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ON A THEOREM OF DOBROWOLSKI ABOUT THE PRODUCT OF CONJUGATE NUMBERS

BY

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1. Let α be an algebraic integer of degree n over Q, different from zero and roots of unity. We consider

$$M(\alpha) = \prod_{i=1}^{n} \max(1, |\alpha_i|),$$

where $\alpha_1, \ldots, \alpha_n$ denote the conjugates of α. Dobrowolski ([1]) has shown that

$$M(\alpha) > 1 + (1-\varepsilon)\left(\frac{\log\log n}{\log n}\right)^3$$

for arbitrary positive ε and $n > n_0(\varepsilon)$. His proof depends on the construction of an auxiliary polynomial with small coefficients, for which purpose a sharpened version of Siegel's lemma is employed. But, instead of the coefficients, it suffices to control the *values* of that polynomial at certain points. This observation enables us to simplify the argument considerably by replacing Siegel's lemma with Minkowski's theorem on linear forms. A slight improvement of the result is obtained too, namely

THEOREM.

$$M(\alpha) > 1 + (2-\varepsilon)\left(\frac{\log\log n}{\log n}\right)^3 \qquad (\varepsilon > 0;\ n > n_0(\varepsilon)).$$

2. We state three lemmas, the first of which is due to Dobrowolski.

LEMMA 1.

(1) $$\alpha_i^r \neq \alpha_j^s \quad \textit{for} \quad r, s \in N,\ r \neq s,\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n;$$

(2) $$\Big| \prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}} (\alpha_i^p - \alpha_j) \Big| \geqslant p^n \quad \textit{for prime numbers } p.$$

LEMMA 2. *Let $Q \subset N$ be a finite set such that*

$$\deg(\alpha^q) = n \quad \text{for} \quad q \in Q \tag{3}$$

and let $R_q \in N$ for $q \in Q$. Moreover, let λ_{qr} be positive real numbers having the property that for arbitrary $F(x) \in Z[x]$ the inequalities

$$\Big|\prod_{i=1}^{n} F^{(r)}(\alpha_i^q)\Big| < \lambda_{qr} \quad (q \in Q;\ r = 0, \ldots, R_q - 1) \tag{4}$$

imply already

$$F(\alpha^q) = F'(\alpha^q) = \ldots = F^{(R_q - 1)}(\alpha^q) = 0 \quad (q \in Q). \tag{5}$$

Then

$$\log M(\alpha) > \frac{\Lambda - \frac{1}{2}(\sum R_q^2)\log(n \sum R_q)}{(\sum R_q)(\sum qR_q)},$$

where the sums are extended over $q \in Q$ and

$$\Lambda = \frac{1}{n} \log \prod_{q \in Q} \prod_{r=0}^{R_q - 1} \lambda_{qr}.$$

Proof. Let $N = n \sum R_q$ and consider a polynomial in x of degree $N-1$ with indeterminate coefficients x_j:

$$\Phi(x; x_j) = \sum_{j=0}^{N-1} x_j \cdot x^j.$$

The terms

$$\frac{d^r}{dx^r} \Phi(x; x_j)\Big|_{x = \alpha_i^q} = \sum_{j=r}^{N-1} j(j-1) \ldots (j-r+1) \alpha_i^{q(j-r)} x_j$$

$$(q \in Q;\ r = 0, \ldots, R_q - 1;\ i = 1, \ldots, n)$$

constitute a system of N linear forms in the x_j's. We denote the absolute value of its determinant by D. Then

$$\prod_{q \in Q} \prod_{r=0}^{R_q - 1} \lambda_{qr} \leqslant D,$$

since otherwise Minkowski's theorem (which is easily extended to cover the case $D = 0$) would supply numbers

$$a_0, \ldots, a_{N-1} \in Z, \quad \text{not all zero,}$$

such that

$$F(x) := \Phi(x; a_j) \in Z[x] \setminus \{0\}$$

would satisfy (4) and hence also (5). This means (cf. (1))

$$\prod_{q\in Q} f_q(x)^{R_q} \mid F(x),$$

$f_q(x)$ signifying the minimal polynomial of α^q. By (3) we obtain a contradiction:

$$N-1 \geqslant \deg F(x) \geqslant \deg \prod_{q\in Q} f_q(x)^{R_q} = N.$$

On the other hand, Hadamard's inequality yields

$$\begin{aligned} D &\leqslant \prod_{q\in Q} \prod_{r=0}^{R_q-1} \prod_{i=1}^{n} \Big\{ \sum_{j=r}^{N-1} |j(j-1)\dots(j-r+1)\alpha_i^{q(j-r)}|^2 \Big\}^{1/2} \\ &< \prod_{q\in Q} \prod_{r=0}^{R_q-1} \prod_{i=1}^{n} \{N^{r+1/2} \max(1, |\alpha_i|)^{qN}\} \\ &= N^{n\sum R_q^2/2} \cdot M(\alpha)^{N\sum qR_q}, \end{aligned}$$

and the assertion follows. □

It remains to prepare a tool for dealing with condition (3). Although Lemma 3 of [1] would suffice for our present purpose, the following lemma may be of independent interest.

LEMMA 3. *Let p be a prime and $\deg(\alpha^p) = d < n$. Then $M(\alpha) = M(\alpha^p)$ or else there is a p-th root of unity ζ such that $\deg(\zeta\alpha) = d$ and $M(\alpha) > M(\zeta\alpha)$.*

This implies that one may assume

$$\deg(\alpha^p) = n \quad \text{for all primes } p \tag{6}$$

in most cases when lower bounds for $M(\alpha)$ are concerned. Indeed, suppose we have proved

$$M(\alpha) > 1 + \Theta(n) \tag{7}$$

for all α subject to (6) and all n, Θ being a positive non-increasing function. Then induction on n yields immediately that (7) generally holds. If (7) is known only for $n > n_0$, we apply the same argument to $\Theta^*(n) = \min(\Theta(n), c)$, where c is some positive constant such that

$$M(\alpha) > 1 + c \quad \text{for} \quad 1 \leqslant n \leqslant n_0,$$

and observe that $\Theta^*(n) = \Theta(n)$ for large n if, additionally, $\Theta(n)$ tends to zero for $n \to \infty$.

Proof of Lemma 3. If $n/d = [Q(\alpha): Q(\alpha^p)] = p$, then each conjugate of α^p occurs exactly p times among the numbers $\alpha_1^p, \dots, \alpha_n^p$; thus

$$M(\alpha)^p = \prod_{i=1}^{n} \max(1, |\alpha_i^p|) = M(\alpha^p)^p.$$

Now let $n/d \neq p$. Then the equation $x^p - \alpha^p = 0$ is reducible over $Q(\alpha^p)$, say

$$x^p - \alpha^p = g(x)h(x), \quad g(x), h(x) \in Q(\alpha^p)[x], \quad 1 \leqslant \deg g(x) =: t < p.$$

Since

$$x^p - \alpha^p = \prod_{s=1}^{p} (x - \alpha e^{2\pi i s/p}),$$

it follows by considering the constant term of $g(x)$ that

$$\xi \alpha^t \in Q(\alpha^p), \quad \xi \text{ a } p\text{th root of unity.}$$

But $kt + lp = 1$ for suitable $k, l \in \mathbf{Z}$, and so

$$\zeta\alpha = (\xi\alpha^t)^k (\alpha^p)^l \in Q(\alpha^p), \quad \text{where } \zeta := \xi^k.$$

Thus we have $Q(\zeta\alpha) = Q(\alpha^p)$, i.e., $\deg(\zeta\alpha) = d$, and $\zeta \in Q(\alpha)$. If $\zeta_1, \ldots, \zeta_n$ are the conjugates of ζ relative to $Q(\alpha)$, then each conjugate of $\zeta\alpha$ over Q occurs exactly n/d times among the numbers $\zeta_1\alpha_1, \ldots, \zeta_n\alpha_n$; hence

$$M(\alpha) = \prod_{i=1}^{n} \max(1, |\zeta_i \alpha_i|) = M(\zeta\alpha)^{n/d} > M(\zeta\alpha). \quad \square$$

3. Proof of the theorem. We assume (6) and choose in Lemma 2 $Q = \{1\} \cup P$, where P is the set of all prime numbers $p \leqslant u$. Further we put $R_1 = R$, $R_p = 1$ for $p \in P$. Then the numbers

$$\lambda_{1r} = 1 \qquad (r = 0, \ldots, R-1),$$
$$\lambda_{p0} = p^{nR} \qquad (p \in P)$$

satisfy the required conditions: (4) implies first

$$F(\alpha) = F'(\alpha) = \ldots = F^{(R-1)}(\alpha) = 0, \quad \text{i.e.,} \quad f_1(x)^R \mid F(x),$$

and then, by (2), $F(\alpha^p) = 0$ for $p \in P$. Hence

$$\log M(\alpha) > \frac{R \sum_{p \leqslant u} \log p - \frac{1}{2}\{R^2 + \pi(u)\} \log(n\{R + \pi(u)\})}{\{R + \pi(u)\}\{R + \sum_{p \leqslant u} p\}}.$$

Finally, setting

$$R = \left[\frac{\log n}{\log\log n}\right], \quad u = \frac{(\log n)^2}{\log\log n},$$

we obtain by means of the prime number theorem

$$\log M(\alpha) > 2\left(\frac{\log\log n}{\log n}\right)^3 (1 + o(1)) > (2 - \varepsilon)\left(\frac{\log\log n}{\log n}\right)^3$$

if n is sufficiently large. This proves the assertion. $\square$

REFERENCE

[1] E. Dobrowolski, *On a question of Lehmer and the number of irreducible factors of a polynomial*, Acta Arithmetica 34 (1979), p. 391–401.

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Annex. In view of the proof of Lemma 2, one will naturally suppose that the determinant considered there can be explicitly expressed by a product of differences, similar to the Vandermondian. This is in fact true, and E. Dobrowolski has just sent me an elegant proof which I shall record here.* As a consequence, the preceding proof may be rearranged so that Minkowski's theorem as well as any reminiscence of transcendence theory is avoided.

Dobrowolski argues as follows: Consider the vector valued function

$$\varphi\colon C \to C^N, \qquad \varphi(z) = (1, z, \ldots, z^{N-1})^T$$

together with its derivatives $\varphi^{(r)}$. The determinant in question is of the form

$$D = \det\left[\varphi(z_1), \ldots, \varphi^{(L_1-1)}(z_1), \ldots, \varphi(z_m), \ldots, \varphi^{(L_m-1)}(z_m)\right]$$

with $\sum_{j=1}^{m} L_j = N$.

(In Lemma 2: $z_1 = \alpha_1^{q_1}, \ldots, z_n = \alpha_n^{q_1}$, $z_{n+1} = \alpha_1^{q_2}, \ldots, z_{2n} = \alpha_n^{q_2}, \ldots$ and $L_1 = \ldots = L_n = R_{q_1}$, $L_{n+1} = \ldots = L_{2n} = R_{q_2}, \ldots$ if $Q = \{q_1, q_2, \ldots\}$.) Now, for given h, let

$$(\Delta_0 \varphi)(z) = \varphi(z), \qquad (\Delta_{r+1} \varphi)(z) = (\Delta_r \varphi)(z+h) - (\Delta_r \varphi)(z) \qquad (r \geqslant 0).$$

Then

$$(\Delta_r \varphi)(z) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} \varphi\left(z + (r-k)h\right) \tag{8}$$

* Editors' note: As pointed out by A. Schinzel, the determinant in question was evaluated by C. Meray in 1867 (cf. M. Shibayama, Tôhoku Mathematical Journal 2 (1912), p. 143–146).

and

$$\lim_{h\to 0} h^{-r}(\Delta_r \varphi)(z) = \varphi^{(r)}(z).$$

Hence

$$D = \lim_{h\to 0} h^{-M} \det[\Delta_0 \varphi(z_1), \ldots, \Delta_{L_1-1} \varphi(z_1), \ldots \\ \ldots, \Delta_0 \varphi(z_m), \ldots, \Delta_{L_m-1} \varphi(z_m)],$$

where $M = \sum_{j=1}^{m} \sum_{i=1}^{L_j-1} i.$

From (8) it follows, on taking linear combinations of the columns, that

$$D = \lim_{h\to 0} h^{-M} \det[\varphi(z_1), \varphi(z_1+h), \ldots, \varphi(z_1+(L_1-1)h), \ldots \\ \ldots, \varphi(z_m), \ldots, \varphi(z_m+(L_m-1)h)],$$

and this is Vandermonde's determinant. So

$$D = \prod_{i>j} (z_i - z_j)^{L_i L_j} \prod_{k=1}^{m} [(L_k-1)!(L_k-2)! \ldots 2!\,1!].$$

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