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A semi-global Taylor formula for manifolds

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Abstract. A version of Taylor formula for sections of a vector bundle over a manifold is proved. An application to symbolic calculus of pseudo-differential operators is sketched.

The aim of this paper is to give a version of Taylor formula for sections of a vector bundle over a manifold as well as to sketch an application of this formula in the theory of pseudo-differential operators (ψdo's) on a manifold.

The basic idea here is Hörmander's characterization of a ψdo as a Fourier integral type operator with a linear phase function. A linear phase function is a substitute of a linear function and it may be used to obtain an asymptotic formula similar to Taylor's formula in a linear space ([5]). Such a Taylor type formula may be used to construct a symbolic calculus of ψdo's, i.e., a homomorphism of the ψdo algebra to an algebra of functions on the cotangent bundle. Widom proved ([4], [5]) that all the properties of the linear phase function, essential in ψdo theory may be formulated in terms of fixed linear connections on the manifold and on the vector bundles between which the operators are acting.

Drager ([1]) showed that any linear connection gives rise to a phase function, roughly speaking, by inversion of the exponential map of the connection.

In this paper a construction of a Taylor formula is given for the Drager linear phase function following Widom. This formula is then applied to derive a formula for the symbol of superposition of ψdo's.

Let M be a paracompact C^{∞}-manifold with a fixed symmetric linear connection given by a covariant derivative ∇ on the tangent bundle TM.

Such a connection determines on M the geodesics, the normal coordinates, and the parallel translation ([2]).

An open set $O_M \subset M \times M$ containing the diagonal is called a *symmetric convex normal* (shortly scn) *neighbourhood* of the diagonal provided

(i) $(x, y) \in O_M$ implies $(y, x) \in O_M$ (symmetry).

(ii) If $(x, y) = O_M$, then there exists a unique geodesic $\gamma_{x,y}$ joining x to y in O_M, in particular $\gamma_{x,y}(t) = \gamma_{y,x}(1-t)$, $t \in [0, 1]$ (geodesic convexity).

(iii) The mapping $W(\cdot, x)\colon y \mapsto \dot{\gamma}_{x,y}(0) \in T_x M$ defines (normal) coordinates in $O_M(x) = \{y\colon (x, y) \in O_M\}$ (normality).

THEOREM 0 (Drager [1]). *For any manifold with a linear connection there exists a scn neighbourhood of the diagonal.*

The proof can be easily deduced from the standard construction of the normal coordinates.

In the following O_M will be a fixed scn neighbourhood of the diagonal. We also fix a function $\chi \in C^\infty(M \times M)$ with support in O_M, equal to 1 in some neighbourhood of the diagonal and satisfying $\chi(x, y) = \chi(y, x)$.

The function $W\colon O_M \to TM$ appearing in the definition of O_M is characterized by the conditions

$$W(y, x) = \dot{\gamma}_{x,y}(0) \in T_x M,$$

$$\gamma_{x,y}\colon [0, 1] \to M, \quad \gamma_{x,y}(0) = x, \quad \gamma_{x,y}(1) = y, \quad \nabla_{\dot{\gamma}} \dot{\gamma} = 0.$$

LEMMA 1.

$$W(\gamma_{x,y}(t_1), \gamma_{x,y}(t_0)) = (t_1 - t_0)\tau_{x,\gamma_{x,y}}(t_0)\, W(y, x)), \qquad t_0, t_1 \in [0, 1],$$

where $\tau_{x,z}$ means the parallel translation $T_x M \to T_z M$ along the geodesic $\gamma_{x,z}$.

Proof. Since O_M is geodesically convex, $\gamma_{x,y}(1-t) = \gamma_{y,x}(t)$ and $\dot{\gamma}_{x,y}(1) = -\dot{\gamma}_{y,x}(0) = -W(x, y)$.

Thus

$$W(x, y) = \dot{\gamma}_{y,x}(0) = -\dot{\gamma}_{x,y}(1) = -\tau_{x,y}(\dot{\gamma}_{x,y}(0)) = -\tau_{x,y}(W(y, x)).$$

Let $\tilde{x} = \gamma_{x,y}(t_0)$, $\tilde{y} = \gamma_{x,y}(t_1)$, $t_0 < t_1$. Then

$$\gamma_{\tilde{x},\tilde{y}}(t) = \gamma_{x,y}(t_0 + t(t_1 - t_0)).$$

Hence

$$W(\tilde{y}, \tilde{x}) = \dot{\gamma}_{\tilde{x},\tilde{y}}(0) = (t_1 - t_0)\dot{\gamma}_{x,y}(t_0) = \frac{t_1 - t_0}{t_0}\dot{\tilde{\gamma}}_{x,y}(1),$$

where $\tilde{\gamma}_{x,y}(t) = \gamma_{x,y}(t_0 t)$. But

$$\dot{\tilde{\gamma}}_{x,y}(1) = \tau_{x,\gamma_{x,y}}(1)(\dot{\tilde{\gamma}}_{x,y}(0)) = \tau_{x,\tilde{x}}(t_0\dot{\gamma}_{x,y}(0)) = t_0\tau_{x,\tilde{x}} W(y, x)$$

and so $W(\tilde{y}, \tilde{x}) = (t_1 - t_0)\tau_{x,\tilde{x}} W(y, x)$. ∎

Now, let E be a fixed vector bundle over M, with a fixed linear connection. The corresponding covariant derivative will be denoted by ∇. As usual, $\Gamma(E)$ denotes the space of C^∞-sections of E.

LEMMA 2. *If γ is a geodesic and $u \in \Gamma(E)$, then*

$$\nabla^k u \dot{\gamma}^k(t) = \nabla^k_{\dot{\gamma}(t)} u = (\nabla/dt)^k (u \circ \gamma),$$

where ∇/dt denotes the covariant derivative along γ and

$$\dot{\gamma}^k(t) = \dot{\gamma}(t) \otimes \ldots \otimes \dot{\gamma}(t) \quad (k\text{-fold tensor product}).$$

Proof. The second equality being the definition of ∇/dt, we only have to prove the first. This may be done by induction on k. According to the definition of the k-th covariant derivative we have

$$\begin{aligned}\nabla^k u \dot{\gamma}^k(t) &= \nabla_{\dot{\gamma}(t)}\big(\nabla^{k-1} u \dot{\gamma}^{k-1}(t)\big) - \sum \nabla^{k-1} u\big(\dot{\gamma}(t) \otimes \ldots \otimes \nabla_{\dot{\gamma}(t)} \dot{\gamma} \otimes \ldots \otimes \dot{\gamma}(t)\big)\\ &= \nabla_{\dot{\gamma}(t)}\big(\nabla^{k-1} u \dot{\gamma}^{k-1}(t)\big)\end{aligned}$$

since $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ for γ a geodesic. ∎

LEMMA 3. *Define for $u \in \Gamma(E)$*

$$\tilde{u}_k(y) = \tau_{x,y} \nabla^k u(x) \cdot W^k(y, x), \qquad y \in O_M(x),$$

where $W^k(y, x) = W(y, x) \otimes \ldots \otimes W(y, x)$ (k-fold product). Let $(x, y) \in O_M$ be fixed and let $\gamma = \gamma_{x,y}$ be the geodesic joining x to y. Then

$$\nabla^l_{\dot{\gamma}(t)} \tilde{u}_k = \begin{cases} \dfrac{k!}{(k-l)!} t^{k-l} \tau_{x,\gamma(t)}\big(\nabla^k u(x) \cdot W^k(y, x)\big) & \text{for } l \leqslant k,\\ 0 & \text{for } l > k.\end{cases}$$

Proof. By Lemma 1

$$\tilde{u}_k(\gamma(t)) = t^k \tau_{x,\gamma(t)}\big(\nabla^k u(x) \cdot W^k(y, x)\big) = t^k \tau_{y,\gamma(t)} \tilde{u}_k(y).$$

The function $t \mapsto \tau_{y,\gamma(t)} \tilde{u}_k(y)$ is a parallel section along γ and so ∇/dt vanishes on it.

Hence by Lemma 2

$$\begin{aligned}\nabla^l_{\dot{\gamma}(t)} \tilde{u}_k &= (\nabla/dt)^l \tilde{u}_k \circ \gamma = (d/dt)^l (t^k) \tau_{y,\gamma(t)} \tilde{u}_k(y)\\ &= \begin{cases} \dfrac{k!}{(k-l)!} t^{k-l} \tau_{y,\gamma(t)} \tilde{u}_k(y), & l \leqslant k,\\ 0, & l > k.\end{cases}\end{aligned}$$ ∎

COROLLARY 1.

$$\nabla^l_{\dot{\gamma}(0)} \tilde{u}_k = \begin{cases} k!\, \nabla^k_{\dot{\gamma}(0)} u & \text{for } l = k,\\ 0 & \text{for } l \neq k.\end{cases}$$

Remark 1. Since $W(y, x) = \dot{\gamma}_{x,y}(0)$, we have by Lemma 2

$$\nabla^k u \cdot W^k(y, x) = \nabla^k_{W(y,x)} u.$$

In the following $S\nabla^k$ will denote the superposition of the k-th covariant derivative with the full symmetrization.

THEOREM 1. *Let* $u \in \Gamma(E)$, $x, y \in M$ *and define*

$$r_k(y) = u(y) - \chi(x, y) \sum_{m=0}^{k} \frac{1}{m!} \tau_{x,y}\left(\nabla^m u(x) \cdot W^m(y, x)\right).$$

Then $S\nabla^l r_k(x) = 0$ *for* $l \leqslant k$.

Proof. It is sufficient to show that $\nabla^l r_k(x)$ vanishes for all tangent l-vectors of the form

$$X \otimes \ldots \otimes X \quad (l\text{-fold product}), \ X \in T_x M.$$

Let $X = \dot{\gamma}_{x,y}(0)$ (any vector εX can be written in such a way for ε small enough).

By Lemma 2 we have to prove that $\nabla_X^l r_k(x) = 0$, and this follows from Corollary 1:

$$\nabla_X^l r_k(x) = \nabla_X^l u(x) - \chi(x, x) \nabla_X^l \left(\sum_{m=0}^{k} \frac{1}{m!} \tilde{u}_k\right)(x)$$
$$= \nabla_X^l u(x) - \nabla_X^l u(x) = 0 \quad \text{for } l \leqslant k.$$

Remark 2. It is easy to see that the vanishing of $S\nabla^l r_k(x)$ for $l \leqslant k$ is equivalent to the vanishing of the partial derivatives of r_k up to order k.

COROLLARY 2 (a semi-global Taylor formula).

$$u(y) = \chi(x, y) \sum_{m=0}^{k} \frac{1}{m!} \tau_{x,y}\left(\nabla^m u(x) W^m(y, x)\right) + r_k(y)$$

with $S\nabla^l r_k(x) = 0$ *for* $l \leqslant k$.

Now, following Widom ([1], [4], [5]) we will give an application of the Taylor formula to the symbolic calculus of ψdo. We will be interested mainly in obtaining a formula for the symbol of the superposition of two operators. Our calculations will be purely formal. All the details can be found in the papers cited above.

We also follow Shubin's ([3]) notation and terminology for the ψdo theory.

Let us define

$$\pi_2 \colon O_M \to M, \quad \pi_2(x, y) = y,$$
$$\varphi \colon \pi_2^* T^* M \to \boldsymbol{R}, \quad \varphi(y, x, \xi) = \xi \cdot W(y, x), \quad \xi \in T_x^* M,$$
$$s_e \colon O_M(x) \to E, \quad s_e(y) = \tau_{x,y} e \quad \text{for } e \in E_x.$$

Let $A \colon \Gamma(E) \to \Gamma(F)$ be a ψdo (E and F are vector bundles over M).

DEFINITION. The *symbol* of A is the mapping $\sigma(A)\colon T^*M \to L(\pi^*E, \pi^*F)$, $\pi\colon T^*M \to M$ being the canonical projection, defined as follows:

$$\sigma(A)(x, \xi)e = A\big(\chi(\cdot, x)s_e(\cdot)e^{i\varphi(\cdot,x,\xi)}\big)(x),$$
$$x \in M,\ \xi \in T_x^*M,\ e \in E_x = (\pi^*E)_\xi.$$

It can be proved ([4]) (or taken for a definition) that if A has order $\leqslant m$, then $\sigma(A) \in S^m(T^*M, L(E, F))$ and that the correspondence $A \mapsto \sigma(A)$ provides an isomorphism $\Psi\mathrm{DO}^m/\Psi\mathrm{DO}^{-\infty} \to S^m/S^{-\infty}$ independent of the choice of χ.

In the following, equality in $S^m/S^{-\infty}$ will be denoted by $\equiv$.

For a fixed $x \in M$, $\sigma(A)|\ T_x^*M$ is a smooth mapping of linear spaces $T_x^*M \to L(E_x, F_x)$. The k-th Fréchet derivative of this mapping at $\xi \in T_x^*M$ will be denoted by $D^k\sigma(A)(x, \xi)$. $D^k\sigma(A)(x, \xi)$ is a symmetric k-linear mapping of T_x^*M to $L(E_x, F_x)$ and will be treated as an element of $\hat{\otimes}^k T_x M \otimes L(E_x, F_x)$, where $\hat{\otimes}$ denotes the symmetric tensor product. Thus $D^k\sigma(A)$ belongs to $\hat{\otimes}^k TM \otimes L(E, F)$ and it can be contracted with $\nabla^k u$, $u \in \Gamma(E)$; the result of such a contraction, denoted by $D^k\sigma(A)\cdot\nabla^k u$, belongs to $\Gamma(F)$. From the definition of the symbol it follows easily that

$$D^k\sigma(A)(x, \xi)e = i^k A\big(\chi(\cdot, x)s_e(\cdot)e^{i\varphi(\cdot,x,\xi)}W^k(\cdot, x)\big)(x).$$

LEMMA 4. *Let* $u \in \Gamma(E)$, $h \in C^\infty(M \times M)$, $h\chi = \chi$. *Then*

$$A\big(h(\cdot, x)e^{i\varphi(\cdot,x,\xi)}u(\cdot)\big)(x) \equiv \sum_{k=0}^{\infty} \frac{i^{-k}}{k!} D^k\sigma(A)\nabla^k u(x).$$

Proof. Using the Taylor formula we get

$$\begin{aligned} &A\big(h(\cdot, x)e^{i\varphi(\cdot,x,\xi)}u(\cdot)\big)(x)\\ &\qquad \equiv \sum \frac{1}{k!} A\Big(h(\cdot, x)\chi(\cdot, x)e^{i\varphi(\cdot,x,\xi)}\tau_{x,(\cdot)}\big(\nabla^k u(x)\cdot W^k(\cdot, x)\big)\Big)(x)\\ &\qquad = \sum \frac{i^{-k}}{k!} D^k\Big(A\big(\chi(\cdot, x)e^{i\varphi(\cdot,x,\xi)}\tau_{x,(\cdot)}\nabla^k u(x)\big)(x)\Big)\\ &\qquad = \sum \frac{i^{-k}}{k!} D^k\sigma(A)(x, \xi)\nabla^k u(x). \quad \blacksquare \end{aligned}$$

The next important lemma is taken from Drager ([1]); its proof is easy.

LEMMA 5 (Drager [1]). *Let* $x = (x_1, \ldots, x_n)$ *be a normal map at* x_0, *let* $e \in E_{x_0}$, $u(x) = \tau_{x_0,x}e$ *and* $f \in C^\infty(M)$. *Then*

$$S\nabla^k(fu)(x_0)\left(\frac{\partial}{\partial x_{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x_{i_k}}\right) = \frac{\partial^k f}{\partial x_{i_1} \ldots \partial x_{i_k}}(x_0)e.$$

LEMMA 6 (Widom [5]). *Let* $\psi \in C^\infty(M)$, $x \in M$, $d\psi(x) \neq 0$, $e \in E_x$. *Then*

$$e^{-i\psi(x)} A(\chi(\cdot, x) e^{i\psi(\cdot)} s_e(\cdot))(x) \equiv \sigma(A)(d\psi(x)) +$$
$$+ \sum_{m_1,\dots,m_k \geq 2} \frac{i^{k-(m_1+\dots+m_k)}}{k!\, m_1! \dots m_k!} D^{m_1+\dots+m_k} \sigma(A)(d\psi(x)) \nabla^{m_1}\psi(x) \otimes \dots \otimes \nabla^{m_k}\psi(x) e.$$

Proof. By the Taylor formula

$$\psi(y) = \psi(x) + \tau_{x,y}(\nabla\psi \cdot W(y, x)) + \psi_1(y, x) \quad \text{for } y \in O_M(x)$$

with

$$\nabla^k \psi_1(\cdot, x)(x) = \begin{cases} 0, & k \leq 1, \\ \nabla^k \psi(x), & k > 1. \end{cases}$$

Writing $\xi = d\psi(x)$ we get

$$\varphi(y, x, \xi) = d\psi(x) \cdot W(y, x) = \tau_{x,y}(d\psi\, W(y, x)) = \tau_{x,y}(\nabla\psi\, W(y, x)),$$
$$e^{-i\psi(x)} A(\chi(\cdot, x) e^{i\psi(\cdot)} s_e(\cdot))(x) = A(\chi(\cdot, x) e^{i\varphi(\cdot, x, \xi)} e^{i\psi_1(\cdot, x)} s_e(\cdot))(x)$$
$$= \sum_{m=0}^{\infty} \frac{i^{-m}}{m!} D^m \sigma(A)(x, \xi) \nabla^m (e^{i\psi_1(\cdot, x)} s_e(\cdot))(x)$$
$$= \sum_{m=0}^{\infty} \frac{i^{-m}}{m!} D^m \sigma(A)(x, \xi) S\nabla^m (e^{i\psi_1(\cdot, x)} s_e(\cdot))(x).$$

The last equality follows from the symmetry of $D^m \sigma(A)(x, \xi)$. To compute $S\nabla^m(e^{i\psi_1(\cdot, x)} s_e(\cdot))(x)$ we use the power series of exp, Lemma 5 and Leibniz formula.

We get

$$S\nabla^m(e^{i\psi_1(\cdot, x)} s_e(\cdot))(x) = \sum_{k=0}^{\infty} \frac{i^k}{k!} S\nabla^m(\psi_1^k(\cdot, x) s_e(\cdot))(x)$$
$$= \sum_{m_1+\dots+m_k=m} \frac{i^k m!}{k!\, m_1! \dots m_k!} S\nabla^{m_1}\psi_1(\cdot, x)(x) \hat{\otimes} \dots \hat{\otimes} S\nabla^{m_k}\psi_1(\cdot, x)(x) e$$
$$= \sum_{\substack{m_1+\dots+m_k=m \\ m_1,\dots,m_k \geq 2}} \frac{i^k m!}{k!\, m_1! \dots m_k!} S\nabla^{m_1}\psi(x) \hat{\otimes} \dots \hat{\otimes} S\nabla^{m_k}\psi(x) e$$
$$= \sum_{\substack{m_1+\dots+m_k=m \\ m_1,\dots,m_k \geq 2}} \frac{i^k m!}{k!\, m_1! \dots m_k!} S(\nabla^{m_1}\psi(x) \otimes \dots \otimes \nabla^{m_k}\psi(x)) e.$$

Substituting the last formula into the previous one and using the symmetry of $D^m \sigma(A)(x, \xi)$ we obtain the required result. ∎

Let $A\colon \Gamma(E)\to\Gamma(F)$, $B\colon \Gamma(F)\to\Gamma(G)$ be properly supported ψdo's. Then the superposition $BA\colon \Gamma(E)\to\Gamma(G)$ is defined.

THEOREM 2.

$$\sigma(BA)(x,\xi)=\sum_{\substack{m_1,\dots,m_p\geq 2\\ k_1,\dots,k_p\geq 1\\ k_0,p\geq 0}}\frac{i^{p-(k_0+\dots+k_p+m_1+\dots+m_p)}}{p!\,k_0!\dots k_p!\,m_1!\dots m_p!}D^{k_0+\dots+k_p}\sigma(B)(x,\xi)\times$$

$$\times\nabla^{k_0}D^{m_1+\dots+m_p}\sigma(A)(x,\xi)\,\nabla^{k_1+m_1}\varphi(\cdot,x,\xi)(x)\otimes\dots\otimes\nabla^{k_p+m_p}\varphi(\cdot,x,\xi)(x),$$

where $\nabla^k D^m\sigma(A)(x,\xi)=\nabla^k D^m\sigma(A)\big(d\varphi(\cdot,x,\xi)\big)(x)$.

Proof. Let $h\chi=\chi$, $x\in M$, $\xi\in T_x^*M$, $e\in E_x$. By Lemma 4 we have

$$\begin{aligned}\sigma(BA)(x,\xi)e&=BA\big(\chi(\cdot,x)e^{i\varphi(\cdot,x,\xi)}s_e(\cdot)\big)(x)\\&\equiv B_z\big(h(z,x)e^{-i\varphi(z,x,\xi)}A_y\big(\chi(y,x)e^{i\varphi(y,x,\xi)}s_e(y)\big)(z)\,e^{i\varphi(z,x,\xi)}\big)(x)\\&=B_z\big(h(z,x)u(z,x,\xi)e^{i\varphi(z,x,\xi)}\big)(x)\\&\equiv\sum_{k=0}^{\infty}\frac{i^{-k}}{k!}D^k\sigma(B)(x,\xi)\,\nabla^k u(\cdot,x,\xi)(x),\end{aligned}$$

where

$$\begin{aligned}u(z,x,\xi)&=e^{-i\varphi(z,x,\xi)}A\big(\chi(\cdot,x)e^{i\varphi(\cdot,x,\xi)}s_e(\cdot)\big)(z)\\&=A\big(\chi(\cdot,x)e^{i[\varphi(\cdot,x,\xi)-\varphi(z,x,\xi)]}s_e(\cdot)\big)(z).\end{aligned}$$

Now we may use Lemma 6 with

$$\psi(y)=\varphi(y,x,\xi)-\varphi(z,x,\xi).$$

Then

$$d\psi(z)=d\varphi(\cdot,x,\xi)(z),\qquad S\nabla^m\psi(z)=S\nabla^m\varphi(\cdot,x,\xi)(z),$$

hence

$$\begin{aligned}u(z,x,\xi)&=\sigma(A)\big(d\varphi(\cdot,x,\xi)(z)\big)e+\\&\quad+\sum_{m_1,\dots,m_p\geq 2}\frac{i^{p-(m_1+\dots+m_p)}}{p!\,m_1!\dots m_p!}D^{m_1+\dots+m_p}\sigma(A)\big(d\varphi(\cdot,x,\xi)\big)(z)\times\\&\quad\times S\nabla^{m_1}\varphi(\cdot,x,\xi)(z)\,\hat{\otimes}\dots\hat{\otimes}\,S\nabla^{m_p}\varphi(\cdot,x,\xi)(z)e.\end{aligned}$$

Next we apply Leibniz formula for the symmetric covariant derivative

$$S\nabla^r(u_1\,\hat{\otimes}\dots\hat{\otimes}\,u_l)=\sum_{r_1+\dots+r_l=r}\frac{r!}{r_1!\dots r_l!}S\nabla^{r_1}u_1\,\hat{\otimes}\dots\hat{\otimes}\,S\nabla^{r_l}u_l,$$

to obtain

$$\sigma(BA)(x,\xi)e \equiv \sum \frac{i^{p-(k_0+\ldots+k_p+m_1+\ldots+m_p)}}{p!\,k_0!\ldots k_p!\,m_1!\ldots m_p!} D^{k_0+\ldots+k_p}\sigma(B)(x,\xi)\times$$
$$\times S\nabla^{k_0} D^{m_1+\ldots+m_p}\sigma(A)(x,\xi) S\nabla^{k_1} S\nabla^{m_1}\varphi(\cdot,x,\xi)(x)\,\hat{\otimes}\ldots$$
$$\ldots\hat{\otimes}\, S\nabla^{k_p} S\nabla^{m_p}\varphi(\cdot,x,\xi)(x)e,$$

where $S\nabla^k D^m\sigma(A) = S\nabla^k D^m\sigma(A)\big(d\varphi(\cdot,x,\xi)\big)(x)$.

The symmetry of the tensors yields the asserted formula. ■

Remark 3. One may prove easily that $\nabla^k\varphi(\cdot,x,\xi)(x)$ only depends upon ξ and the curvature tensor ([5]).

References

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