ON COVERINGS OF A UNIFORMITY

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In Isbell's book [1], p. 52, there is raised a question whether or not the coverings of a given uniformity, the cardinalities of which are less than a given cardinal number \( m \), form a base for a uniformity. Shirota [3] was the first who answered positively to this question in the case of a fine uniformity for the given topology. The answer is known to be positive for \( m = \aleph_0 \) (cf. [1], exercise, p. 52); the answer is also positive for uniformities having bases consisting of point-finite coverings (cf. [1], Theorem 28, p. 69, or, in a more precise form, the paper [2] of Kulpa) and for uniformities having bases consisting of \( \sigma \)-point-finite coverings (cf. Vidossich [4]).

In this note we shall give a positive solution of Isbell's question, assuming the generalized continuum hypothesis (i.e., if \( n < m \), then \( 2^n \leq m \) for all cardinal numbers).

A covering \( \mathcal{V} \) is a point-star-refinement of a covering \( \mathcal{U} \) if, for each \( x \) from \( X \), there exists a \( U \) in \( \mathcal{U} \) such that \( \text{st} (x, \mathcal{V}) \subseteq U \); it is a star-refinement of \( \mathcal{U} \) if, for each \( V \) in \( \mathcal{V} \), there exists a \( U \) in \( \mathcal{U} \) such that \( \text{st} (V, \mathcal{V}) \subseteq U \); recall that a point-star-refinement, if applied two times, gives a star-refinement.

THEOREM. Let \( \mu \) be a uniformity on \( X \) and let \( m \) be an infinite cardinal number. Then the family of coverings from \( \mu \), the cardinalities of which are less than \( m \), is a base for a uniformity \( \nu, \nu \subseteq \mu \).

Proof. It suffices to show that, for each \( \mathcal{U} \) from \( \mu \), there exists in \( \mu \) a covering \( \mathcal{W} \) being a point-star-refinement of \( \mathcal{U} \) and such that \( \text{card} \mathcal{W} \leq \text{card} \mathcal{U} \) or \( \mathcal{W} \) is finite if \( \mathcal{U} \) is finite.

Let \( \lambda \) be an initial ordinal number for the cardinality of \( \mathcal{U} \). We can assume that \( \mathcal{U} = \{ U_\alpha : \alpha < \lambda \} \). Let \( \mathcal{V} \) be an arbitrary star-refinement of \( \mathcal{U} \) belonging to \( \mu \). Define \( p(V) \) for \( V \in \mathcal{V} \) to be the least \( a \) such that \( U_\alpha \ni \text{st}(V, \mathcal{V}) \), and let \( \mathcal{V}_a = \{ V : p(V) = a \} \). We have

\[
(1) \quad \text{st}(V, \mathcal{V}) \subseteq U_\alpha \quad \text{for} \quad V \in \mathcal{V}_a.
\]
For each \( a \) we define a partition of the collection \( \mathcal{V}_a \) as follows: elements \( V \) and \( V' \) of \( \mathcal{V}_a \) are in the same element of the partition iff

\[
V \subset U_\gamma \iff V' \subset U_\gamma \quad \text{for each } \gamma, \text{ where } \gamma \leq a.
\]

The set of all elements of the partition of \( \mathcal{V}_a \) is of the cardinality not greater than the cardinality of the family of all subsets of the set \( \{ \gamma : \gamma \leq a \} \), and so, in virtue of the generalized continuum hypothesis, it is not greater than card \( \mathcal{W} \) or is finite if \( \mathcal{W} \) is finite.

Let \( \mathcal{W}_a \) be the collection consisting of unions of elements of the partition of \( \mathcal{V}_a \). Let \( \mathcal{W} = \bigcup \{ \mathcal{W}_a : a < \lambda \} \).

It is obvious that \( \mathcal{W} \) is a covering belonging to \( \mu \) (since \( \mathcal{V} \) is a refinement of \( \mathcal{W} \) and \( \mathcal{V} \subset \mu \)) and that card \( \mathcal{W} \leq \text{card } \mathcal{W} \) or \( \mathcal{W} \) is finite.

It remains to prove that \( \mathcal{W} \) is a point-star-refinement of \( \mathcal{W} \). To do this take \( x \in X \). Let \( a(x) = \min \{ a : x \in W \in \mathcal{W}_a \} \). We shall show that \( \text{st}(x, \mathcal{W}) = U_{a(x)} \). From the definition of \( a(x) \) it follows that there exists a \( V \in \mathcal{V}_{a(x)} \) such that \( x \in V \). By (1), we get \( x \in V \subset \text{st}(V, \mathcal{V}) \subset U_{a(x)} \).

Let \( W \) be such that \( x \in W \in \mathcal{W}_a \). There exists \( V' \in \mathcal{V}_a \) such that \( x \in V' \) and such that \( V' \) belongs to that element of the partition of \( \mathcal{V}_a \) the union of which is \( W \).

But \( x \in V \cap V' \). Hence \( V' \subset \text{st}(V, \mathcal{V}) \subset U_{a(x)} \). By definition (2) of the partition and the inequality \( a(x) \leq a \), we infer that \( W \) is contained in \( U_{a(x)} \).

Since \( W \) is an arbitrary element of \( \mathcal{W} \) such that \( x \in W \), we infer that \( \text{st}(x, \mathcal{W}) \subset U_{a(x)} \).

REFERENCES


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