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Remarks on the preceding paper of P. Anandani

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1. In the preceding paper [1] P. Anandani has established some formulae for the derivative of the H-function, and some recurrence formulae for this function. In the first kind of these formulae the derivative of H is expressed in terms of two H-functions which are different in parameters. The H-function has been studied rather extensively in the last seven years (see [3], [2] and [7]). This function plays an important role in mathematics and can be widely applied in physics.

In this note I assume notation of the preceding paper and find for each formula of P. Anandani concerned with the derivative of $H = H_{p,q}^{m,n}(z^\delta)$ a number ε such that the derivative $(d/dz)\{z^\varepsilon H_{p,q}^{m,n}(z^\delta)\}$ can be expressed by only one function $H = H_{p,q}^{m,n}(z^\delta)$, certainly different in parameters. Such formulae are more convenient for applications. I use them in order to establish four expansion formulae (10), (11), (12) and (13), which are the main result of this note.

The obtained results contain the results of C. S. Meijer [5] as particular cases, and the present method is an analogue of the method applied there.

2. Lemma. *Suppose that m, n, p, q are integers satisfying*

$$(1) \qquad 1 \leqslant m \leqslant q, \quad 0 \leqslant n \leqslant p\,.$$

Moreover, suppose that a_j $(j = 1, \ldots, p)$, β_j $(j = 1, \ldots, q)$ are positive numbers and a_j $(j = 1, \ldots, p)$, b_j $(j = 1, \ldots, q)$ are complex numbers satisfying

$$(2) \qquad a_j(b_h+\nu) \neq \beta_h(a_j-1-\lambda) \quad \text{for} \quad \nu, \lambda = 0, 1, \ldots;\ h = 1, \ldots, m;\ j = 1, \ldots, n\,,$$

$$(3) \qquad \mu = \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}a_j \geqslant 0\,.$$

Finally, suppose that z ***and*** δ ***are complex numbers such that***

$$z \neq 0 \quad \text{for} \quad \mu > 0\,,$$

(4)
$$0 < |z^\delta| < \prod_{j=1}^{p} \alpha_j^{-\alpha_j} \prod_{j=1}^{q} \beta_j^{\beta_j} \quad \text{for} \quad \mu = 0\,.$$

Then

(5)
$$\frac{d^r}{dz^r}\left\{z^{-\delta b_1/\beta_1} H_{p,q}^{m,n}\left[z^\delta \middle| \begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]\right\}$$
$$= (-\delta/\beta_1)^r z^{-r-\delta b_1/\beta_1} H_{p,q}^{m,n}\left[z^\delta \middle| \begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\ (r+b_1, \beta_1), (b_2, \beta_2), \ldots, (b_q, \beta_q)\end{matrix}\right]$$
$$(r = 1, 2, \ldots;\ \delta = \beta_1 \text{ for } r > 1)\,,$$

(6)
$$\frac{d^r}{dz^r}\left\{z^{-\delta b_q/\beta_q} H_{p,q}^{m,n}\left[z^\delta \middle| \begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]\right\}$$
$$= (\delta/\beta_q)^r z^{-r-\delta b_q/\beta_q} H_{p,q}^{m,n}\left[z^\delta \middle| \begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots, (b_{q-1}, \beta_{q-1}), (r+b_q, \beta_q)\end{matrix}\right]$$
$$(m < q;\ r = 1, 2, \ldots;\ \delta = \beta_q \text{ for } r > 1)\,,$$

(7)
$$\frac{d^r}{dz^r}\left\{z^{-\delta(1-a_1)/\alpha_1} H_{p,q}^{m,n}\left[(1/z)^\delta \middle| \begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]\right\}$$
$$= (-\delta/\alpha_1)^r z^{-r-\delta(1-a_1)/\alpha_1} H_{p,q}^{m,n}\left[(1/z)^\delta \middle| \begin{matrix}(-r+a_1, \alpha_1), (a_2, \alpha_2), \ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]$$
$$(n > 0;\ r = 1, 2, \ldots;\ \delta = \alpha_1 \text{ for } r > 1)\,,$$

(8)
$$\frac{d^r}{dz^r}\left\{z^{-\delta(1-a_p)/\alpha_p} H_{p,q}^{m,n}\left[(1/z)^r \middle| \begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]\right\}$$
$$= (\delta/\alpha_p)^r z^{-r-\beta(1-a_p)/\alpha_p} H_{p,q}^{m,n}\left[(1/z)^\delta \middle| \begin{matrix}(a_1, \alpha_1), \ldots, (a_{p-1}, \alpha_{p-1}), (-r+a_p, \alpha_p)\\ (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]$$
$$(n < p;\ r = 1, 2, \ldots;\ \delta = \alpha_p \text{ for } r > 1)\,.$$

Proof. Formulae (5), (6), (7), (8) with $r = 1$ are immediate consequences of the formulae (2.3), (2.4), (2.1), (2.2) in the preceding paper [1], respectively. Now (1), (2), (3), (4) with $r = 2, 3, \ldots$ follow by a simple induction argument.

3. THEOREM. *Suppose that* m, n, p, q *are integers satisfying* (1); α_j $(j = 1, \ldots, p)$, β_j $(j = 1, \ldots, q)$ *are positive numbers and* a_j $(j = 1, \ldots, p)$, b_j $(j = 1, \ldots, q)$ *are complex numbers satisfying* (2) *and*

(9)
$$\sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j > 0\,.$$

Furthermore, suppose that ω and η are complex numbers such that $\omega \neq 0$ and $\eta \neq 0$. Then

$$(10)\qquad H_{p,q}^{m,n}\left[\eta\omega\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$= \eta^{b_1/\beta_1}\sum_{r=0}^{\infty}(1/r!)(1-\eta^{1/\beta_1})^r H_{p,q}^{m,n}\left[\omega\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(r+b_1,\beta_1),(b_2,\beta_2),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$(\eta \text{ arbitrary for } m=1, \text{ and } |\eta^{1/\beta_1}-1|<1 \text{ for } m>1),$$

$$(11)\qquad H_{p,q}^{m,n}\left[\eta\omega\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$= \eta^{b_q/\beta_q}\sum_{r=0}^{\infty}(1/r!)(\eta^{1/\beta_q}-1)^r H_{p,q}^{m,n}\left[\omega\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_{q-1},\beta_{q-1}),(r+b_q,\beta_q)\end{matrix}\right.\right]$$

$$(m<q,\ |\eta^{1/\beta_q}-1|<1),$$

$$(12)\qquad H_{p,q}^{m,n}\left[\eta\omega\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$= \eta^{-(1-a_1)/\alpha_1}\sum_{r=0}^{\infty}(1/r!)(1-\eta^{-1/\alpha_1})^r H_{p,q}^{m,n}\left[\omega\left|\begin{matrix}(-r+a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$(n>0,\ \operatorname{Re}\eta^{1/\alpha_1}>\tfrac{1}{2}),$$

$$(13)\qquad H_{p,q}^{m,n}\left[\eta\omega\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$= \eta^{-(1-a_p)/\alpha_p}\sum_{r=0}^{\infty}(1/r!)(\eta^{-1/\alpha_p}-1)^r \times$$

$$\times H_{p,q}^{m,n}\left[\omega\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_{p-1},\alpha_{p-1}),(-r+a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$(n<p,\ \operatorname{Re}\eta^{1/\alpha_p}>\tfrac{1}{2}).$$

Here the values of the many-valued functions $H = H_{p,q}^{m,n}(\eta\omega)$ and $H^* = H_{p,q}^{m,n}(\omega)$ are connected in the following way: if the value of $\arg\omega$ is chosen, then the value of $\arg(\eta\omega)$ is determined by:

$$(14)\qquad \arg(\eta\omega) = \begin{cases} \beta_1\arg\eta^{1/\beta_1}+\arg\omega & \textit{and}\quad -\tfrac{1}{2}\pi<\arg\eta^{1/\beta_1}<\tfrac{1}{2}\pi \quad \textit{in case of } (10),\\ \beta_q\arg\eta^{1/\beta_q}+\arg\omega & \textit{and}\quad -\tfrac{1}{2}\pi<\arg\eta^{1/\beta_q}<\tfrac{1}{2}\pi \quad \textit{in case of } (11),\\ \alpha_1\arg\eta^{1/\alpha_1}+\arg\omega & \textit{and}\quad -\tfrac{1}{2}\pi<\arg\eta^{1/\alpha_1}<\tfrac{1}{2}\pi \quad \textit{in case of } (12),\\ \alpha_p\arg\eta^{1/\alpha_p}+\arg\omega & \textit{and}\quad -\tfrac{1}{2}\pi<\arg\eta^{1/\alpha_p}<\tfrac{1}{2}\pi \quad \textit{in case of } (13).\end{cases}$$

Proof. It is well known (see e.g. [2], pp. 239-240) that the H-function has, in general, branch-points at 0 and ∞, and that, in case of (9), these points are the only singularities of H. Hence, if we take an arbitrary number $\omega \neq 0$ and make a choice for $\arg\omega^{1/\beta_h}$ $(h = 1, q)$, $\arg\omega^{1/\beta_h}$ $(h = 1, p)$ (not necessary the principal values), then

$$H_h^{(1)} = H_{p,q}^{m,n}(z^{\beta_h}) \quad (h = 1, q), \qquad H_h^{(2)} = H_{p,q}^{m,n}(z^{a_h}) \quad (h = 1, p)$$

are analytic in the domains

$$\Delta_h^{(1)} = \{z\colon\ |z-\omega^{1/\beta_h}| < |\omega^{1/\beta_h}|\} \quad (h = 1, q),$$

$$\Delta_h^{(2)} = \{z\colon\ |z^{-1}-\omega^{-1/a_h}| < |\omega^{-1/a_h}|\} = \{z\colon\ \mathrm{Re}(\omega^{-1/a_h}z) > \tfrac{1}{2}\} \quad (h = 1, p),$$

respectively. Moreover, the functions are holomorphic in the corresponding domains provided that in each case the value of $\arg z$ is determined uniquely by an appropriate agreement, e.g. by:

$$(15)\qquad \begin{aligned} &-\tfrac{1}{2}\pi < \arg z - \arg\omega^{1/\beta_h} < \tfrac{1}{2}\pi \quad \text{in case of } H_h^{(1)} = H_{p,q}^{m,n}(z^{\beta_h}) \quad (h = 1, q),\\ &-\tfrac{1}{2}\pi < \arg z - \arg\omega^{1/a_h} < \tfrac{1}{2}\pi \quad \text{in case of } H_h^{(2)} = H_{p,q}^{m,n}(z^{a_h}) \quad (h = 1, p). \end{aligned}$$

Now, the functions

$$\hat{H}_h^{(1)} = z^{-b_h}H_{p,q}^{m,n}(z^{\beta_h}) \quad (h = 1, q), \qquad \hat{H}_h^{(2)} = z^{1-a_h}H_{p,q}^{m,n}(z^{a_h}) \quad (h = 1, p)$$

can be expanded in the Taylor series in $z-\omega^{1/\beta_h}$ $(z \in \Delta_h^{(1)};\ h = 1, q)$, $z^{-1}-\omega^{-1/a_h}$ $(z \in \Delta_h^{(2)};\ h = 1, p)$, respectively. Applying (5), (6), (7) and (8) we reduce the above described power series to the form (10), (11), (12) and (13), respectively. Clearly, since (11) is a consequence of (6), we have to assume $m < q$ there, and, in case of (12) and (13), $n > 0$ and $n < p$, respectively. The conditions

$$z \in \Delta_h^{(1)} \quad (h = 1, q), \qquad z \in \Delta_h^{(2)} \quad (h = 1, p)$$

should be replaced by

$$|\eta^{1/\beta_h}-1| < 1 \quad (h = 1, q), \qquad \mathrm{Re}\,\eta^{1/a_h} > \tfrac{1}{2} \quad (h = 1, p),$$

respectively; the conditions (15) may be replaced by (14).

In case of (10) the condition $|\eta^{1/\beta_1}-1| < 1$ can be removed provided $m = 1$. In fact, here we have (see [2], p. 279)

$$z^{-b_1}H_{p,q}^{1,n}\left[z^{\beta_1}\middle|\begin{matrix}(a_1, a_1), \ldots, (a_p, a_p)\\(b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]$$

$$= (1/b_1)\sum_{\nu=0}^{\infty}\frac{(-1)^\nu\prod_{j=1}^{n}\Gamma\big(1-a_j+a_j(b_1+\nu)/\beta_1\big)}{\nu!\prod_{j=2}^{q}\Gamma\big(1-b_j+\beta_j(b_1+\nu)/\beta_1\big)\prod_{j=n+1}^{p}\Gamma\big(a_j-a_j(b_1+\nu)/\beta_1\big)}\,z^\nu.$$

Consequently, $z = 0$ is regular for the function considered, and (10) holds for an arbitrary η. The proof is completed.

Remark. Since the H-function is symmetrical in the pairs $(a_1, \alpha_1), \ldots, (a_n, \alpha_n)$; likewise in $(a_{n+1}, \alpha_{n+1}), \ldots, (a_p, \alpha_p)$; in $(b_1, \beta_1), \ldots$ $\ldots, (b_m, \beta_m)$ and in $(b_{m+1}, \beta_{m+1}), \ldots, (b_q, \beta_q)$, so the results established can be written in various other forms.

4. Finally, I would like to remark that the expansion formulae of [5] were generalized by C. S. Meijer in [6], and by myself in [4]. These results deal with the G-function and naturally there arises a problem to generalize them for the H-function.

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