IMBEDDING LOCALLY CONVEX LATTICES
INTO COMPACT LATTICES

BY

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Let $\mathcal{L}$ be the class of locally convex, distributive topological lattices. All those distributive topological lattices which are compact [5], or locally compact and connected [1], or discrete belong to $\mathcal{L}$. Relying on the imbedding theorem in [2], it was shown in [6] that if $L \in \mathcal{L}$ and has finite breadth $n$, then $L$ can be imbedded in a product of $n$ compact chains. The condition of local convexity thus serves to characterize sublattices of finite products of compact chains. In general, it is not even true that compact members of $\mathcal{L}$ can be imbedded in products of chains [3].

In this note we are concerned with the question of when members of $\mathcal{L}$ — without the assumption of finite breadth — can be imbedded in compact lattices. We first show that if $L \in \mathcal{L}$ and $L$ is locally compact and connected, then $L$ can be so imbedded. Next, we give an example of a member of $\mathcal{L}$ — which has the discrete topology — which cannot be imbedded in a compact lattice.

Recall that a topological lattice is a Hausdorff space $L$ with a pair of continuous maps $\wedge, \vee : L \times L \to L$ such that $(L, \wedge, \vee)$ is a lattice. A subset $A$ of a lattice is convex if whenever $a, b \in A$ and $a \leq x \leq b$, then $x \in A$. A topological lattice is locally convex if its topology has a base of convex sets. As noted above, $\mathcal{L}$ will be the class of locally convex, distributive topological lattices. For $L \in \mathcal{L}$ and $a, b \in L$ with $a < b$, the interval from $a$ to $b$, $[a, b]$, is $\{x \in L ; a \leq x \leq b\}$ and $[a, b]^\#$ is the natural continuous homomorphism of $L$ onto $[a, b]$, i.e., for $x \in L$,

$$[a, b]^\#(x) = (a \vee x) \wedge b = a \vee (x \wedge b).$$

By an imbedding we shall mean an open, injective, continuous homomorphism. For a set $W$, $\partial(W)$ will be its boundary and $W^*$ will be its closure.

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1. Locally compact connected lattices. Throughout this section \( L \) will be an arbitrary but fixed member of \( \mathcal{L} \) which is, in addition, locally compact and connected. The topology on \( L \) will have a neighborhood base of compact convex sets. \( \mathcal{F} \) will denote the set of compact intervals of \( L \). The following lemma will be used often and usually without reference.

**Lemma 1.1.** Let \( W \) be a compact, convex neighborhood of the point \( p \) of \( L \), let \( w_1, w_2 \in W \), and \( q, r \in L/W \) with \( w_1 < w_2 \) and \( q < p \). Then

(i) \([q, p] \cap \partial(W) \neq \emptyset\);
(ii) \([w_1, w_2] \subseteq W\);
(iii) \([w_1, w_2] \in \mathcal{F}\);
(iv) either \( p \land r \notin W \) or \( p \lor r \notin W \).

**Proof.** (i) holds, because \([q, p]\) is connected. (ii) and (iii) result from the fact that \( W \) is both compact and convex. If both \( p \land r \) and \( p \lor r \) belong to \( W \), then, because \( W \) is convex, \( r \) would belong to \( W \). Hence (iv) holds.

Now endow \( \mathbb{X} \mathcal{F} \) with coordinate-wise operations and the Tychonov topology. \( \mathbb{X} \mathcal{F} \) becomes a compact distributive topological lattice. The parametric map \( \mathcal{F}^\# : L \to \mathbb{X} \mathcal{F} \) defined by

\[
(\mathcal{F}^\#(x))_{[a, b]} = [a, b]^\#(x)
\]

is a continuous homomorphism. From Mrówka's imbedding theorem in [4] and the discussion in section 3 of [6] we have the following

**Lemma 1.2.** If \( \mathcal{F} \) separates points in \( L \) (i.e., for \( x, y \in L \) with \( x \neq y \), there is \([a, b] \in \mathcal{F} \) such that \([a, b]^\#(x) \neq [a, b]^\#(y)\)), then \( \mathcal{F}^\# \) is injective. \( \mathcal{F}^\# \) is an imbedding if, in addition, given \( p \in L \) and \( F \) a closed subset of \( L \) not containing \( p \), there are \([a_1, b_1], \ldots, [a_n, b_n] \in \mathcal{F} \) and subsets \( F_1, \ldots, F_n \) of \( L \) such that

\[
F = F_1 \cup \ldots \cup F_n \quad \text{and} \quad [a_i, b_i]^\#(p) \notin ([a_i, b_i]^\#(F_i))^*.
\]

With this result we are now prepared to prove our imbedding theorem.

**Theorem 1.3.** \( \mathcal{F}^\# : L \to \mathbb{X} \mathcal{F} \) is an imbedding. Thus every locally compact, connected distributive topological lattice can be imbedded in a compact distributive topological lattice.

**Proof.** We begin by showing that \( \mathcal{F}^\# \) is injective. Let \( x, y \in L \) with \( x \neq y \). Select a compact, convex neighborhood \( W \) of \( x \) which excludes \( y \). Then either \( x \lor y \notin W \) or \( x \land y \notin W \). We assume the latter holds. Then

\[
[x \land y, x] \cap \partial(W) \neq \emptyset.
\]

Let \( w \) be any point of that set. \([w, x] \in \mathcal{F} \) and

\[
[w, x]^\#(y) = w \neq x = [w, x]^\#(x).
\]

Hence \( \mathcal{F}^\# \) is injective.
Now suppose that \( p \in L \) and \( F \) is a closed subset of \( L \) with \( p \notin F \). There is a compact, convex neighborhood \( W \) of \( p \) which is contained in \( L \setminus F \). The sets

\[
B = \partial(W) \cap (p \wedge L) \quad \text{and} \quad T = \partial(W) \cap (p \vee L)
\]

are compact. This implies that there are a subset \( T_0 = \{t_1, \ldots, t_n\} \subseteq T \) and open sets \( U(t_i), \ldots, U(t_n) \) of \( T \) such that \( t_i \in U(t_i) \),

\[
\bigcup_{i=1}^{n} U(t_i) = T \quad \text{and} \quad [p, t_i]^{\#}(p) \notin ([p, t_i]^{\#}(U(t_i)))^* \vee [p, t_i].
\]

Similarly, there are a subset \( B_0 = \{b_1, \ldots, b_m\} \subseteq B \) and open sets \( U(b_1), \ldots, U(b_m) \) of \( B \) such that \( b_i \in U(b_i) \),

\[
\bigcup_{i=1}^{m} U(b_i) = B \quad \text{and} \quad [b_i, p]^{\#}(p) \notin ([b_i, p]^{\#}(U(b_i)))^* \wedge [b_i, p].
\]

Next, we define a map \( \gamma : B \cup T \to B_0 \cup T_0 \) which assigns to an element \( x \) of \( B \cup T \) an element \( \gamma(x) \) of \( B_0 \cup T_0 \) with the property that \( x \in U(\gamma(x)) \). For \( f \in F \), either \( p \wedge f \notin W \) or \( p \vee f \notin W \). Hence

\[
([p \wedge f, p] \cup [p, p \vee f]) \cap \partial(W) \neq \emptyset.
\]

Let \( \delta(f) \) be any point of this set and let \( F_i = \delta^{-1}(\gamma^{-1}(b_i)) \) for \( i = 1, \ldots, m \) and \( F_{m+1} = \delta^{-1}(\gamma^{-1}(t_i)) \) for \( i = 1, \ldots, n \). Then \( F = F_1 \cup \ldots \cup F_{m+n} \). Now, let \( f \in F_i \). We may assume that \( i = 1 \). Then \( b_1 = \gamma(\delta(f)) \) and we have

\[
[b_1, p]^{\#}(f) = b_1 \vee (p \wedge f) \leq b_1 \vee \delta(f).
\]

Hence

\[
[b_1, p]^{\#}(f) \notin ([b_1, p]^{\#}(U(b_i)))^* \wedge [b_1, p].
\]

Therefore, \( [b_1, p]^{\#}(p) \notin ([b_1, p]^{\#}(F_i))^* \). It then follows that \( \mathcal{F}^{\#} \) is an imbedding.

Remark. The methods used in this section can be modified to show that if \( L \) is a locally compact, connected (not necessarily distributive) topological lattice such that every pair of distinct points of \( L \) can be separated by a continuous homomorphism into a compact topological lattice, then \( L \) can be imbedded in a compact lattice. In particular, if \( L \in \mathcal{L} \), \( L \) is locally compact and connected and \( \text{Hom}(L, I) \) separates points, then \( L \) can be imbedded in a product of copies of \( I \). (\( I \) is the real interval \( [0, 1] \) with the usual operations.)

2. An example. In this section we present an example of a member of \( \mathcal{L} \) which cannot be imbedded in a compact lattice. \( 2 \) will be the lattice on \( \{0, 1\} \). Let \( Y \) be the cartesian product of a countable number of copies of \( 2 \) with coordinate-wise operations and the \textit{discrete} topology. \( Y \), obviously, belongs to \( \mathcal{L} \). \( \text{Hom}(Y, 2) \) separates points, so \( Y \) has a continuous injective
homomorphism into a compact lattice; in fact, $Y$ has a continuous injective homomorphism onto a compact topological lattice. Suppose that $Y$ is a sublattice of a compact topological lattice $K$. We may assume that $O_Y = O_K$ and $1_K = 1_Y$. Since $\{O_Y\}$ is an open set in $Y$, it cannot be a limit of a sequence of points of $Y$. However, define $a_n$ to be that point of $Y$ whose first $n$ coordinates are zero and all subsequent coordinates are one. Then $\{a_n\}$ is an infinite decreasing sequence in $K$, so it must have a limit point $k$ and $k \leq a_n$ for all $n$. Next, we define $e_n$ to be that point of $Y$ whose $n$-th coordinate is one and all other coordinates are zero. $e_n \wedge a_m = 0$ for $m \neq n$. Then, since $K$ is a compact topological lattice, the infinite distributivity law holds, so

$$O_Y = \bigvee_{n=1}^{\infty} (e_n \wedge k) = k \wedge \left( \bigvee_{n=1}^{\infty} e_n \right) = k \wedge 1_Y = k.$$ 

This is a contradiction. Hence $Y$ cannot be imbedded in a compact lattice. (See also [7].)

REFERENCES


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