformations [Re]; [Pa]; [Wi]; then in the case of piecewise C^2 maps [La–Y]. Finally, the existence of an a.c.i.p.m. was obtained under conditions weaker than C^2 [W]; [Co]. The problem is therefore: is the C^1 character of piecewise expanding maps of the interval sufficient for the existence of an a.c.i.p.m.?

We give a condition on the modulus of continuity of f' which ensures the existence of an a.c.i.p.m.; we prove that it is ergodic and Bernoulli in one particular case. Then we construct a piecewise continuous map differentiable except on a countable set of points, which does not fulfil this condition and has no a.c.i.p.m. [Sc].

1. Definitions; notation

DEFINITION. A mapping f of the interval $[0, 1]$ into itself is called *piecewise C^1 expanding* if there exist a finite partition $([a_i, a_{i+1}])_{i=0,\ldots,p-1}$ of the interval $[0, 1[$, $0 = a_0 < a_1 < \ldots < a_p = 1$, and a real constant $\varrho > 1$ such that

(i') $f|_{]a_i,a_{i+1}[}$ is C^1 and extends to a C^1 function on $[a_i, a_{i+1}]$,

(ii) $|(f|_{]a_i,a_{i+1}[})'| \geqslant \varrho$.

A probability measure defined on the Borel σ-algebra of $[0, 1]$ is called *f-invariant* if for all Borel sets B in $[0, 1]$ we have

$$\mu(f^{-1}(B)) = \mu(B).$$

A measure is an a.c.i.p.m. if it is f-invariant and there exists a function h defined on $[0, 1]$, integrable with respect to the Lebesgue measure λ on $[0, 1]$, such that

$$\int_{[0,1]} h(x)\,d\lambda(x) = 1,$$

and for all Borel subset B of $[0, 1]$

$$\mu(B) = \int_B h(x)\,d\lambda(x).$$

The problem of existence of a.c.i.p.m. for C^1 transformations of the interval has been the object of numerous articles in recent years: [La–Y]; [Wa]; [L–Y]; [B–S]; [Co]; [B]; [Le]; [Mi]; [W], etc... In the case of piecewise expanding maps it leads to the examination of the modulus of continuity of $|f'|$ on each interval $[a_i, a_{i+1}]$.

We denote by $L^1_\lambda([0, 1])$ the real Banach space of functions h defined on $[0, 1]$ integrable with respect to the Lebesgue measure λ. We define the *Perron–Frobenius operator* P on $L^1_\lambda([0, 1])$ by its action on h:

$$Ph(x) = \sum_{y \in f^{-1}(\{x\})} \frac{h(y)}{|f'(y)|}.$$

It is easy to verify that for all Borel subsets B of $[0, 1]$ we have

$$\int_{f^{-1}(B)} h(x)\,d\lambda(x) = \int_B Ph(x)\,d\lambda(x),$$

which implies that the existence of an a.c.i.p.m. is equivalent to the existence of a fixed point in $L^1([0, 1])$ for the operator P.

We denote by $\mathscr{P}$ the partition modulo 0 of $[0, 1]$ consisting of the intervals $(]a_i, a_{i+1}[)_{i=0,\ldots,p-1}$ and by $\mathscr{P}^{(n)}$ the partition

$$\mathscr{P}^{(n)} = \bigvee_{i=0}^{n-1} f^{-i}(\mathscr{P}).$$

For an atom $A^{(n)}$ of $\mathscr{P}^{(n)}$, we define the real number

$$d(A^{(n)}) = \operatorname*{Sup}_{A^{(n)}} |f'| - \operatorname*{Inf}_{A^{(n)}} |f'|.$$

We consider the sequence $(d_n)_{n\in N}$,

$$d_n = \operatorname*{Sup}_{A^{(n)}\in\mathscr{P}^{(n)}} d(A^{(n)}),$$

and inspect the condition

(iii) $$\sum_{n\geq 1} d_n < +\infty.$$

Remarks. (1) Condition (iii) generalizes various conditions studied before by other authors.

(2) Let Ω be the function

$$\Omega(h) = \operatorname*{Sup}_{i\in[0,p-1]} \operatorname*{Sup}_{\substack{0<|x-y|<h\\ x,y\in]a_i,a_{i+1}[}} |f'(x)-f'(y)|.$$

Condition (iii) is equivalent to the integrability in the sense of Riemann (improper integral) of the function $\dfrac{\Omega(h)}{h}$ in a neighbourhood of zero.

Finally, we say that a probability measure μ defined on Borel subsets of $[0, 1]$ is ergodic if all f-invariant Borel sets have μ measure zero or one.

2. Markov transformations, piecewise C^1 and expanding

Definition. Let f be a transformation satisfying conditions (i′) and (ii); it is Markov relatively to the partition $([a_i, a_{i+1}])$ of the interval $[0, 1[$ if for all pairs of integers (i, j) from $[0, p-1]$ such that

$$f(]a_i, a_{i+1}[) \cap]a_j, a_{j+1}[\neq \emptyset,$$

we have

$$f(]a_i, a_{i+1}[) \supset]a_j, a_{j+1}[.$$

The existence of an a.c.i.p.m. for a Markov transformation is equivalent to the existence of an a.c.i.p.m. for an iterate f^k on an f^k-invariant interval. We do not restrict the problem when we replace condition (i′) by:

(i) f satisfies (i′) and $f([a_i, a_{i+1}] = [0, 1]$ for $i = 0, 1, \ldots, p-1$.

THEOREM 2.1. *Every mapping f of the interval* $[0, 1]$ *into itself satisfying conditions* (i), (ii), (iii) *admits a unique a.c.i.p.m.*

To prove this theorem, we show that the Perron–Frobenius operator P admits a unique fixed point. We remind the main properties of this operator in the following proposition.

PROPOSITION 2.1. (1) *P is a positive and sub-Markov operator on* $L^1_\lambda([0, 1])$.

(2) *If h belongs to* $L^1_\lambda([0, 1])$ *then*

$$\int_{[0,1]} Ph(x)\,d\lambda(x) = \int_{[0,1]} h(x)\,d\lambda(x).$$

(3) *If g belongs to* $L^\infty_\lambda([0, 1])$ *and h belongs to* $L^1_\lambda([0, 1])$ *then*

$$\int_{[0,1]} g \circ f^n(x)\,h(x)\,d\lambda(x) = \int_{[0,1]} g(x)\,P^n h(x)\,d\lambda(x).$$

(4) *If A is a Borel subset of* $[0, 1]$ *then*

$$\lambda(f^{-n}(A)) = \int_A (P^n 1)(x)\,d\lambda(x).$$

To prove the Theorem 2.1, we introduce a cone K in $C^+([0, 1])$, P invariant, whose P orbits are relatively compact in $L^1_\lambda([0, 1])$ with the usual norm. To define the cone K we need the following concept.

DEFINITION. *Two points x and y of* $[0, 1]$ *are n-neighbouring if x and y belong to the same atom of $\mathscr{P}^{(n)}$; in symbols, $x \overset{n}{\sim} y$.*

We define K as the set of continuous functions h for which there exists a real constant $k(h)$ such that, given any integer $n \geqslant 0$ and any x, y with $x \overset{n}{\sim} y$, we have:

$$\frac{h(x)}{h(y)} \leqslant \exp(k\,R_{n+1}),$$

where

$$R_{n+1} = \sum_{m \geqslant n+1} d_m.$$

LEMMA 2.1. (1) *The set K is a convex nonempty cone in* $C^+([0, 1])$.

(2) *If g, $h \in K$, then $g \cdot h \in K$.*

(3) *L, the linear span of K, is an algebra, it is either dense in $C([0, 1])$ with the topology of uniform convergence, or reduces to the constants.*

Remark. If we replace the series (d_n) by suitable convergent series (d'_n) such that $d_n \leqslant d'_n$ for all $n \in N$, we may suppose that the algebra L is dense in $C([0, 1])$.

The condition of relative compactness in $L^1_\lambda([0, 1])$ of the orbits $(P^n h, n \in N)$ for h belonging to K follows from the following uniform property.

LEMMA 2.2. *If h belongs to K, then there exists a real constant M depending only on h and satisfying the following condition, for any integers n, p: if $x \overset{\mathcal{L}}{\sim} y$, then*

$$\frac{P^n h(x)}{P^n h(y)} \leqslant \exp(MR_{p+1}).$$

The proof of this lemma is based on an idea used by P. Collet [Co] and implies the next lemma, in which $Q_n = \frac{1}{n} \sum_{i=0}^{n-1} P^i$.

LEMMA 2.3. *If h belongs to K, then the family $(Q_n h)_{n \in N}$ is relatively compact in $L^\lambda_1([0, 1])$.*

We then deduce the existence of a fixed point h for the operator P, and we can easily show that it belongs to K.

Theorem 2.1 hence follows; in fact, if we denote by μ the a.c.i.p.m. $hd\lambda$, we have the following property, resulting from the fact that L is dense in $C([0, 1])$, in the uniform convergence norm:

PROPOSITION 2.2. *Every mapping f of the interval into itself satisfying, conditions* (i), (ii), (iii) *admits a unique ergodic a.c.i.p.m.*

We say that a finite partition Q of $[0, 1]$ defines a *weakly Bernoulli process* for the dynamical system $([0, 1]), f, \mu)$ if for all $\varepsilon > 0$ there exists a positive integer n such that for all positive integers m and p we have:

$$\sum |\mu(A \cap B) - \mu(A)\mu(B)| < \varepsilon,$$

summation spreading over $A \in \bigvee_{i=0}^{m-1} f^{-i} Q$, $B \in \bigvee_{i=m+n}^{m+n+p} f^i Q$.

Moreover, if the partition Q is generating, then the natural extension of dynamical system $([0, 1], f, \mu)$ is isomorphic to a Bernoulli scheme.

Using the results of P. Walters [Wa; Th. 13 p. 26] we obtain.

PROPOSITION 2.3. *If f fulfils conditions* (i), (ii), (iii), *then the unique a.c.i.p.m. is ergodic for f^k, for any positive integer k, and the partition defines a weakly Bernoulli process.*

3. The general case

THEOREM 3.1. *Every mapping f of the interval* [0, 1] *into itself, satisfying conditions* (i′), (ii), (iii) *admits an a.c.i.p.m.*

The proof of this theorem is based on a criterion of precompactness in $L_1^\lambda([0, 1])$ due to Fréchet–Kolmogorov [Yo]. It uses the inequality resulting from the proof of Lemma 2.2 if $x \overset{n+p}{\sim} y$ then

$$\frac{|(f^n)'(x)|}{|(f^n)'(y)|} \leqslant \exp\left[\frac{1}{P}\sum_{i=0}^{n-1} d_{n+p-i}\right].$$

This proposition relies on two lemmas, which are adapted from those given in [Co].

LEMMA 3.1. *There exist two reals* $\varepsilon_0 > 0$ *and* $\gamma > 0$ *such that, for any integer n, any x belonging to* [0, 1] *and any real* ε, $0 < \varepsilon < \varepsilon_0$, *we have*

$$\int_{[0,1]\cap[x-\varepsilon,x+\varepsilon]} P^n 1(x)\, d\lambda(x) \leqslant \varepsilon^\gamma.$$

LEMMA 3.2. *If f satisfies conditions* (i′), (ii), (iii) *then*

$$\lim_{\eta\to 0}\int_{[0,1]} |P^n 1(x) - P^n 1(x+\eta)\, 1_{[0,1]}(x+\eta)|\, d\lambda(x) = 0.$$

The proofs of these lemmas are omitted here; the reader can refer to [Co]; [Sc].

The assertion of Lemma 3.2 is also satisfied by the sequence $Q_n 1 = \frac{1}{n}\sum_{l=0}^{n-1} P^l 1$.

All that remains is to apply the Fréchet–Kolmogorov criterion for precompactness to obtain Theorem 3.1.

In this general case, the problem of the number of a.c.i.p.m. measures, as well as the problem of their ergodicity remains open.

4. The counterexample

(a) *Construction of a Cantor set of Lebesgue measure zero.*

Consider the interval $I = [0, 1]$. We define the following open intervals:

– $I_1^1 =]a_1^1, b_1^1[$ is the interval centered at $\frac{1}{2}$, of length $\frac{1}{2}$.

– In the two connected components of $K_1 = I \setminus I_1^1$ we define the intervals $I_2^1 =]a_2^1, b_2^1[$ and $I_2^2 =]a_2^2, b_2^2[$ centered at the midpoints of there components, so that

$$a_2^1 < b_2^1 < a_2^2 < b_2^2 \quad \text{and} \quad \lambda(I_2^1) = \lambda(I_2^2) = \frac{\frac{1}{3}\left(1-\frac{1}{2}\right)}{2}.$$

– Suppose that the intervals $I_p^{i_p}$ are constructed for $p = 1, 2, \ldots, n$ and $i_p = 1, 2, \ldots, 2^{p-1}$.

The set $K_n = I \setminus \bigcup_{p=1}^{n} \left(\bigcup_{i_p=1}^{2^{p-1}} I_p^{i_p} \right)$ is compact and it is the union of 2^n connected components; we define the intervals $I_{n+1}^{i_{n+1}} =]a_{n+1}^{i_{n+1}}, b_{n+1}^{i_{n+1}}[$, $i_{n+1} = 1, 2, \ldots, 2^n$ centered of the midpoints of these components so that

$$a_{n+1}^1 < b_{n+1}^1 < a_{n+1}^2 < b_{n+1}^2 < \ldots < a_{n+1}^{2^n} < b_{n+1}^{2^n},$$

$$\lambda(I_{n+1}^{i_{n+1}}) = \frac{\frac{1}{n+2}\left(1-\frac{1}{n+1}\right)\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{1}{2}\right)}{2^n}.$$

The intervals $(I_n^{i_n})_{n \geq 1}$ are disjoint and the set $K = \bigcap_{n=1}^{\infty} K_n$ is a Cantor set. By construction we have:

$$\lambda(K_n) = \left(1-\frac{1}{2}\right)\ldots\left(1-\frac{1}{n+1}\right),$$

and hence $\lambda(K) = 0$.

(b) *Construction and properties of the map.*

(b.α) *Construction.* Le b be a real number greater than 1, and let $J = [0, b]$. We construct a transformation f of J onto itself in the following way:

C.1. $f(x) = f(1-x)$ for all x belonging to $\left[0, \frac{1}{2}\right]$.

C.2. $f_0 = f|_{]1,b[}$ is the linear increasing map of $]1, b[$ onto $]0, b[$.

C.3. $f_n^{i_n} = f|_{I_n^{i_n}}$ is the linear increasing map of $]a_n^{i_n}, b_n^{i_n}[$ onto $]a_{n-1}^{i_n} b_{n-1}^{i_n}[$, for $i_n = 1, 2, \ldots, 2^{n-2}$ and $n \geq 2$.

C.4. $f_1 = f|_{I_1^1}$ is the linear increasing map of $\left]a_1^1, \frac{1}{2}\right[$ onto $]1, b[$.

C.5. We choose b so that f_1' be strictly greater than 2.

LEMMA 4.1. *Suppose that* $b > 1$ *is fixed and satisfies* C.5. *There is a unique mapping* f *of* J *into itself fulfilling conditions* C.1, C.2, C.3, C.4. *It is Lipschitz with constant* 6, *differentiable and expanding except on a countable subset of* J; *it is Markov for the partition* $\mathscr{P} = \left\{\left]0, \frac{1}{2}\right[; \left]\frac{1}{2}, 1\right[;]1, b[\right\}$ *and onto on each atom of* $\mathscr{P}$.

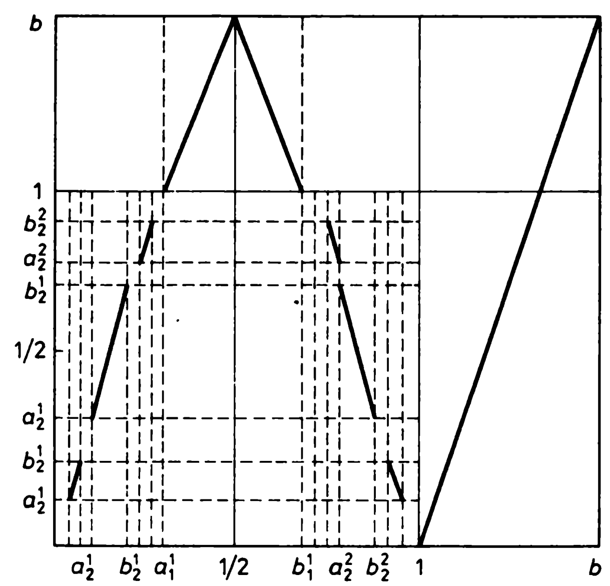


Fig. 1

(b.β) *A property of the derivative.*
We define the reals

$$d_n = \sup_{A^{(n)} \in \mathscr{A}^{(n)}} \left(\sup_{A^{(n)}} |f'| - \inf_{A^{(n)}} |f'|\right)$$

LEMMA 4.2. *The series* $\sum_{n>1} d_n$ *is divergent.*

It is obvious that for each atom $A^{(n)} \subset [1, b]$.

$$\sup_{A^{(n)}} |f'| - \inf_{A^{(n)}} |f'| = 0$$

On the other hand, according to conditions C.1 and C.3, the intervals $I_p^{j_p}$ ($j_p = 1, \dots, 2^{p-1}$) satisfy the inclusion

$$f^j(I_p^{j_p}) \subset \left]0, \frac{1}{2}\right[\cup \left]\frac{1}{2}, 1\right[\quad \text{for } p \geqslant n+1 \text{ and } j = 0, 1, \dots, n.$$

Then, for each atom $A^{(n)}$ of $\mathscr{P}^{(n)}$ contained in [0, 1], there is an interval $I_p^{i_p}$, $p \geqslant n+1$, such that

$$I_p^{i_p} \subset A^{(n)}.$$

Therefore,

$$\sup_{A^{(n)}[0,1]} |f'| - \inf_{A^{(n)}[0,1]} |f'| \geqslant 2\frac{n+2}{n} - 2\frac{p+1}{p-1}, \quad \forall p \geqslant n+1.$$

We conclude that $d_n \geqslant 2\frac{n+2}{n} - 2 = \frac{4}{n}$. The lemma is proved.

(b.γ) *Transformation f does not admit any* a.c.i.p.m. We write

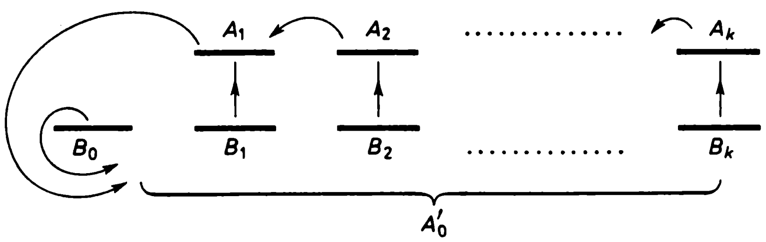
$$A_0 = [1, b]; \quad A_k = \bigcup_{i_k=1}^{2^{k-1}} I_k^{i_k}; \quad \lambda_0 = \frac{1}{f_0'}; \quad \lambda_i = \frac{2}{f_i'}, \quad i = 1, 2 \ldots$$

Let $A_0' = A_0 \backslash f^{-1}(K)$ and $B_n = f^{-1}(A_n) \cap A_0'$ be the sets of points which return to A_0' in exactly $n+1$ steps. We have:

$$A_0' = \bigcup_{n \geqslant 0} [f^{-1}(A_n) \cap A_0'].$$

LEMMA 4.3. *The induced map $f_{A_0'}$ admits the Lebesgue measure $\lambda_{A_0'}$, on A_0 as the unique a.c.i.p.m.; the dynamical system $(A_0', f_{A_0'}, \lambda_{A_0'})$ is Bernoulli.*

We can now describe the mapping f together with its induced transformation $f_{A_0'}$ and one floor towers in the following way:



We have:

$$f^{-1}(A_k) = B_k \cup A_{k+1}, \tag{2}$$
$$f^{-1}(A_0) = B_0 \cup A_1.$$

LEMMA 4.4. *The measure induces on A_0' by an a.c.i.p.m. for f, is an a.c.i.p.m. for $f_{A_0'}$.*

PROPOSITION 4.1. *The mapping f either has no a.c.i.p.m. or has an infinite and σ-finite absolutely continuous invariant measure.*

Proof. Using relation (2), we see that in μ is an a.c.i.p.m. then

$$\mu(A_0') = \mu(B_0) + \mu(A_1).$$

Using Lemmas 4.3 and 4.4, we infer that the measure induced by μ on A_0' is equal to A_0' and therefore

$$\mu(A_1) = \mu(A_0') - \lambda_{A_0'}(B_0) = \mu(A_0')(1 - \lambda_{A_0'}(B_0))$$
$$= \mu(A_0') \cdot \sum_{i \geqslant 1} \lambda_{A_0'}(B_i).$$

By induction,

$$\mu(A_k) = \mu(A'_0) \sum_{i \geqslant k} \lambda_{A'_0}(B_i).$$

Therefore,

$$\mu\big(J \setminus K \cup f^{-1}(K)\big) = \mu(A'_0)\,[\lambda_{A'_0}(B_0) + \sum_{k \geqslant 1} k\lambda_{A'_0}(B_k),$$

and

$$\mu(J) = \frac{\mu(A'_0)}{\lambda(A'_0)}[\lambda(B_0) + \sum_{k \geqslant 1} k\lambda(B_k). \tag{3}$$

Since $B_k = \bigcup_{i_k=1}^{2^k-1} f^{-1}(I_k^{i_k}) \cap A_0$ and f_{A_0} restricted to $f^{-1}(I_k^{i_k}) \cap A_0$ is onto, piecewise linear with two pieces, each of them with derivative $|f'_0 f_1 \ldots f_k|$, we have

$$(b-1) = \frac{1}{2}|f'_0 f'_1 \ldots f'_k| \cdot \lambda\big(f^{-1}(I_k^{i_k}) \cap A_0\big).$$

Therefore,

$$\lambda(B_k) = \frac{2(b-1)\lambda_0 \lambda_1}{k(k+1)}.$$

The series $\sum_{k \geqslant 1} (k\lambda(B_k))$ being divergent, in view of (3) we have two possibilities: $\mu(A'_0) = 0$ and f has no a.c.i.p.m. (Lemma 4.4) or $\mu(A'_0) \neq 0$ and the invariant measure μ is infinite and σ-finite.

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