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# POLYNOMIAL FLOWS ON $R^n$

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## 1. Definitions and notation

Consider a class $C^1$ vector field $X$ on $\boldsymbol{R}^n$ and the associated (autonomous) system of ordinary differential equations

(1) $$\dot{x} = X(x) \qquad (\dot{x} \equiv dx/dt).$$

According to the established theory of such equations (see e. g. [1], [7], [10], [13], [15], or [23]) for each initial-condition vector

(2) $$x(0) = z \in \boldsymbol{R}^n,$$

there exists a unique maximal local solution (flow)

(3) $$x = \varphi(t, z) = \varphi_z(t) = \varphi^t(z)$$

defined and satisfying (1),

(1′) $$\dot{\varphi}(t, z) = X(\varphi(t, z)),$$

for all real $t$ in a maximal open interval $J(z)$ about $t = 0$, and also satisfying the initial-condition equation (2),

(2′) $$\varphi(0, z) = z.$$

The solution $\varphi_z$ through $z$ is called *complete* if $J(z) = \boldsymbol{R}$, and the flow $\varphi$ is called *complete* (or *global*) if $J(z) = \boldsymbol{R}$ for each $z$ in $\boldsymbol{R}^n$.

But even local flows satisfy the group property

(4) $$\varphi(s, \varphi(t, z)) = \varphi(s+t, z)$$

locally; that is, for all $s, t$ sufficiently near $t = 0$. See [15].

For precision and clarity in our definitions and results stated below it is helpful to consider the following additional sets determined by the system (1):

The (necessarily symmetric) *system interval*

$$I = I(X) = \bigcup \{J(z)\colon\ z \in \mathbf{R}^n\}; \tag{5}$$

for each $t \in I$, the domain of $\varphi^t$,

$$U_t = \operatorname{dom} \varphi^t = \{z \in \mathbf{R}^n\colon\ t \in J(z)\}, \tag{6}$$

which is an open subset of $\mathbf{R}^n$; and

$$\Omega = \operatorname{dom} \varphi = \{(t, z) \in \mathbf{R} \times \mathbf{R}^n\colon\ t \in J(z)\}, \tag{7}$$

which is an open subset of $\mathbf{R} \times \mathbf{R}^n$.

We do not assume *a priori* that $\operatorname{dom} \varphi = I \times \mathbf{R}^n$, nor even that $\Omega$ contains a subset of the form $(-\varepsilon, \varepsilon) \times \mathbf{R}^n$.

The classical theorems about dependence on initial conditions and parameters state that smoothness of the vector-field $X$ entails the same smoothness of the flow $\varphi(t, z)$ in both $t$ and $z$. In particular, $C^1 X \Rightarrow C^1 \varphi$, $C^\infty X \Rightarrow C^\infty \varphi$, and (see [10]) $C^\omega X \Rightarrow C^\omega \varphi$.

A vector-field $X$ is called *polynomial* if each of the $n$ components of $X\colon \mathbf{R}^n \to \mathbf{R}^n$ is a polynomial in the components of $x \in \mathbf{R}^n$. Simple and well-known examples, such as $\dot{x} = x^2$ with $x = \varphi(t, z) = z/(1 - tz)$, show that the flow $\varphi$ of a polynomial vector field need be neither global nor polynomial in the initial-condition vector $z$. We certainly do not expect $\varphi(t, z)$ to be polynomial in $t$: e.g., if $\dot{x} = 1 + x^2$, then $x = \varphi(t, z) = (z \cos t + \sin t)/(\cos t - z \sin t)$.

Definition 1 (from [4]). A local flow $\varphi(t, z)$ is called a *polynomial flow* (or a *poly flow*, for short) if $\varphi(t, z)$ is polynomial in $z$ for each fixed $t$. More precisely, $\varphi$ is a *polynomial flow* if, for each multi-index $r \in \mathbf{N}^n$, there is a function $a_r\colon I \to \mathbf{R}^n$ such that, for each $t$ in $I$, $a_r(t) = 0$ for all but finitely many $r$, and

$$\varphi(t, z) = \Sigma_r a_r(t) z^r \quad \text{for all } z \in U_t, \tag{8}$$

where $z^r = z_1^{r_1} z_2^{r_2} \ldots z_n^{r_n}$.

Definition 2. A class $C^1$ vector-field $X$ whose local flow $\varphi$ is a polynomial flow will be called a *polynomial-flow-vector-field* (or a PF-vector-field, for short).

Question 1. Which $C^1$ vector fields $X$ have poly flows $\varphi$? That is, which $C^1$ vector fields are PF-vector fields?

*Remark.* It follows from (1′) and (2′) that

$$X(z) = \dot{\varphi}(0, z) \quad \text{for all } z \in \mathbf{R}^n. \tag{9}$$

Thus it would seem to be almost obvious that a necessary condition for $X$ to be a PF-vector field is that $X$ be a polynomial vector field. Indeed this is true (see Theorem 1 part (iii), below), but its proof does not follow immediately from (9) because there exist $C^\infty$ maps $\varphi$: $\boldsymbol{R}\times\boldsymbol{R}^1\to\boldsymbol{R}^1$ such that $\varphi(t, z)$ is polynomial in $z$ for each $t$, and yet $\dot{\varphi}(0, z)$ is not even analytic (let alone polynomial). See Example 7.1 below and [33].

## 2. Polyomorphisms of $\boldsymbol{R}^n$

DEFINITION 3. By a *polyomorphism* $\psi$ of $\boldsymbol{R}^n$ we mean a diffeomorphism of $\boldsymbol{R}^n$ such that the components of both $\psi$ and $\psi^{-1}$ are polynomials in the components of their $n$-dimensional vector variables.

If $\varphi$ is a global flow, then each $\varphi^t$ is a diffeomorphism of $\boldsymbol{R}^n$. If $\varphi$ is both global and polynomial, then each $\varphi^t$ is a polyomorphism because $(\varphi^t)^{-1} = \varphi^{-t}$. It is shown in Theorem 1 below that polynomial flows are always global.

The group $\mathscr{P}(\boldsymbol{R}^n)$ of all polyomorphisms of $\boldsymbol{R}^n$ is denoted $\mathrm{GA}_n(\boldsymbol{R})$ in [4] and is sometimes called the *affine Cremona*, or *ganze Cremona*, group. The group $\mathscr{P}(\boldsymbol{R}^1)$ of one-dimensional polyomorphisms is identical to the affine group $\mathrm{Af}(\boldsymbol{R}^1)$: That is, $T\in\mathscr{P}(\boldsymbol{R}^1)$ iff $T(x) = ax+b$. But for $n\geqslant 2$, the group $\mathscr{P}(\boldsymbol{R}^n)$ is much larger than the affine group $\mathrm{Af}(\boldsymbol{R}^n)$ and contains nonlinear polyomorphisms of every degree $\geqslant 2$. For $n\geqslant 3$, one does not even know what constitutes a set of generators for $\mathscr{P}(\boldsymbol{R}^n)$. But the structure of 2-dimensional polyomorphisms is fairly well-understood by virtue of the

THEOREM OF VAN DER KULK–JUNG (see [16], [21], [22], [30]). *The group $\mathscr{P}(\boldsymbol{R}^2)$ is the free product of its two subgroups, the affine group $\mathrm{Af}(\boldsymbol{R}^2)$ and the triangular group $\mathscr{T}\mathscr{P}(\boldsymbol{R}^2)$, amalgamated over their intersection, the group of triangular linear maps. This is written $\mathscr{P}(\boldsymbol{R}^2) = \mathrm{Af}(\boldsymbol{R}^2) *_\Delta \mathscr{T}\mathscr{P}(\boldsymbol{R}^2)$.*

*The elements $T$ of the triangular group $\mathscr{T}\mathscr{P}(\boldsymbol{R}^2)$ have the form*

$$T\colon\ u = ax+\alpha,\quad v = by+f(x),\quad ab\neq 0, \tag{10}$$

*where $f(x)$ is a polynomial in one variable.*

The content of the theorem of van-der-Kulk–Jung can be expressed more concretely as follows: Each element $T$ of $\mathscr{P}(\boldsymbol{R}^2)$ can be written as a composition

$$T = L\circ S(f_1, v_1)\circ\ldots\circ S(f_l, v_l) \tag{11}$$

of one linear (triangular) map $L$: $\boldsymbol{R}^2\to\boldsymbol{R}^2$ and a finite number $l$ (called the *length* of $T$) of special *nonlinear shifts* $S(f, v)$ defined by

$$S(f, v)(x) = x+f(x\cdot v^\perp)v,\quad x\in\boldsymbol{R}^2, \tag{12}$$

where $f$ is a polynomial in one real variable, $v = (v_1, v_2)$ is a unit vector in $\boldsymbol{R}^2$, and $v^\perp = (-v_2, v_1)$. There is no analogous result known for $\mathscr{P}(\boldsymbol{R}^n)$ when $n\geqslant 3$.

The amalgamated product $\mathscr{P}(\boldsymbol{R}^2) = \text{Af} *_{\Delta} \mathscr{T}\mathscr{P}$ means that for every group $G$ and every pair of homomorphisms $f$: $\text{Af}(\boldsymbol{R}^2) \to G$ and $g$: $\mathscr{T}\mathscr{P}(\boldsymbol{R}^2) \to G$ which agree on $\Delta(\boldsymbol{R}^2) = \text{Af}(\boldsymbol{R}^2) \cap \mathscr{T}\mathscr{P}(\boldsymbol{R}^2)$, there exists a unique homomorphism $F$: $\mathscr{P}(\boldsymbol{R}^2) \to G$ such that $F$ agrees with $f$ on $\text{Af}(\boldsymbol{R}^2)$ and $F$ agrees with $g$ on $\mathscr{T}\mathscr{P}(\boldsymbol{R}^2)$.

For diffeomorphisms $S$, $T$ of $\boldsymbol{R}^n$ it follows from the chain rule

$$D(S \circ T) = (DS)(T) \cdot DT, \qquad DS = S' = \text{Jacobian derivative of } S, \tag{13}$$

with $S = T^{-1}$ that

$$1 = (\det D(T^{-1})) \cdot (\det DT). \tag{14}$$

If, in addition, $S$ and $T$ are polyomorphisms then both $\det DT$ and $\det D(T^{-1})$ are polynomials so that (14) implies that they must be *nonzero constants* (reciprocals), and then (13) shows that the map $T \to \det DT$ is a group homomorphism of $\mathscr{P}(\boldsymbol{R}^n)$ into the multiplicative group of nonzero real numbers. We will later (in Theorem 1) make use of this fact, specifically that

$$T \in \mathscr{P}(\boldsymbol{R}^n) \text{ implies } \det DT \equiv \text{const} \neq 0. \tag{15}$$

The so-called *Jacobian Conjecture* states that, conversely, if $T$ is a polynomial transformation of $\boldsymbol{R}^n$ and if $\det DT$ is a nonzero constant, then $T \in \mathscr{P}(\boldsymbol{R}^n)$.

Since volume ($n$-dimensional Lebesgue measure) of small regions $U \subset \boldsymbol{R}^n$ transforms by the formula

$$\text{vol}(T(u)) = \int_{T(U)} du = \int_U (\text{Det } DT) \cdot dx, \tag{16}$$

the Jacobian Conjecture can be stated as follows:

Polynomial transformations $T$: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ which are (locally) volume-preserving ($\det DT \equiv 1$) are necessarily globally one-to-one and onto with polynomial inverse.

That *analytic* volume-preserving transformations of $\boldsymbol{R}^2$ need not be globally one-to-one is shown by examples such as the following (shown to me by Brian Coomes):

$$T \qquad \begin{aligned} u &= \sqrt{2}\, e^{x/2} \cos(y e^{-x}), \\ v &= \sqrt{2}\, e^{x/2} \sin(y e^{-x}). \end{aligned}$$

For further discussion and information on the Jacobian Conjecture see [3], [11], [19], [25], and [31]. For further discussion of polyomorphisms see [2], [4], [12], [16], [17], [19], [21], [22], [24], [30], and [32].

The Hénon map (see [9] and [14])

$$H \qquad \begin{aligned} u &= y+1-ax^2, \\ v &= bx \end{aligned}$$

is an example of a 2-dimensional nonlinear polyomorphism.

## 3. The fundamental theorem for polynomial flows on $R^n$

FUNDAMENTAL THEOREM (from [4]). *If $X$: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ is a* PF-*vector-field with polynomial flow* $x = \varphi(t, z)$, *then*

(i) $\varphi$ *is a complete flow. Hence* $\{\varphi^t: t \in \boldsymbol{R}\}$ *is a one-parameter group of polyomorphisms of* $\boldsymbol{R}^n$ *so that* (4) *holds for all* $s$, $t$ *in* $\boldsymbol{R}$.

(ii) $\varphi$ *has bounded degree. That is, there exists an integer* $d \geq 0$ *such that*

$$\varphi(t, z) = \sum_{|r| \leq d} a_r(t) z^r \text{ for all } t \in \boldsymbol{R}, z \in \boldsymbol{R}^n. \tag{17}$$

(iii) *The functions* $a_r$: $\boldsymbol{R} \to \boldsymbol{R}^n$ *which occur in* (17) *are real-analytic at each* $t \in \boldsymbol{R}$. (*In fact, it has been pointed out by Brian Coomes* [8] *that these coefficient functions must be entire functions of the complex time-variable* $t$.)

(iv) $X$ *is a polynomial vector-field on* $\boldsymbol{R}^n$.

(v) $\operatorname{div} X \equiv$ *constant*.

*Remarks on the proof.* For the proof of (i), (ii), and (iii) see [4] and [8]. Statement (iv) follows directly from (ii) and (9). Statement (v) follows from the classical formula

$$\frac{d}{dt} \det D_z \varphi(t, z) = \operatorname{div} X(\varphi(t, z)) \cdot \det D_z \varphi(t, z) \tag{18}$$

at $t = 0$, because of (15) and the fact that each $\varphi^t \in \mathscr{P}(\boldsymbol{R}^n)$ as established in (i).

This theorem provides the first steps toward an answer to our Question 1 above since it gives

Some nevessary conditions for PF-vector-fields:

N.C.I. $X$ must be a polynomial vector field.

N.C.II. $\operatorname{div} X$ must be identically constant.

N.C.III. $X$ must be complete (i.e., its flow must be a global flow defined for all real $t$).

PROBLEM 1.1. Find further necessary conditions for PF-vector-fields.

N.C.IV (B. Coomes [8]). The flow $\varphi(t, z)$ must be (extendable to) an *entire* function of *complex* time $t$.

Remark 1. The Lorenz Equations [28] satisfy the first three necessary conditions, but not the fourth. See Example 7.2 below.

*Remark* 2. There exist vector-fields on $\boldsymbol{R}^2$ which satisfy all four of these necessary conditions, but which are, nevertheless, not PF-vector-fields: Their flows are not polynomial in the initial conditions $x_0, y_0$. See Example 7.3 below.

## 4. Classification of polynomial flows in dimensions 1 and 2

CLASSIFICATION THEOREM (from [4]).

*Dimension* 1. *Every polynomial flow* $\varphi(t, z)$ *on* $\boldsymbol{R}^1$ *has the form* $\varphi(t, z) = ze^{at} + (b/a)(e^{at} - 1)$ *if* $a \neq 0$, or $\varphi(t, z) = z + bt$ if $a = 0$, *and has the vector-field* $X(x) = ax + b$.

*Dimension* 2. *Every polynomial flow* $\varphi(t, z)$ *on* $\boldsymbol{R}^2$, *after a change-of-coordinates by means of a polyomorphism of* $\boldsymbol{R}^2$, *has one of the following (inequivalent) forms*:

| | *Flow* $\varphi$. | *Vector-field* $X$. |
|---|---|---|
| (i) | $u = e^{at}(u_0 \cos bt + v_0 \sin bt)$,<br>$v = e^{at}(v_0 \cos bt - u_0 \sin bt)$, | $\dot{u} = au + bv, \quad b > 0$,<br>$\dot{v} = av - bu$, |
| (ii) | $u = u_0 e^{at}, \quad ab \neq 0$,<br>$v = v_0 e^{bt}$, | $\dot{u} = au$,<br>$\dot{v} = bv$, |
| (iii) | $u = u_0$,<br>$v = v_0 e^{bt}$, | $\dot{u} = 0$,<br>$\dot{v} = bv$, |
| (iv) | $u = u_0 + t$,<br>$v = v_0 e^{bt}$, | $\dot{u} = 1$,<br>$\dot{v} = bv$, |
| (v) | $u = u_0 e^{at}, \quad a \neq 0$,<br>$v = e^{amt}(v_0 + u_0^m t)$,<br>$m = 1, 2, 3, 4, \ldots$ | $\dot{u} = au$,<br>$\dot{v} = amv + u^m$, |
| (vi) | $u = u_0$,<br>$u = v_0 + p(u_0)t$,<br>$\deg p \geqslant 1$, | $\dot{u} = 0$,<br>$\dot{v} = p(u)$. |

A polynomial change-of-coordinates

(19) $$x = P(u), \quad P \in \mathscr{P}(\boldsymbol{R}^n)$$

transforms the polynomial autonomous system

(20) $$\dot{x} = X(x), \quad x(0) = x_0$$

with solution

(21) $$x = \varphi(t, x_0), \quad \varphi(0, x_0) = x_0$$

into the polynomial autonomous system

(22) $$\dot{u} = \tilde{X}(u), \quad u(0) = u_0 = p^{-1}(x_0)$$

with solution

(23) $$u = \tilde{\varphi}(t, u_0)$$

where

(24) $$\tilde{X}(u) \equiv P'(u)^{-1} X(P(u)),$$

and where

(25) $$\tilde{\varphi}(t, u_0) \equiv P^{-1}\big(\varphi(t, P(u_0))\big).$$

Thus $\varphi^t$ and $\tilde{\varphi}^t$ are *conjugates* in $\mathscr{P}(\boldsymbol{R}^n)$:

(26) $$\tilde{\varphi}^t = P^{-1} \circ \varphi^t \circ P.$$

Thus we will say that two flows $\varphi^t$ and $\tilde{\varphi}^t$ are *polyomorphic flows* if they are conjugate one-parameter subgroups in $\mathscr{P}(\boldsymbol{R}^n)$.

*Remark.* Flows which are merely diffeomorphic to polynomial flows need not be polynomial flows, even if their vector fields satisfy the four necessary conditions of § 3. See Example 7.4 below.

QUESTION 2. If a vector-field $X$ is polynomial with constant divergence and its flow $\varphi(t, z)$ is complete and (extends to) an *entire* function of *complex* time $t$ (that is, if it satisfies all four of the necessary conditions of § 3), must it then be diffeomorphic (or at least homeomorphic) to a polynomial-flow vector-field?

QUESTION 3. Which (polynomial) vector-fields are diffeomorphic (or homeomorphic) to polynomial-flow vector fields?

*Remark.* The *divergence* of a vector-field transforms under diffeomorphisms $T\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ according to the following equations: Given a vector-field $X(x)$ we write $y = T(x)$ and $x = T^{-1}(y) = G(y)$. Then the new (transformed) vector-field is

$$Y(y) = T'(G(y)) \cdot X(G(y))$$

$$\text{or} \quad Y_i = (\partial T_i / \partial x_k) X_k \quad (\text{summation on } k)$$

and

$$\operatorname{div} Y = \operatorname{div} X + X_k \cdot \frac{\partial^2 T_i}{\partial x_l \, \partial x_k} \cdot \frac{\partial G_l}{\partial y_i} \quad (\text{summation on } i,\ k, \text{ and } l).$$

Evidently, the statement "$\operatorname{div} X \equiv \text{constant}$" is invariant under polyomorphisms of polyflows.

## 5. Outline of the proof of the classification theorem for polynomial flows on $R^2$

A. It follows from the Fundamental Theorem of Bass and Meisters [4] which was stated in § 3 above that a polynomial flow $\varphi$ defines a one-parameter subgroup $\{\varphi^t: t \in \boldsymbol{R}\}$ of $\mathscr{P}(\boldsymbol{R}^n)$ and this subgroup has bounded degree.

B. By the Theorem of van der Kulk and Jung (discussed in § 2 above) we know that $\mathscr{P}(\boldsymbol{R}^2)$ is the free product of its two subgroups $\mathrm{Af}(\boldsymbol{R}^2)$ and $\mathscr{T}\mathscr{P}(\boldsymbol{R}^2)$ amalgamated over their intersection $\Delta(\boldsymbol{R}^2)$:

$$\mathscr{P}(\boldsymbol{R}^2) = \mathrm{Af}(\boldsymbol{R}^2) *_{\Delta} \mathscr{T}\mathscr{P}(\boldsymbol{R}^2). \tag{27}$$

C. Elements $P$ of such amalgamated products have unique *lengths* $l$ as words (11), or words

$$P = L \circ T_1 \circ A_1 \circ \ldots \circ T_l \circ A_l, \tag{28}$$

where $L \in \Delta$ and the $T_i$ and the $A_i$ are representatives of the nontrivial $\Delta$-cosets of $\mathscr{T}\mathscr{P}(\boldsymbol{R}^2)$ and $\mathrm{Af}(\boldsymbol{R}^2)$, respectively. See [16], [18], [21], [22], [27], [30].

D. It follows from A, B, C above that, for a *two-dimensional* polynomial flow $\varphi$, the lengths of the elements of

$$\{\varphi^t: t \in \boldsymbol{R}\} \subset \mathrm{Af}(\boldsymbol{R}^2) *_{\Delta} \mathscr{T}\mathscr{P}(\boldsymbol{R}^2)$$

are bounded. See [4] and [32].

E. Theorem 8 in § 4.3 on page 36 of Jean-Pierre Serre's book [27] states that: Every bounded subgroup of an amalgamated product $G_1 *_A G_2$ is contained in a conjugate of either $G_1$ or $G_2$. Serre defines a subset $\Sigma$ of an amalgamated product to be "bounded" if there is a bound on the lengths of the reduced decompositions (words) of the elements of $\Sigma$.

F. It now follows from D and E that to each polynomial flow $\varphi$ on $\boldsymbol{R}^2$ there exists a polyomorphism $P \in \mathscr{P}(\boldsymbol{R}^2)$ such that

either $\quad P \circ \varphi^t \circ P^{-1} \in \mathrm{Af}(\boldsymbol{R}^2) \quad$ for all $t \in \boldsymbol{R}^2$

or $\quad P \circ \varphi^t \circ P^{-1} \in \mathscr{T}\mathscr{P}(\boldsymbol{R}^2) \quad$ for all $t \in \boldsymbol{R}^2$.

In either case it is possible, by means of further polyomorphisms if necessary, to conjugate $\varphi^t$ into one of the six types listed in the classification theorem stated in § 4 above.

For further details see [4].

## 6. Vector fields with constant divergence

THEOREM 6.1. $\operatorname{div} X \equiv$ *constant* $\alpha$ *implies*

$(n = 1)$ $X(x) = \alpha x + \beta$ *for some constant* $\beta$.
$(n \geqslant 2)$ $X(x) = Ax + (\partial H(x))^T$, $x \in \boldsymbol{R}^n$,

*where* $A$ *is any* $n \times n$ *constant matrix whose trace equals* $\operatorname{div} X$, *and where* $H$ *can be chosen* (*but depends on the choice of* $A$) *to be a skew-symmetric* ($H^{\perp} = -H$) $n \times n$ *matrix function of* $x \in \boldsymbol{R}^n$. *Here* $(\partial H)^T$ *denotes the column vector whose k-th component is*

$$\sum_{i=1}^{n} (\partial H_{ik}/\partial x_i). \tag{29}$$

*The notation* $\partial H$ *is intended to suggest the matrix product of the row*

$$\partial = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$$

*times the* $n \times n$ *matrix function* $H$ *to produce the row vector whose components are given in* (29) *above.*

*Proof of Theorem* 6.1. Let $A$ be any constant $n \times n$ matrix (or linear map of $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$) such that

$$\operatorname{trace} A = \alpha = \operatorname{div} X = \sum_{i=1}^{n} (\partial X_i/\partial x_i).$$

Define $Y$: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ by $Y(x) = X(x) - Ax$. Then

$$\operatorname{div} Y = \operatorname{div} X - \operatorname{trace} A = 0.$$

Define the $(n-1)$-form

$$\omega = \sum_{k=1}^{n} (-1)^{k-1} Y_k (dx_1 \wedge \ldots \wedge d\hat{x}_k \wedge \ldots \wedge dx_n) \tag{30}$$

where $d\hat{x}_k$ means "omit the factor $dx_k$". Then $d\omega = (\operatorname{div} Y) \cdot dx_1 \wedge \ldots \wedge dx_n = 0$, since $\operatorname{div} Y = 0$. That is, the $(n-1)$-form $\omega$ is a *closed* differential form on (the simply connected set) $\boldsymbol{R}^n$. If $n = 1$, then $Y(x) \equiv$ constant (say $\beta$). If $n \geqslant 2$, then by the Poincaré Lemma there is an $(n-2)$-form

$$\theta = \sum_{1 \leqslant i < j \leqslant n} G_{ij} dx_1 \wedge \ldots \wedge d\hat{x}_i \wedge \ldots \wedge d\hat{x}_j \wedge \ldots \wedge dx_n$$

so that $\omega = d\theta$. That is, $\omega$ is *exact*. But then

$$\omega = d\theta = \sum_{i<j}\sum_{k=1}^{n} \frac{\partial G_{ij}}{\partial x_k} dx_k \wedge (dx_1 \wedge \ldots \wedge d\hat{x}_i \wedge \ldots \wedge d\hat{x}_j \wedge \ldots \wedge dx_n)$$

$$= \sum_{i<j}\Bigg[(-1)^{i-1}\frac{\partial G_{ij}}{\partial x_i} dx_1 \wedge \ldots \wedge d\hat{x}_j \wedge \ldots \wedge dx_n$$

$$+(-1)^{j}\frac{\partial G_{ij}}{\partial x_j} dx_1 \wedge \ldots \wedge d\hat{x}_i \wedge \ldots \wedge dx_n\Bigg].$$

So

(31)

$$\omega = \sum_{k=1}^{n}\sum_{1<k}(-1)^{i-1}\frac{\partial G_{ik}}{\partial x_i} - \sum_{i>k}(-1)^{i-1}\frac{\partial G_{ki}}{\partial x_i}(dx_1 \wedge \ldots \wedge d\hat{x}_k \wedge \ldots \wedge dx_n);$$

comparing (30) and (31) we see that

$$Y_k = \sum_{i<k}(-1)^{i+k}\frac{\partial G_{ik}}{\partial x_i} - \sum_{i>k}(-1)^{i+k}\frac{\partial G_{ki}}{\partial x_i}.$$

So if we define a skew-symmetric matrix $H = -H^T$ by the equations

$$H \qquad \begin{aligned} &H_{ik} = (-1)^{i+k} G_{ik} && \text{for } i<k,\\ &H_{ii} = 0 && \\ &H_{ik} = -H_{ki} = -(-1)^{i+k} G_{ki} && \text{for } i>k \end{aligned}$$

then $Y_k = \sum_{i=1}^{n} \partial H_{ik}/\partial x_i$ for $k = 1, 2, \ldots, n$, so the *column vector* $Y = (\partial H)^T$. Therefore,

$$\begin{aligned} X(x) &= Ax + Y(x)\\ \text{or} \qquad &= Ax + (\partial H)^T. \quad \blacksquare \end{aligned}$$

*Special cases of Theorem* 6.1 *when* $n = 1, 2, 3$:

$n = 1$, $\qquad \dot{x} = X(x) = ax + b,$

$n = 2$, $\qquad \dot{x}_1 = A_{11}x_1 + A_{12}x_2 - \partial G_{21}/\partial x_2,$

$\qquad \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + \partial G_{21}/\partial x_1.$

$\qquad \dot{x} = \textit{Linear} + \textit{Hamiltonian},$

$n = 3$, $\qquad \dot{x} = Ax + \operatorname{curl} G,$

$\qquad G = (G_{23}, G_{13}, G_{12}).$

*Remark.* The converse of Theorem 6.2 is also true: Vector fields of the form

$$X(x) = Ax + (\partial H)^T \quad \text{with } H^T = -H$$

automatically satisfy $\operatorname{div} X = \text{const.} = (\operatorname{tr} A)$.

THEOREM 6.2. *If* $\operatorname{div} X =$ *constant* $\alpha$, *then the determinant of the Jacobian of the flow* $x = \varphi(t, z)$, $\varphi(0, z) = z$, *satisfies*

$$\det D_z\, \varphi(t, z) = \exp\{(\operatorname{div} X)\cdot t\}. \tag{32}$$

*Proof.* It follows from (18) that

$$\det D_z\, \varphi(t, z) = \exp\Big\{\int_0^t (\operatorname{div} X)\big(\varphi(s, z)\big)\, ds\Big\} \tag{33}$$

from which (32) is an immediate consequence when $\operatorname{div} X \equiv \text{constant}$.

COROLLARY. *If* $\operatorname{div} X =$ *constant* $\alpha$, *then* $\operatorname{vol}\big(\varphi^t(u)\big) = \operatorname{vol}(U)\cdot\exp(\alpha t)$, $t \in \boldsymbol{R}$.

*Proof.* It follows directly from (16) and (32).

*Remark.* If $\operatorname{div} X \equiv \text{constant } \alpha < 0$, then the flow $\varphi(t, z)$ is *volume-crunching*. If $\operatorname{div} X \equiv \text{constant } \alpha > 0$, then the flow $\varphi(t, z)$ is *volume-expanding*. If $\operatorname{div} X \equiv 0$, then the flow $\varphi(t, z)$ is *volume-preserving*.

## 7. Examples

EXAMPLE 7.1. There exist $C^1$ function $\varphi\colon \boldsymbol{R}_t \times \boldsymbol{R}_z \to \boldsymbol{R}_x$ such that $\varphi(t, z)$ is polynomial in $z$ for each fixed $t$, but $\dot\varphi(0, z)$ is not analytic.

*Proof* (personal communication from Robert M. McLeod).

(1) Let $f \in C^\infty(\boldsymbol{R})$ but not analytic. For each $n$ there is a polynomial $Q_n(z)$ such that $|Q_n(z) - f(z)| < 2^{-n}$ for $|z| \leqslant n$. Set $P_1(z) = Q_1(z)$ and $P_n(z) = Q_n(z) - Q_{n-1}(z)$ for $n \geqslant 2$. Then

$$f(z) = \sum_{n=1}^{\infty} P_n(z), \qquad \text{for all } z \text{ in } \boldsymbol{R}.$$

(Convergence is uniform and absolute on compact sets.)

(2) Let $g\colon \boldsymbol{R} \to \boldsymbol{R}$ be an odd, infinitely differentiable function with $g'(0) = 1$ and $g(t) = 0$ for $|t| \geqslant h$. Set $A_n(t) = g(2^n t)/2^n$. Then $A_n(t) = 0$ for $|t| \geqslant h/2^n$ and $A_n'(t) = g'(2^n t)$. Hence $A_n'(0) = 1$ for all $n$. Also there is a constant $G$ such that $|A_n'(t)| \leqslant G$ for all $t$.

(3) Let $\varphi(t, z) = \sum_{n=1}^{\infty} A_n(t) P_n(z)$. Now every term is 0 when $t = 0$ since $g(0) = 0$; and when $t \neq 0$ all terms are zero for $n$ satisfying $2^n |t| \geqslant h$. Thus $\varphi(t, z)$ is a polynomial in $z$ for each fixed $t$.

(4) Now we want to show that $\dot{\varphi}(0, z) = f(z)$. First,

$$\left|\sum_{n=1}^{\infty} A_n'(t)\, P_n(z)\right| \leqslant \sum_{n=1}^{\infty} |A_n'(t)\, P_n(z)| \leqslant \sum_{n=1}^{\infty} G\cdot|P_n(z)|.$$

Thus the series converges uniformly in $t$. Let

$$h(t, z) = \sum_{n=1}^{\infty} A_n'(t)\, P_n(z).$$

Then

$$\begin{aligned}\int_0^t h(s, z)\, ds &= \sum_{n=1}^{\infty} \int_0^t A_n'(s)\, P_n(z)\, ds \\ &= \sum_{n=1}^{\infty} (A_n(t) - A_n(0))\, P_n(z) \\ &= \sum_{n=1}^{\infty} A_n(t)\, P_n(z) = \varphi(t, z).\end{aligned}$$

Differentiate both sides and set $t = 0$. The result is

$$h(0, z) = \dot{\varphi}(0, z).$$

But

$$h(0, z) = \sum_{n=1}^{\infty} A_n'(0)\, P_n(z) = \sum_{n=1}^{\infty} P_n(z) = f(z).$$

Thus $\dot{\varphi}(0, z) = f(z)$. ∎

Example 7.2. The Lorenz equations (see [28])

$$(L)\qquad \begin{aligned}\dot{x} &= \sigma(x - y) & \sigma &= \text{pos. const.},\\ \dot{y} &= rx - y - xz & r &= \text{pos. const.},\\ \dot{z} &= xy - bz & b &= \text{pos. const.}\end{aligned}$$

satisfy the first three necessary conditions (given above in § 3) for a polynomial-flow vector-field, but they do not satisfy the fourth necessary condition

N.C.I. $(L)$ is a polynomial vector field. Obvious.
N.C.II. $\operatorname{div} L = -\sigma - 1 - b =$ negative constant.
N.C.III. $(L)$ is *complete*.

(a) It is proved in Appendix C of Sparrow's book [28] that the solutions of $(L)$ remain bounded as $t \to +\infty$. Hence they must exist for all positive time.

(b) To see that solutions exist for all negative time, replace $(L)$ by $(-L)$ and consider the Liapunov function

$$v = rx^2 + \sigma y^2 + \sigma(z-2r)^2$$

as $t \to +\infty$. It can then be easily verified that along solutions curves of $(-L)$ for $t > 0$

$$\dot{V} \leqslant BV + C$$

where $B = 2 + 2\sigma + 2b$ and $c \geqslant 2\sigma b^2 r^2/(1+\sigma)$. It follows that $t$ tends to $+\infty$ with $x^2 + y^2 + z^2$ so solutions of $(-L)$ must exist for all positive time, and hence solutions of $(L)$ must exist for all negative time. But N.C.IV. fails to hold true for $(L)$ because evidently (see [8] and [29]) the Lorenz flow $\varphi(t, x, y, z)$ has singularities in the complex $t$-plane, off the real $t$ line.

EXAMPLE 7.3. There exist vector fields on $\boldsymbol{R}^2$ which satisfy all four necessary conditions (given in § 3 above) for a polynomial-flow vector-field, but which are, nevertheless, not PF-vector fields: That is, the components $\varphi(t, x_0, y_0)$, $\psi(t, x_0, y_0)$ are not polynomials in $x_0, y_0$.

Let $P(u)$ be a real polynomial in one real variable $u$, and let $a$ and $b$ be any two real numbers. Consider the two-dimensional system

$$X \qquad \begin{aligned} \dot{x} &= ax - \partial P(xy)/\partial y, \\ \dot{y} &= by + \partial P(xy)/\partial x. \end{aligned}$$

N.C.I. Clearly $X$ is a polynomial vector-field.
N.C.II. $\operatorname{div} X = a + b \equiv$ constant.
N.C.III. That $X$ is *complete* can be seen by direct examination of the solution which can be expressed explicitly as follows:

$$x = x_0 \exp\left\{at - \int_0^t P'(x_0 y_0 e^{(a+b)s})\, ds\right\},$$

$$y = y_0 \exp\left\{bt + \int_0^t P'(x_0 y_0 e^{(a+b)s})\, ds\right\}.$$

N.C.IV. is also seen to hold by direct examination of the explicitly given solution $\varphi$.

EXAMPLE 7.4. The following vector-field $X$ satisfies all four necessary conditions for a PF-vector-field and, in addition, it is diffeomorphic to a PF-vector-field; but, nevertheless, it itself is not a PF-vector-field.

$$X \qquad \begin{aligned} \dot{u} &= u - 2v(u^2+v^2) = u - \partial H/\partial v. \\ \dot{v} &= v + 2u(u^2+v^2) = v + \partial H/\partial u. \end{aligned}$$

$$H(u, v) \equiv (u^2+v^2)^2/2.$$

But its flow is not a polynomial-flow:

$$\varphi \quad \begin{aligned} u &= u_0 e^t \cos[(u_0^2+v_0^2)(1-e^{2t})] + v_0 e^t \sin[(u_0^2+v_0^2)(1-e^{2t})].\\ v &= -u_0 e^t \sin[(u_0^2+v_0^2)(1-e^{2t})] + v_0 e^t \cos[(u_0^2+v_0^2)(1-e^{2t})]. \end{aligned}$$

However, the diffeomorphism

$$T \quad \begin{aligned} u &= x\cos r^2 - y\sin r^2, \quad r^2 = x^2+y^2,\\ v &= x\sin r^2 + y\cos r^2, \quad\quad = u^2+v^2 \end{aligned}$$

with inverse

$$T^{-1} \quad \begin{aligned} x &= u\cos r^2 + v\sin r^2,\\ y &= -u\sin r^2 + v\cos r^2 \end{aligned}$$

transforms $X$ into the system $\tilde{X}$: $\dot{x} = x$, $\dot{y} = y$ with the polynomial flow $\tilde{\varphi}$: $x = x_0 e^t$, $y = y_0 e^t$.

## 8. The dynamics of polynomial flows

It follows easily from the Classification Theorem in dimensions 1 and 2 that

A. *In dimension* 1. A polynomial flow can have either one isolated stationary point $(at-b/a)$ which is stable when $a<0$ and unstable when $a>0$; or no stationary point (when $a=0$ and $b\neq 0$); or else all points are stationary points (when $a=b=0$).

B. *In dimension* 2. The stationary points are either none, one, or infinitely many (consisting of a finite number of lines each point of which is a stationary point). The periodic orbits are either none or all points (center). There can be no isolated limit cycles in a two-dimensional polynomial flow.

QUESTION 4. How many isolated stationary points can a three-dimensional polynomial flow have? Same Question in each dimension $n>3$. Exactly what is the nature of the stationary points in dimensions $n \geqslant 3$ for polynomial flows?

QUESTION 5. How many periodic orbits can a polynomial flow have in dimensions $n \geqslant 3$? Can there be isolated periodic orbits? See [5].

QUESTION 6. Exactly which two-dimensional polynomial vector-fields of the form

$$\dot{x} = ax - \partial H/\partial y,$$
$$\dot{y} = by + \partial H/\partial x$$

are *complete*? (Here $H = H(x, y)$ is a polynomial in $x$ and $y$.) That is, which two-dimensional polynomial vector-fields with constant divergence are *complete*? See [6].

QUESTION 7. What type of attractors can a polynomial flow have? See [20].

QUESTION 8. How is the degree of a polynomial flow $\varphi(t, z)$ related to the degree of its vector-field $\dot{\varphi}(0, z)$? See [4].

QUESTION 9. (Question 1 re-phrased): *the recognition problem.* Given a polynomial vector field $X$ on $\boldsymbol{R}^n$, how can one *decide* whether or not its flow $\varphi(t, z)$ is a polynomial flow? Note that, in spite of the Classification Theorem, this question is unanswered even in dimension 2.

A flow $\varphi(t, z)$ is said to have *sensitive dependence on initial conditions* if there exists a $\delta > 0$ such that for each $x_0 \in \boldsymbol{R}^n$ and each neighborhood $N$ of $x_0$, there exists a $y \in N$ and a real number $t > 0$ such that

$$\|\varphi(t, x_0) - \varphi(t, y)\| > \delta.$$

QUESTION 10. Can a polynomial flow exhibit sensitive dependence on initial conditions?

A flow $\varphi(t, z)$ is said to be *topologically transitive* if for every pair of open sets $U$, $V \subset \boldsymbol{R}^n$ there exists a $t > 0$ such that

$$\varphi(t, U) \cap V \neq \emptyset.$$

QUESTION 11. Can a polynomial flow be *topologically transitive*?

QUESTION 12. What kind of sets $\Omega \subset \boldsymbol{R}^n$ can be nonwandering sets for polynomial flows?

**Added in proof.** One can construct a $C^\infty$ function $\varphi$ such as one in Example 7.1. The proof will be published elsewhere.

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