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# On piecewise flat surfaces in the sense of Toralballa

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In this work we shall be concerned with the continous non-parametric surface $S: z = f(x, y)$, $(x, y) \in E$, where $E$ is a closed region on the $Oxy$ plane bounded by a simple closed polygon. Toralballa in [2] introduced the notion of piecewise flatness and gave a geometric definition of the surface area $A(S)$ for a piecewise flat surface $S$. He proved also, that for such a surface partial derivatives $f'_x$ and $f'_y$ exist almost everywhere (in the sense of the Lebesgue plane measure) and $A(S) = \iint_E \sqrt{1+f'^2_x+f'^2_y}\,dxdy$. From this equality we infer that the function $f$ from the representation of a piecewise flat surface is absolutely continuous in the sense of Tonelli (Rado [1], p. 519).

In this work we shall give a necessary and sufficient condition for a surface $S$ to be piecewise flat. Throughout the work we shall use the following notions: $\operatorname{proj}(A)$ means the projection of the set $A$ on the $Oxy$ plane, the polyhedron $\Pi$ inscribed on $S$ means a polyhedron such that each face $T \in \Pi$ is a triangle, $\Pi$ has a finite number of faces, each vertex of $\Pi$ lies on $S$, $\operatorname{proj}\big(\bigcup_{T\in\Pi} T\big) = E$, $\operatorname{int}\operatorname{proj}(T_1) \cap \operatorname{int}\operatorname{proj}(T_2) = \emptyset$, when $T_1, T_2 \in \Pi$, $T_1 \neq T_2$; $\theta(Oz, T)$ means the acute (or right) angle between the $Oz$-axis and the normal line to a triangle $T$, $\operatorname{int}_A B$ means the interior of the set $B$ relative to the set $A$, namely $\operatorname{int}_A B = A - \overline{A-B}$.

The following definitions are due to Toralballa [2]:

DEFINITION 1. A triangle $T$ inscribed on $S$ is called *$a$-admissible* $(0 < a \leqslant \frac{1}{3}\pi)$ when one angle of $T$ lies between $a$ and $\pi - a$.

DEFINITION 2. A polyhedron $\Pi$ inscribed on $S$ is called $(a, M)$-*admissible* if for each $T \in \Pi$ $\sec\theta(Oz, T) \leqslant M$ and $T$ is $a$-admissible.

DEFINITION 3. Let $T_1, T_2$ be $a$-admissible triangles inscribed on $S$. Let $\Phi(T_1, T_2)$ be the acute (or right) angle between the normal lines to $T_1$ and $T_2$. The angle $\Phi_a(T_1) = \sup\Phi(T_1, T_2)$, where $\operatorname{proj}(T_2) \subset \operatorname{proj}(T_1)$ is called the *$a$-deviation* of $T_1$. If $\Pi$ is an $(a, M)$-admissible polyhedron inscribed on $S$, then the angle $\Phi_a(\Pi) = \max_{T\in\Pi}\Phi_a(T)$ is termed the $a$-deviation norm of $\Pi$.

DEFINITION 4. A surface $S$ is called *$(a, M)$-regular* if there exists a sequence $\{\Pi_n\}$ of $(a, M)$-admissible polyhedra such that $\lim_{n\to\infty} \Phi_n(\Pi_n) = 0$. A surface $S$ is termed piecewise flat if there exist $a$ and $M$ such that $S$ is $(a, M)$-regular.

LEMMA 1. *If $T_1, T_2$ are $a$-admissible triangles inscribed on $S$ and* $\operatorname{proj}(T_1) \subset \operatorname{proj}(T_2)$, *then* $\Phi_a(T_1) \leqslant 2\Phi_a(T_2)$.

Proof. For each $a$-admissible triangle $T$ inscribed on $S$ for which $\operatorname{proj}(T) \subset \operatorname{proj}(T_1)$, we have

$$\Phi(T, T_1) \leqslant \Phi(T, T_2)+\Phi(T_2, T_1) \leqslant \Phi_a(T_2)+\Phi_a(T_2) = 2\Phi_a(T_2).$$

Hence $\Phi_a(T_1) \leqslant 2\Phi_a(T_2)$.

LEMMA 2. *If $S$ is an $(a, M)$-regular surface, then there exists an $(a, M)$-admissible polyhedron $\Pi$ inscribed on $S$ such that* $\arccos M^{-1}+\Phi_a(\Pi) < \frac{1}{2}\pi$.

Proof. Let $\{\Pi_n\}$ be a sequence of $(a, M)$-admissible polyhedra such that

$$\lim_{n\to\infty} \Phi_a(\Pi_n) = 0. \tag{1}$$

Of course, we have also

$$\sec\theta(Oz, T_{n,i}) \leqslant M \tag{2}$$

for $n = 1, 2, \ldots, i = 1, 2, \ldots, k_n$, where $T_{n,1}, \ldots, T_{n,k_n}$ are all faces of the polyhedron $\Pi_n$. From (2) we have

$$\theta(Oz, T_{n,i}) \leqslant \arccos M^{-1} = \theta_0 < \tfrac{1}{2}\pi \quad \text{for each } n, i, \tag{3}$$

and the proof of Lemma 2 follows immediately form (1) and (3).

COROLLARY. *If $S$ is $(a, M)$-regular, then $f$ fulfils locally the Lipschitz condition.*

Proof. Let us observe that if $T$ is an $a$-admissible triangle inscribed on $S$ such that $\operatorname{proj}(T) \subset \operatorname{proj}(T_{n,i})$ for some $n, i$, then we have

$$\theta(Oz, T) \leqslant \theta_0+\Phi_a(T_{n,i}) \leqslant \theta_0+\Phi_a(\Pi_n). \tag{4}$$

From (1) and (3) for arbitrary $\theta' : \theta_0 < \theta' < \frac{1}{2}\pi$ there exists an $N$ such that for each $n > N$

$$\theta_0+\Phi_a(\Pi_n) < \theta'. \tag{5}$$

If the Lipschitz condition is not fulfilled locally, then there exists a point $p_0 = (x_0, y_0)$ such that for each $\delta > 0$ there exists a point $p_1 = (x_1, y_1) \neq p_0$ such that

$$\varrho(p_0, p_1) < \delta, \tag{6}$$

and

$$\frac{|f(x_1, y_1)-f(x_0, y_0)|}{\varrho(p_0, p_1)} = K > \sec\theta'. \tag{7}$$

Let us consider the polyhedron $\Pi_n$, $n > N$. If $p_0 \in \operatorname{int}\operatorname{proj}(T_{n,i_1})$ for some $T_{n,i_1} \in \Pi_n$; then we can choose $p_1 \in \operatorname{int}\operatorname{proj}(T_{n,i_1})$ and (7) is fulfilled. If

$$p_0 \bigcup_{i=1}^{k_n} \operatorname{Fr}\operatorname{proj}(T_{n,i}), \qquad \text{where } \bigcup_{i=1}^{k_n} T_{n,i} = \Pi_n,$$

then we can choose $p_1$ such that $p_0$ and $p_1$ belong to the projection of one of these triangles, say $T_{n,i_1}$, but $p_1$ is not a vertex of $\operatorname{proj}(T_{n,i})$. In both cases we can choose without difficulty a third point $p_2 = (x_2, y_2) \in \operatorname{proj}(T_{n,i_1})$ such that the triangle $T$ with vertices $(x_0, y_0, f(x_0, y_0))$, $(x_1, y_1, f(x_1, y_1))$, $(x_2, y_2, f(x_2, y_2))$ has a right angle in the vertex $(x_1, y_1, f(x_1, y_1))$. Then, of course, $T$ is $a$-admissible (for each $a$) and $\operatorname{proj}(T) \subset \operatorname{proj}(T_{n,i_1})$. From (4) and (5) we have $\theta(Oz, T) < \theta'$. But $T$ has one side with vertices $(x_0, y_0, f(x_0, y_0))$, $(x_1, y_1, f(x_1, y_1))$, so the normal line to this triangle lies on the plane perpendicular to the vector $[x_1-x_0, y_1-y_0, f(x_1, y_1)-f(x_0, y_0)]$. It is not difficult to see that

$$\theta(Oz, T) \geqslant \arctan K. \tag{8}$$

Consequently,

$$\sec\theta(Oz, T) \geqslant \tan\theta(Oz, T) \geqslant K > \sec\theta'. \tag{9}$$

This contradicts (4) and (5).

LEMMA 3. *If $S\colon z = f(x, y)$, $(x, y) \in T_0$ is a continuous surface, where $T_0$ is a triangle on an $Oxy$ plane, then there exists a polyhedron $\Pi_0$ inscribed on $S$ such that every triangle $T \in \Pi_0$ is $\frac{1}{3}\pi$-admissible.*

**Proof.** Let $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (x_3, y_3)$ be vertices of $T_0$. Let us consider the triangle $T$ with vertices $(x_1, y_1, f(x_1, y_1))$, $(x_2, x_2, f(x_2, y_2))$, $(x_3, y_3, f(x_3, y_3))$. If $T$ is $\frac{1}{3}\pi$-admissible, then we may assume that $\Pi_0$ is a polyhedron consisting of one triangle $T$. If $T$ is not $\frac{1}{3}\pi$-admissible, then $T$ has an angle $\varphi > \frac{2}{3}\pi$ in one vertex, say in $(x_1, y_1, f(x_1, y_1))$. Let $P$ denote the bisector plane of $\varphi$. This plane, of course, has a non-empty intersection with the image of the segment $\overline{p_2p_3}$. Let $(x_0, y_0, f(x_0, y_0))$ belong to this intersection, let $T_1$ be the triangle with vertices $(x_0, y_0, f(x_0, y_0))$, $(x_1, y_1, f(x_1, y_1))$, $(x_2, y_2, f(x_2, y_2))$ and $T_2$ — the triangle with vertices $(x_0, y_0, f(x_0, y_0))$, $(x_1, y_1, f(x_1, y_1))$, $(x_3, y_3, f(x_3, y_3))$. If $\varphi_1, \varphi_2$ are angles in $T_1$ and $T_2$ respectively with the vertex $(x_1, y_1, f(x_1, y_1))$, then

$$\varphi_1 = \varphi_2, \tag{10}$$

$$\varphi_1+\varphi_2 \geqslant \varphi > \tfrac{2}{3}\pi, \tag{11}$$

$$\varphi_1+\varphi_2+\varphi \leqslant 2\pi. \tag{12}$$

From (10), (11), (12) it follows immediately that $\frac{1}{3}\pi \leqslant \varphi_1 \leqslant \frac{2}{3}\pi$, $\frac{1}{3}\pi \leqslant \varphi_2 \leqslant \frac{2}{3}\pi$; so $T_1$ and $T_2$ are $\frac{1}{3}\pi$-admissible and $\Pi_0$ consisting of $T_1$ and $T_2$ fulfils the required condition.

Let us introduce the following definitions:

Definition 4. If $p = (x, y) \in A$, where $A \subset E$ is a closed region bounded by a simple closed polygon, then by a deviation of a surface $S$ at the point $p$ with regard to $A$ we shall mean $\Phi_a(p\,|\,A) = \inf \Phi_a(T)$, where $T$ is an $a$-admissible triangle inscribed on $S$ such that $p \in \operatorname{int}_A \operatorname{proj}(T)$.

As an immediate corollary to this definition and Lemma 1 we have

Lemma 4. *If $p \in A_1 \subset A_2$, then $\Phi_a(p\,|\,A_1) \leqslant 2\Phi_a(p\,|\,A_2)$ for each $a$.*

Definition 5. By a *regular point* we shall mean a point $p \in E$ for which there exists an $a$ such that $\Phi_a(p\,|\,E) = 0$.

Definition 6. By an *$a_0$-irregular point* we shall mean a point $p \in E$ such that, for each $a$, $\Phi_a(p\,|\,E) > 0$ and for each $\varepsilon > 0$ there exist $\delta > 0$ and $a_0$-admissible triangles $T_1, \ldots, T_n$ inscribed on $S$ such that $p \in \bigcap_{i=1}^{n} \operatorname{proj}(T_i)$ $\operatorname{int}\operatorname{proj}(T_i) \cap \operatorname{int}\operatorname{proj}(T_j) = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{n} \operatorname{proj}(T_i) \supset K(p, \delta) \cap E$, where $K(p, \delta)$ means a circular neighbourhood of $p$ with radius $\delta$ and $\max_{1 \leqslant i \leqslant n} \Phi_{a_0}(T_i) < \varepsilon$.

Definition 7. By an *irregular point* we shall mean a point $p \in E$ such that, for each $a$, $\Phi_a(p\,|\,E) > 0$ and $p$ is not $a$-irregular.

Theorem. *A surface $S$ is piecewise flat if and only if there exist $a_0$ and $M_0$ such that*

1° *there exists at least one $(a_0, M_0)$-admissible polyhedron $\Pi$ inscribed on $S$ for which $\arccos M_0^{-1} + \Phi_{a_0}(\Pi) < \frac{1}{2}\pi$,*

2° *each point $p \in E$ is either regular or $a_0$-irregular,*

3° *for each $\varepsilon > 0$ there exists a finite set of straight lines $L_1, \ldots, L_m$ such that $C_\varepsilon = \{p : \Phi_{a_0}(p\,|\,E) > \varepsilon\} \subset \bigcup_{i=1}^{m} L$ and if $E_1, \ldots, E_k$ are closed regions with disjoint interiors for which straight lines $L_1, \ldots, L_m$ divide the whole region $E$, then $D_\varepsilon = \{p : \max_{1 \leqslant i \leqslant k} \Phi_{a_0}(p\,|\,E_i) > \varepsilon\}$ is a finite set (if $p \notin E_i$, then we put $\Phi_{a_0}(p\,|\,E_i) = 0$).*

Proof. *Necessity.* Let $S$ be piecewise flat. Then there exist $a$ and $M$ such that $S$ is $(a, M)$-regular. We shall prove that conditions 1°-3° are fulfilled for $a = a_0$ and $M = M_0$.

1. Condition 1° results at once form Lemma 2.

2. Let $p_0 \in E$ be an irregular point. We shall consider two cases. Let $\Pi$ be an arbitrary $(a, M)$-admissible polyhedron inscribed on $S$:

1° there exists a $T \epsilon \Pi$ such that $p_0 \epsilon \operatorname{int}\operatorname{proj}(T)$. Then $\Phi_a(\Pi) \geqslant \Phi_a(T) \geqslant \Phi_a(p_0|E) > 0$.

2° there exists a $T \epsilon \Pi$ such that $p_0 \epsilon \operatorname{Fr}\operatorname{proj}(T)$. Then from the definition of an irregular point for $a$ there exists an $\varepsilon_a > 0$ such that, for some $T_0 \epsilon \Pi$ for which $p_0 \epsilon \operatorname{Fr}\operatorname{proj}(T_0)$, we have $\Phi_a(T_0) > \varepsilon_a$. Hence for an arbitrary $(a, M)$-admissible polyhedron $\Pi$ inscribed on $S$ we have $\Phi_a(\Pi) \geqslant \min\big(\Phi_a(p_0|E), \varepsilon_a\big)$; and so $S$ is not piecewise flat.

3. If the first part of this condition is not fulfilled, then there exists an $\varepsilon_{a_0} > 0$ with the property that for each finite set $L_1, \ldots, L_m$ of straight lines there exists a $p_0 \epsilon C_{\varepsilon_{a_0}} - \bigcup_{i=1}^{m} L_i$. Consequently, for each $(a_0, M_0)$-admissible polyhedron $\Pi$ inscribed on $S$ there exist $T \epsilon \Pi$ and $p_0 \epsilon \operatorname{int}\operatorname{proj}(T)$ such that $\Phi_{a_0}(p|E) > \varepsilon_{a_0}$. Hence $\Phi_{a_0}(\Pi) \geqslant \Phi_{a_0}(T) \geqslant \Phi_{a_0}(p|E) > \varepsilon_{a_0}$ so that $S$ is not $(a_0, M_0)$-regular.

Now let us suppose that for some $\varepsilon_0$ the set $D_{\varepsilon_0}$ is not finite. Then for each $(a_0, M_0)$-admissible polyhedron $\Pi$ inscribed on $S$ there exists a triangle $T \epsilon \Pi$ such that the set $D_{\varepsilon_0} \cap \operatorname{proj}(T)$ is not finite. From Lemma 4 and the first part of this condition we conclude that $D_{\varepsilon_0} \subset C_{\varepsilon_0/2} \subset \bigcup_{i=1}^{m} L_i$, where $L_1, \ldots, L_m$ are straight lines corresponding to $\varepsilon_0/2$. Let us observe that the set of point belonging to $D_{\varepsilon_0} \cap \operatorname{proj}(T)$ which do not belong to $\operatorname{int}_{E_j}\operatorname{proj}(T)$ for any $j = 1, \ldots, k$ (where $E_1, \ldots, E_k$ are closed regions generated by $L_1, \ldots, L_m$) is finite. In fact, this set consists of one-point intersections of lines $L_i$, $i = 1, \ldots, m$ and sides of $\operatorname{proj}(T)$ and (possibly) of some vertices of $\operatorname{proj}(T)$ if one or more sides of $\operatorname{proj}(T)$ lie on some lines from the set $L_1, \ldots, L_m$. Hence there exist $p_0 \epsilon D_{\varepsilon_0} \cap \operatorname{proj}(T)$ and $E_{i_0}$, $1 \leqslant i_0 \leqslant k$, such that $p_0 \epsilon \operatorname{int}_{E_{i_0}}\operatorname{proj}(T)$; then $\Phi_{\varepsilon_0}(T) > \varepsilon_0$ and so $\Phi_{a_0}(\Pi) > \varepsilon_0$ and $S$ is not $(a_0, M_0)$-regular.

*Sufficiency*. Let 1°-3° be fulfilled. Let $\varepsilon > 0$ be a given number. We shall construct an $(a_0, M)$-admissible polyhedron $\Pi_\varepsilon$ inscribed on $S$ such that $\Phi_{\varepsilon_0}(\Pi_\varepsilon) < \varepsilon$ and $M$ does not depend on $\varepsilon$. From condition 1° we have an $(a_0, M_0)$-admissible polyhedron $\Pi$ inscribed on $S$ such that $\arccos M_0^{-1} + \Phi_{a_0}(\Pi) < \frac{1}{2}\pi$. If $\Phi_{a_0}(\Pi) < \varepsilon$, then we put $\Pi = \Pi_\varepsilon$. If $\Phi_{a_0}(\Pi) \geqslant \varepsilon$, then for $S_T : z = f(x, y)$, $(x, y) \epsilon \operatorname{proj}(T)$, where $T \epsilon \Pi$, we shall construct a polyhedron $\Pi_T$ fulfilling all the requirements and at last we shall put $\Pi_\varepsilon = \bigcup_{T \epsilon \Pi} \Pi_T$.

If $T \epsilon \Pi$, then $\operatorname{proj}(T) \subset E$ and conditions 2° and 3° are fulfilled for $\operatorname{proj}(T)$ instead of $E$. So there exist straight lines $L_1, \ldots, L_m$ such that $C_{\varepsilon/2} = \big\{p : \Phi_{a_0}\big(p|\operatorname{proj}(T)\big) > \frac{1}{2}\varepsilon\big\} \subset \bigcup_{i=1}^{m} L_i$. Let $E_1, \ldots, E_k$ be closed regions generated by these lines and let $E_j$ be one of them. We shall construct a polyhedron $\Pi_j$ inscribed on $S_j: z = f(x, y)$, $(x, y) \in E_j$, and next we shall put $\Pi_T = \bigcup_{j=1}^{k} \Pi_j$. Let $p_1, \ldots, p_s \in E_j$ be all points such that $p_l \in E_j \cap D_{\varepsilon/2}$, $l = 1, \ldots, s$. From the second condition each of these points is $\alpha_0$-irregular. For arbitrary $l$ we can choose $\alpha_0$-admissible triangles $T_{l,1}, \ldots, T_{l,r_l}$ inscribed on $S$ such that all the conditions of Definition 6 are fulfilled for $\varepsilon/4$. For each $l, i$, $\operatorname{proj}(T_{l,i}) \cap E_j$ is a closed region bounded by a closed polygon. From Lemma 3 dividing, if necessary, $\operatorname{proj}(T_{l,i}) \cap E_j$ into triangles, we can construct $\alpha_0$-admissible triangles $T_{l,i,t}$, $t = 1, 2, \ldots$, such that

$$\bigcup_{l=1}^{s} \bigcup_{i=1}^{r_l} \bigcup_{t} \operatorname{proj}(T_{l,i,t}) = \bigcup_{l=1}^{s} \bigcup_{i=1}^{r_l} \operatorname{proj}(T_{l,i}) \cap E_j$$

and $\operatorname{proj}(T_{l,i,t})$ have disjoint interiors. From Lemma 1 we have $\Phi_{\alpha_0}(T_{l,i,t}) < \frac{1}{2}\varepsilon < \varepsilon$ for each $l, i, t$. Now let us consider the set

$$E_j' = \overline{E_j - \bigcup_{l=1}^{s} \bigcup_{i=1}^{r_l} \operatorname{proj}(T_{l,i})},$$

$E_j'$ is a closed region bounded by a closed polygon or a finite sum of such regions. For each $p \in E_j'$ we have $\Phi_{\alpha_0}(p \mid E_j) < \varepsilon/2$, so for each $p \in E_j'$ there exists a triangle $T(p)$ inscribed on $S_j$ such that $p \in \operatorname{int}_{E_j'} \operatorname{proj}(T(p))$ and $\Phi_{\alpha_0}(T(p)) < \varepsilon/2$. The class of open (relatively in $E_j'$) sets $\{\operatorname{int}_{E_j'} \operatorname{proj}(T(p)) : p \in E_j'\}$ covers the compact set $E_j'$. In virtue of the Heine–Borel theorem there exists a finite covering of $E_j'$, say

$$E_j' \subset \bigcup_{i=1}^{k} \operatorname{int}_{E_j'} \operatorname{proj}(T(p)) \subset \bigcup_{i=1}^{k} \operatorname{proj}(T(p_i)).$$

From Lemma 3 we can again construct $\alpha_0$-admissible triangles $T_{i,t}$ inscribed on $S_j$ such that $\bigcup_{i=1}^{k} \bigcup_{t} \operatorname{proj}(T_{i,t}) = E_j'$ and $\operatorname{proj}(T_{i,t})$ have disjoint interiors. From Lemma 1, $\Phi_{\alpha_0}(T_{i,t}) < \varepsilon$ for each $i, t$.

Let $\Pi_j$ be a polyhedron consisting of all triangles $T_{l,i,t}$ and $T_{i,t}$. It is clear that $\Pi_j$ is inscribed on $S$ and $\Phi_{\alpha_0}(\Pi_j) < \varepsilon$. At the same time for each $T \in \Pi_j$ we have $\theta(Oz, T) \leqslant \arccos M_0^{-1} + \Phi_{\alpha_0}(\Pi)$. Hence $\Pi_j$ is an $(\alpha_0, M)$-admissible polyhedron, where

$$M = \sec\big(\arccos M_0^{-1} + \Phi_{\alpha_0}(\Pi)\big).$$

Then $\Pi_\varepsilon = \bigcup_{T=\Pi} \bigcup_{j} \Pi_j$ is an $(\alpha_0, M)$-admissible polyhedron inscribed on $S$ such that $\Phi_{\alpha_0}(\Pi_\varepsilon) < \varepsilon$. From the arbitrariness of $\varepsilon$, $S$ is $(\alpha_0, M)$-regular.

EXAMPLE. Let $E$ be a square $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$, $p_n = (2^{-n}, 2^{-n})$, $K_n = K(p_n, 2^{-n-3})$. Of course, $K_n \cap K_m = \emptyset$ for $n \neq m$. Let

$$f(x, y) = \begin{cases} 0, & (x, y) \in E - \bigcup_{n=1}^{\infty} K_n, \\ 2^{-n-3} - \sqrt{(x-2^{-n})^2 + (y-2^{-n})^2}, & (x, y) \in K_n, \; n = 1, 2, \ldots \end{cases}$$

The surface $S: z = f(x, y)$, $(x, y) \in E$ is not piecewise flat for two reasons: the point $(0, 0)$ is irregular and condition 3° is not fulfilled. By a slight modification of this example (the rounding of each cone at the top and bottom) we can obtain a surface which is not piecewise flat but for which $f'_x$ and $f'_y$ exist everywhere.

### References

[1] T. Rado, *Length and area*, New York 1948.

[2] L. V. Toralballa, *Piecewise flatness and surface area*, Ann. Polon. Math. 21 (1969), p. 223-230.

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