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EXOTIC LOGICS

BY

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We call a logic *exotic* (resp., *nearly exotic*) if it has no states (resp., exactly one state). We show that any logic can be embedded into an exotic logic with given center. If a logic has a two-valued state, then it can be embedded into a nearly exotic logic with given center.

Introduction. In the axiomatic quantum mechanics we sometimes identify the event structure of a quantum system with an orthomodular partially ordered set. This orthomodular poset is called a *logic* (of the quantum system). The states of the system then correspond to the (probability) measures on the logic and the set of "absolutely comparable" elements of the system corresponds to the Boolean algebra of all absolutely compatible elements of the logic (see [3] and [9]).

In this paper we consider logics with extremally small sets of states. We embed, as announced in the abstract, any logic into an "exotic" type logic and, moreover, we leave the center as a free parameter. We thus extend the results of Greechie [2] and Shultz [7]. As a by-product we exhibit an elementary example of a (nontrivial) logic with exactly one state (cf. [7]).

As indicated above, our considerations have certain interpretations in the foundations of quantum theory. Since the exotic (resp., nearly exotic) extension of an orthomodular lattice is again a lattice, our technique can also find an application in the lattice theory.

1. Preliminaries. Let us first recall basic definitions.

DEFINITION 1.1. A *logic* is a set L endowed with a partial ordering $\leqslant$ and a unary operation $'$ such that

(i) $0, 1 \in L$;

(ii) $a \leqslant b \Rightarrow b' \leqslant a'$ for any $a, b \in L$;

(iii) $(a')' = a$ for any $a \in L$;

(iv) $a \vee a' = 1$ and $a \wedge a' = 0$ for any $a \in L$ (the symbols $\vee$ and $\wedge$ mean here the lattice-theoretic operations given by $\leqslant$);

(v) $\bigvee_{n=1}^{p} a_n$ exists in L whenever $a_n \in L$ for any n, $1 \leqslant n \leqslant p$, and $a_n \leqslant a_k'$ for $n \neq k$;

(vi) $b = a \vee (b \wedge a')$ whenever $a, b \in L$ and $a \leqslant b$.

In what follows we reserve the letter L for logics. An example of a logic is the lattice of projectors of a Hilbert space or, of course, a Boolean algebra.

DEFINITION 1.2. Two elements $a, b \in L$ are called *compatible* if there are three elements $c, d, e \in L$ such that $c \leqslant d'$, $d \leqslant e'$, $e \leqslant c'$ and $a = c \vee d$, $b = c \vee e$.

PROPOSITION 1.1. (i) *If $a \leqslant b$, then a, b are compatible.*

(ii) *If a, b are compatible, then $a \vee b$ and $a \wedge b$ exist and we have $a \wedge b = 0$ if and only if $a \leqslant b'$.*

(iii) *If a, b are compatible, then a, b' are compatible (and, as a consequence, vice versa).*

The proof is obvious (see also [3]).

DEFINITION 1.3. An element $a \in L$ is called *central* if a is compatible with any element of L. We denote by $C(L)$ the set of all central elements of L and call $C(L)$ the *center* of L. The logic L is said to have a *trivial center* if $C(L) = \{0, 1\}$.

PROPOSITION 1.2. *The set $C(L)$ with the operations $'$, $\vee$, and $\wedge$ inherited from L is a Boolean algebra. The logic L is a Boolean algebra if and only if $L = C(L)$.*

Proof. The set $C(L)$ can be extended to a maximal Boolean subalgebra of L (see, e.g., [1]). Since $C(L)$ is obviously the intersection of all maximal Boolean subalgebras of L, the proof follows. (The rest of Proposition 1.2 is obvious.)

In the sequel we shall need the construction of a product of logics. Let us recall the definition.

DEFINITION 1.4. Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of logics. Denote by $\prod_{\alpha\in I} L_\alpha$ the ordinary Cartesian product of the sets L_α and endow the set $\prod_{\alpha\in I} L_\alpha$ with the relation $\leqslant$ and the unary operation $'$ as follows. If

$$k = \{k_\alpha \mid \alpha \in I\} \in \prod_{\alpha\in I} L_\alpha \quad \text{and} \quad h = \{h_\alpha \mid \alpha \in I\} \in \prod_{\alpha\in I} L_\alpha,$$

then $k \leqslant h$ (resp., $k' = h$) if and only if $k_\alpha \leqslant h_\alpha$ (resp., $k_\alpha' = h_\alpha$) for any $\alpha \in I$. The set $\prod_{\alpha\in I} L_\alpha$ with the above-defined relation $\leqslant$ and the operation $'$ is called the *product of the collection* $\{L_\alpha \mid \alpha \in I\}$.

PROPOSITION 1.3. *Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of logics. Then $\prod_{\alpha\in I} L_\alpha$ is a* *logic. If* $C(L_\alpha) = \{0, 1\}$ *for any* $\alpha \in I$, *then* $C(\prod_{\alpha\in I} L_\alpha)$ *is Boolean isomorphic to the Boolean algebra of all subsets of* I.

The proof is obvious (see also [4]).

DEFINITION 1.5. A *state on a logic* L is a mapping $s\colon L \to \langle 0, 1\rangle$ such that

(i) $s(1) = 1$,

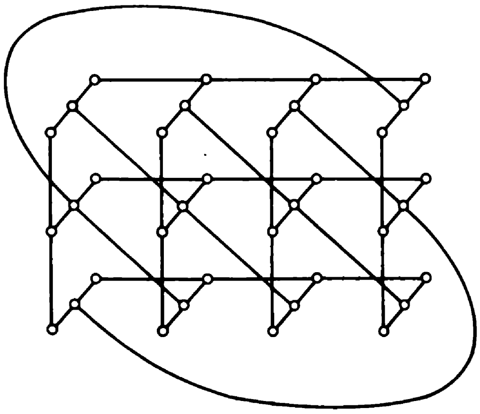
(ii) if $a, b \in L$ and $a \leqslant b'$, then $s(a \vee b) = s(a) + s(b)$.

PROPOSITION 1.4. *If* $a, b \in L$ *and* $a \leqslant b$, *then* $s(a) \leqslant s(b)$ *for any state* s *on* L.

Proof. One uses the orthomodular law (Definition 1.1 (vi)).

The set of states on a logic may generally look more complex than that of a Boolean algebra. The key results on the set of states can be found for example in the papers [2] and [5]–[7]. Here we concentrate on a rather "exotic" situation – on the type of logics with no or exactly one state.

2. Exotic and nearly exotic logics exist. A finite and relatively simple example of an exotic logic was constructed by Greechie (see [2]). Since we shall implicitly use it for exhibiting a (finite) nearly exotic logic, let us briefly recall the construction (see the figure below). One first takes 20 Boolean algebras with three atoms each and 3 Boolean algebras with four atoms each. One then "identifies" suitably chosen atoms in the manner illustrated in the figure. What comes into existence is a (finite lattice) logic L which has only three- and four-atom Boolean algebras as maximal Boolean subalgebras of L. Now the logic L is exotic since 1. any state on L would mean a state on any maximal Boolean subalgebra of L, and 2. the set of atoms of L



The exotic logic of Greechie

admits two partitions into maximal Boolean subalgebras such that the former partition has 12 classes and the latter has 11 classes. This contradicts the existence of a state on L (for details see [2]).

Let us now make the following observation:

PROPOSITION 2.1. *If L is an exotic logic, then the product of L with the two-point Boolean algebra $\{0, 1\}$ is a nearly exotic logic.*

Proof. Let s be a state on $L \times \{0, 1\}$. Since $(1, 0) \vee (0, 1) = (1, 1)$, we see that $s(1, 0) + s(0, 1) = 1$. If $s(1, 0)$ is positive, then we can easily construct a state on L. Therefore, $s(1, 0) = 0$, and also $s(x, 0) = 0$ for any $x \in L$. Consequently, $s(0, 1) = 1$, and also $s(x, 1) = 1$ for any $x \in L$. We thus forced the necessary values for s. One verifies easily that this evaluation really comes to a state on $L \times \{0, 1\}$. The logic $L \times \{0, 1\}$ has then exactly one state.

COROLLARY 2.1. *There exists a (finite) nearly exotic logic.*

We should mention here that there is another way of constructing a nearly exotic logic (see [7]). Our procedure is considerably simpler.

For the following considerations we need exotic and nearly exotic logics with trivial centers.

PROPOSITION 2.2. *There exist exotic and nearly exotic logics with trivial centers.*

Proof. The Greechie example of exotic logic has obviously the trivial center. If L is a nearly exotic logic, then we obtain a nearly exotic logic with trivial center by taking two copies of L, say L_1 and L_2, forming the disjoint union $L_1 \cup L_2$ and identifying 0, resp. 1, in L_1 with 0, resp. 1, in L_2.

3. Embeddings of logics into the exotic and nearly exotic ones.

DEFINITION 3.1. A mapping $f\colon L_1 \to L_2$ of two logics is called an *embedding* if

(i) $f(0) = 0$;

(ii) $f(a') = (f(a))'$ for any $a \in L_1$;

(iii) $a \leqslant b$ if and only if $f(a) \leqslant f(b)$, $a, b \in L_1$;

(iv) if $a \leqslant b$, then $f(a \vee b) = f(a) \vee f(b)$, $a, b \in L_1$.

A logic L_1 is called a *sublogic* of L_2 if there exists an embedding of L_1 into L_2.

PROPOSITION 3.1. *Any logic can be embedded into an exotic logic with trivial center.*

Proof. Let L_1 be a logic and let M be an exotic logic. Take the logic L which is obtained by forming the disjoint union of L_1 and M and identifying 0, resp. 1, in L_1 with 0, resp. 1, in M. Then L is obviously exotic and L_1 is a sublogic of L.

THEOREM 3.1. *Let L_1 be a logic and let B be a Boolean algebra. Then there exists an exotic logic L such that*

(i) *L_1 is a sublogic of L,*

(ii) *$C(L) = B$.*

Proof. There is a representation of the Boolean algebra B by a collection Δ of subsets of a set I. Thus $B = (I, \Delta)$. Denote further by K the exotic extension of L_1 with $C(K) = \{0, 1\}$ (Proposition 3.1). Put $K_\alpha = K$ for any $\alpha \in I$ and take the logic

$$P = \prod_{\alpha \in I} K_\alpha.$$

The logic L we look for will be a sublogic of P determined by the following requirement. An element $r \in P$ belongs to L if (and only if) there exists a finite partition $\mathscr{P}$ of I, $\mathscr{P} = \{A_i \mid 1 \leqslant i \leqslant n\}$, such that $A_i \in \Delta$ for any i, $i \leqslant n$, and $r_p = r_q$ provided $\{p, q\} \subset A_j$ for an index j, $1 \leqslant j \leqslant n$.

Evidently, $0 \in L$, and if $k \in L$, then $k' \in L$. If $k, h \in L$ and $k \leqslant h$, then $k = h \vee (k \wedge h')$. This follows from the definition of L, because if $\mathscr{P}$ and $\mathscr{R}$ are the partitions corresponding to k and h, then $\mathscr{P} \cap \mathscr{R}$ is the partition corresponding to $k \wedge h'$. Therefore L is a logic.

Further, since $C(K) = \{0, 1\}$ for any $\alpha \in I$, we see that any central element of L must have only the elements 0 and 1 for the coordinates. One checks easily that $k = \{k_\alpha \mid \alpha \in I\}$, where any k_α is either 0 or 1, is an element of L if and only if the set $A = \{\alpha \mid k_\alpha = 1\}$ belongs to Δ. This implies that $C(L) = B$.

Finally, since the mapping $f\colon K \to L$, $f(k) = (k, k, k, \ldots)$, is an embedding of K into L, we see that L is exotic. Indeed, if s is a state on L, then sf is a state on K. But K is exotic. Since L_1 can be embedded in K, we have an embedding of L_1 into L and the proof is complete.

Let us now take up the case of nearly exotic logics. We start again with an auxiliary proposition.

PROPOSITION 3.2. *Let L_1 be a logic and let L_1 have a two-valued state. Then L_1 can be embedded into a nearly exotic logic L with $C(L) = \{0, 1\}$.*

Proof. Embed first L_1 into an exotic logic K. Take then the logic $K \times \{0, 1\}$, which is nearly exotic (Proposition 2.1), and form the disjoint union $L_1 \cup K \times \{0, 1\}$. Determine now the logic L by identifying elements in $L_1 \cup K \times \{0, 1\}$ as follows. Take a two-valued state s on L_1 and identify $k \in L_1$ with $(k, 1) \in K \times \{0, 1\}$ provided $s(k) = 1$; otherwise, identify $k \in L_1$ with $(k, 0) \in K \times \{0, 1\}$. One checks easily that L is a nearly exotic logic containing L_1. (If $C(L) \neq \{0, 1\}$, we form the disjoint union of two L's and obtain the desired logic by identifying 0's and 1's.)

THEOREM 3.2. *Let L_1 be a logic having a two-valued state and let B be a Boolean algebra. Then there is a nearly exotic logic L such that*

(i) *L_1 is a sublogic of L,*

(ii) *$C(L) = B$.*

Proof. By Proposition 3.2, we can embed L_1 into a nearly exotic logic K with $C(K) = \{0, 1\}$. By Proposition 3.1, we can embed K into an exotic logic

M with $C(M) = \{0, 1\}$. Let (I, Δ) be a set representation of B. Choose a point $a \in I$ and put $L_a = K$. Put further $L_b = M$ for any $b \in I - \{a\}$. Set

$$V = \prod_{c \in I} L_c$$

and consider the sublogic L of V whose elements are determined as follows. An element $r \in V$ belongs to L if (and only if) there is a finite partition $\mathscr{P}$ of I, $\mathscr{P} = \{A_i \mid 1 \leqslant i \leqslant n\}$, such that $A_i \in \Delta$ for any i, $1 \leqslant i \leqslant n$, and $r_p = r_q$ provided $\{p, q\} \subset A_j$ for an index j, $1 \leqslant j \leqslant n$.

One can show in the same manner as in Theorem 3.1 that L is indeed a logic and $C(L) = B$. Evidently, L_1 is a sublogic of L – one takes the natural mapping f: $L_1 \to L$, $f(k) = (k, k, k, \ldots)$. It remains to prove that L is nearly exotic. Since K is nearly exotic, it suffices to show that L has as many states as K. Let s be a state on L and let $r \in L$. Then there is a partition $\mathscr{P}$ of I, $\mathscr{P} = \{A_i \mid 1 \leqslant i \leqslant n\}$, such that $r_p = r_q$ whenever $\{p, q\} \subset A_j$ for some j, $1 \leqslant j \leqslant n$. For any i, $1 \leqslant i \leqslant n$, define an element $s^i \in L$ by requiring that $s^i_c = 1$ if $c \in A_i$ and $s^i_c = 0$ if $c \in I - A_i$. Put $r^i = r \wedge s^i$. Then

$$r = \bigvee_{i \leqslant n} r^i$$

and all r^i are mutually orthogonal in L. This implies that

$$s(r) = \sum_{i=1}^{n} s(r^i).$$

Suppose now that $a \in A_{i_0}$, $1 \leqslant i_0 \leqslant n$. If $s(r^i) \neq 0$ for some $i \neq i_0$, then we would obtain a state on a product of exotic logics. This is absurd. Then $s(r) = s(r^{i_0})$, and therefore any state on L is determined by a state on K. Since there is exactly one state on K, the proof is complete.

As a consequence of Theorem 3.2 we infer that any Boolean algebra can be embedded into a nearly exotic logic (with an arbitrary center). This among others rules out any chance for extending states on Boolean sublogics over the entire logic.

4. Appendix on σ-complete logics. A logic L is called *σ-complete* if L is closed under the formation of the upper bounds of the countable mutually orthogonal subsets of L. Let us again call L *exotic* or *nearly exotic* if it has no or exactly one σ-additive state. One can again show that any σ-complete logic can be (σ-completely) embedded into an exotic one and any σ-complete logic with a two-valued state can be embedded into a nearly exotic one. (In the particular case of σ-Boolean algebras, one can show that any σ-Boolean algebra can be embedded into an exotic σ-Boolean algebra, and analogously for nearly exotic σ-Boolean algebras.)

The center of a σ-complete logic is a σ-Boolean algebra and we therefore may meaningfully translate our questions into the σ-complete case. Unfortunately, the constructions of Theorems 3.1 and 3.2 do not always produce a σ-

complete logic. We have not been able to alter the construction to achieve the positive result in general. (The question may be related to the existence of the least σ-product of σ-Boolean algebras – a problem posed as open in the book by Sikorski (see [8], p. 136).)

On the other hand, the construction works, or may be simply altered, for the types of σ-complete logics which have occupied an important place in the axiomatic foundations of quantum theories. One example may be logics with the center equal to a σ-algebra of all subsets of a set (see, e.g., [9]), another example – the logics with all Boolean subalgebras finite (see, e.g., [2], [3], and [7]). We conclude by stating the results for the above classes of logics.

THEOREM 4.1. (i) *Let L_1 be a σ-complete logic and let B be a σ-algebra of all subsets of a set. Then L_1 can be σ-completely embedded into an exotic logic L with $C(L) = B$. If, moreover, L_1 has a two-valued σ-additive state, then L_1 can be σ-completely embedded into a nearly exotic logic L with $C(L) = B$.*

(ii) *Let L_1 be a σ-complete logic with all Boolean sublogics finite. Let B be an arbitrary σ-Boolean algebra. Then L_1 can be σ-completely embedded into an exotic logic L with $C(L) = B$. If, moreover, L_1 and B have a two-valued σ-additive state, then L_1 can be σ-completely embedded into a nearly exotic logic L with $C(L) = B$.*

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