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On superposition of quasianalytic functions

by W. PLEŚNIAK (Kraków)

Let E be a compact set in the space C^n of n complex variable-$z = (z_1, \ldots, z_n)$. Let $\mathscr{C}(E)$ denote the Banach algebra of all complex funcs tions continuous in E with the norm

$$\|f\|_E = \max_{z \in E} |f(z)|, \quad f \in \mathscr{C}(E).$$

Denote by $\mathscr{E}_\nu(f, E)$ the ν-th measure of the best Čebyšev approximation to $f \in \mathscr{C}(E)$ on E by polynomials in z, i.e.

$$\mathscr{E}_\nu(f, E) = \inf \|f - P_\nu\|_E,$$

where inf is spread over all the polynomials P_ν of degree $\leqslant \nu$. Let $\mathscr{B}(E)$ denote the subset of $\mathscr{C}(E)$ consisting of all the functions satisfying the following condition:

$$\liminf_{\nu \to \infty} \sqrt[\nu]{\mathscr{E}_\nu(f, E)} < 1.$$

The functions of $\mathscr{B}(E)$ are called *quasianalytic* on E in Bernstein's sense. In the case $n = 1$, if E is a compact interval of the real axis R, the basic properties of functions $f \in \mathscr{B}(E)$ may be found in [1] or [6]. The term "quasianalytic" arises from the following identity principle given by Bernstein:

If E and I are compact intervals in R and $I \subset E$, then every function $f \in \mathscr{B}(E)$ vanishing on I is identically equal to zero.

In the case $n \geqslant 1$, a generalization of this result has been given in [4].

Let $\{\nu_k\}$ be an increasing sequence of positive integers. Denote by $[\{\nu_k\}]$ the set of all increasing sequences $\{\mu_k\}$ of positive integers such that $1/M < \mu_k/\nu_k < M$ for $k \geqslant 1$, M being a positive constant dependent on $\{\mu_k\}$. Denote by $\mathscr{B}(E, \{\nu_k\})$ the set of all functions $f \in \mathscr{B}(E)$ such that

$$\lim_{k \to \infty} \sqrt[\nu_k]{\mathscr{E}_{\nu_k}(f, E)} < 1$$

and define $\mathscr{B}(E, [\{\nu_k\}]) = \{f \in \mathscr{B}(E, \{\mu_k\}) \colon \{\mu_k\} \in [\{\nu_k\}]\}$. One can check that $\mathscr{B}(E, [\{\nu_k\}])$ is a ring with respect to the ordinary pointwise addition and multiplication of functions. If E satisfies the assumptions of the identity theorem in [4], then the ring $\mathscr{B}(E, [\{\nu_k\}])$ is a domain of integrity.

The main purpose of this paper is to answer in the negative the following natural question:

(I) Let $f \in \mathscr{B}(E, [\{\nu_k\}])$ and $g \in \mathscr{B}(f(E), [\{\nu_k\}])$. Does this imply that $g \circ f \in \mathscr{B}(E, [\{\nu_k\}])$?

It follows from a result of Siciak (cf. Lemma 1) that every function φ holomorphic in a neighbourhood of a polynomially convex compact set F in C^n is a member of $\mathscr{B}(F, \{\nu_k\})$ for any $\{\nu_k\}$. Hence, in order to answer question (I) we may consider the following problem:

(II) Let $f \in \mathscr{B}(E, [\{\nu_k\}])$ and let φ be a holomorphic function in a neighbourhood of $f(E)$. What conditions are necessary and sufficient for $\varphi \circ f \in \mathscr{B}(E, [\{\nu_k\}])$?

If E is polynomially convex, we give some sufficient condition for (II). It is also a necessary one if φ is rational and the extremal function $\Phi(z, E)$ (introduced by J. Siciak in [5]) is continuous in E (see Theorem 1). On the other hand, given $f \in \mathscr{B}(E, [\{\nu_k\}])$, we may always assume that E is polynomially convex (see Lemma 3). Hence, Counter-examples 1, 2 and 3 give us a negative reply to (I).

Nevertheless, in accordance with Examples 4 and 5, there exist functions $f \in \mathscr{B}(E, [\{\nu_k\}])$ and $g \in \mathscr{B}(f(E), [\{\nu_k\}])$, $g(w) \not\equiv w$, such that $g \circ f \in \mathscr{B}(E, [\{\nu_k\}])$.

Finally, we note that a simple characterization of *essentially quasi-analytic* functions on E (i.e. functions not continuable to holomorphic functions in any neighbourhood of E) is given by Lemma 2. In the case $n = 1$, because of the Montel theorem, that lemma can be formulated as follows.

A complex function f defined and bounded on a compact set E is the restriction to E of a function F holomorphic in a neighbourhood of the polynomially convex envelope $\hat{E}$ of E if and only if there exist polynomials $\{P_k\}$, an open set U, $E \subset U$, and at least two distinct points $a, b \in C$ such that

$$\lim_{k\to\infty} \|f - P_k\|_E = 0$$

and

$$P_k(U) \subset C \setminus \{a, b\}, \quad k \geqslant 1.$$

We start from some lemmas. The first one is a slight modification of the well-known result of Siciak [5].

LEMMA 1. *Let $\{f_k\}$ be a sequence of bounded holomorphic functions in an open set Ω in C^n. Write $M_k = \sup_{z \in \Omega} |f_k(z)|$. Then for every polynomially* *convex compact set E, $E \subset \Omega$, there exist positive constants M and ϱ, $\varrho \in (0, 1)$, independent of k and such that*

$$\mathscr{E}_\nu(f_k, E) \leqslant M M_k \varrho^\nu, \quad \nu \geqslant 1, k \geqslant 1.$$

Proof. Fix a polynomially convex compact set E, $E \subset \Omega$, and a number $R > 1$. It follows from the definition of the polynomial convexity and from the Borel–Lebesgue theorem that there exist polynomials $P_1, \ldots, P_m$ such that $\|P_j\|_E \leqslant 1$ $(j = 1, \ldots, m)$ and

$$E \subset \operatorname{int} L \subset L = \{z \in C^n\colon |P_j(z)| \leqslant R, \ j = 1, \ldots, m\} \subset \Omega.$$

Put $g_k = f_k/M_k$. Applying the Weil integral formula in L, by the same reasoning as in [5], p. 345, for every g_k we find a sequence of polynomials $\{Q_\nu^k\}_{\nu \geqslant 1}$, $\deg Q_\nu^k \leqslant \nu$, such that

$$\|g_k - Q_\nu^k\|_E \leqslant M \varrho^\nu, \quad \nu \geqslant 1,$$

where the constants M and ϱ are independent of k, $\varrho \in (0, 1)$. Hence putting $R_\nu^k = M_k Q_\nu^k$ gives

$$\|f_k - R_\nu^k\|_E \leqslant M M_k \varrho^\nu, \quad \nu \geqslant 1,$$

for $k = 1, 2, \ldots$ The proof is completed.

Given a compact set E in C^n, we shall denote by $\hat{E}$ the polynomially convex envelope of E. By Lemma 1 (for $f_k = f$) one can easily prove the following

LEMMA 2. *Let f be a complex function defined and bounded on E. A necessary and sufficient condition that f be the restriction to E of a function $\tilde{f}$ holomorphic in a neighbourhood of $\hat{E}$ is that there exist polynomials $\{P_k\}$ and an open set U, $\hat{E} \subset U$, such that*

$$\lim_{k \to \infty} \|f - P_k\|_E = 0$$

and the sequence $\{P_k\}$ forms a normal family in U.

LEMMA 3. *Suppose that $f \in \mathscr{B}(E, \{\nu_k\})$. Then there exist a function $\tilde{f} \in \mathscr{B}(\hat{E}, \{\nu_k\})$ such that $\tilde{f}|_E = f$.*

Proof. Take polynomials $\{P_{\nu_k}\}$, $\deg P_{\nu_k} \leqslant \nu_k$, such that

$$\|f - P_{\nu_k}\|_E \leqslant M \varrho^{\nu_k}, \quad k \geqslant 1, \tag{1}$$

where M and ϱ are constants independent of k, $\varrho \in (0, 1)$. Because of (1) and the triangle inequality, the function f can be expanded into the series

$$f(z) = P_{\nu_1}(z) + \sum_{k=1}^{\infty} [P_{\nu_{k+1}}(z) - P_{\nu_k}(z)] \tag{2}$$

convergent uniformly in E. By the definition of $\hat{E}$ series (2) is uniformly convergent in $\hat{E}$ to a function $\tilde{f}$. Moreover,

$$\|\tilde{f}-P_{\nu_k}\|_{\hat{E}} = \Big\| \sum_{l=k}^{\infty} (P_{\nu_{l+1}}-P_{\nu_l}) \Big\|_{\hat{E}} \leqslant \sum_{l=k}^{\infty} \|P_{\nu_{l+1}}-P_{\nu_l}\|_E$$

$$\leqslant M \sum_{l=k}^{\infty} (\varrho^{\nu_{l+1}}+\varrho^{\nu_l}) \leqslant \frac{2M}{1-\varrho}\varrho^{\nu_k}.$$

This implies that $\tilde{f}\in \mathscr{B}(\hat{E}, \{\nu_k\})$ as asserted.

LEMMA 4. *Let E be a compact set in C^n and let $\mathscr{F}$ be a family of polynomials satisfying the following conditions*:

(i) $|f(z)| \geqslant m > 0$, $z\in E$, $f\in \mathscr{F}$,

(ii) *there exists an open set U in C^n such that $E\subset U$ and $f(z)\neq 0$ for $z\in U$, $f\in\mathscr{F}$.*

Then, for every $\omega > 1$ there exists an open set V in C^n such that $E\subset V$ and

$$|f(z)|\,\omega^{\deg f} \geqslant m, \quad z\in V, \quad f\in\mathscr{F} \text{ (}^1\text{)}.$$

Proof. Fix a number $\omega > 1$ and a point $a = (a_1, \dots, a_n)\in E$ and put $\theta = \omega^{-1/n}$. Given a polynomial $f\in\mathscr{F}$, we write

$$g(z_1) = f(z_1, a_2, \dots, a_n) = \beta(z_1-\alpha_1)\dots(z_1-\alpha_l), \tag{1}$$

where the numbers β, α_j $(j = 1, \dots, l)$ may depend on f and on the point a, $0\leqslant l\leqslant \deg f$. By assumption (i) we obtain

$$|\beta| \geqslant \frac{m}{|a_1-\alpha_1|\dots|a_1-\alpha_l|}. \tag{2}$$

Hence

$$|g(z_1)| \geqslant m\left|\frac{z_1-\alpha_1}{a_1-\alpha_1}\right|\dots\left|\frac{z_1-\alpha_l}{a_1-\alpha_l}\right|. \tag{3}$$

Take $\delta = \operatorname{dist}(E, C^n\setminus U)/2\sqrt{n}$. By (ii), $f(z)\neq 0$ for $z\in P(a,\delta) = K(a_1,\delta)\times\dots\times K(a_n,\delta)$, where $K(a_k,\delta) = \{z_k\in C: |z_k-a_k|\leqslant\delta\}$. Hence $|a_1-\alpha_j| > \delta$ for $j = 1, \dots, l$. So, setting $\delta_\omega = (1-\theta)\delta$ gives

$$\left|\frac{z_1-\alpha_j}{a_1-\alpha_j}\right| \geqslant 1-\left|\frac{a_1-z_1}{a_1-\alpha_j}\right| \geqslant 1-\frac{\delta_\omega}{\delta} = \theta \quad \text{for } z_1\in K(a_1,\delta_\omega). \tag{4}$$

Hence by (1) and (3) we obtain

$$|f(z_1, a_2, \dots, a_n)| \geqslant m\theta^l \geqslant m\theta^{\deg f} \quad \text{for } z_1\in K(a_1,\delta_\omega). \tag{5}$$

([1]) If $n = 1$ and E is connected, this lemma is due to Leja [2]. He proved it by means of his well-known Polynomial Lemma. Our proof is a direct one.

By induction, in order to complete the proof it is enough to show that:

if

(6) $|f(z)| \geqslant m\theta^{k\deg f}$

$$\text{for } z \in K(a_1, \delta_\omega) \times \ldots \times K(a_k, \delta_\omega) \times a_{k+1} \times \ldots \times a_n,$$

then

$$|f(z)| \geqslant m\theta^{(k+1)\deg f}$$

$$\text{for } z \in K(a_1, \delta_\omega) \times \ldots \times K(a_{k+1}, \delta_\omega) \times a_{k+2} \times \ldots \times a_n.$$

To this end fix a point $b^k = (b_1, \ldots, b_k) \in K(a_1, \delta_\omega) \times \ldots \times K(a_k, \delta_\omega)$. Then the polynomial of one variable z_{k+1}

$$g(z_{k+1}) = f(b^k, z_{k+1}, a_{k+2}, \ldots, a_n)$$

may be written in the form

$$g(z_{k+1}) = \beta(z_{k+1} - \alpha_1) \ldots (z_{k+1} - \alpha_l),$$

where β, α_j $(j = 1, \ldots, l)$ are numbers dependent on f, b^k and $a_{k+2}, \ldots$ $\ldots, a_n$, $0 \leqslant l \leqslant \deg f$. By the same reasoning as in the proof of (5) one can show that

$$|g(z_{k+1})| \geqslant m\theta^{(k+1)\deg f} \quad \text{for } z_{k+1} \in K(a_{k+1}, \delta_\omega)$$

independently of the choice of $b^k \in K(a_1, \delta_\omega) \times \ldots \times K(a_k, \delta_\omega)$, which ends the proof of (6). By (5) and (6) we obtain

$$|f(z)|\,\omega^{\deg f} \geqslant m \quad \text{for } z \in P(a, \delta_\omega).$$

Since δ_ω does not depend on the choice of the point $a \in E$ and $f \in \mathcal{F}$, we get the assertion of the lemma with $V = \bigcup_{a \in E} \operatorname{int} P(a, \delta_\omega)$.

LEMMA 5. *Let f be a complex function defined and bounded on a compact set E in C^n. Let φ be a function holomorphic in a neighbourhood Ω of the set $F = f(E)$. If for a sequence of polynomials $\{P_k\}$*

$$\lim_{k \to \infty} \|f - P_k\|_E = 0,$$

then there exist constants $M > 0$ and $K_0 > 0$ such that

$$|\varphi(f(z)) - \varphi(P_k(z))| \leqslant M|f(z) - P_k(z)|, \quad z \in E,\ k \geqslant k_0.$$

Proof. Fix an δ, $0 < \delta < \operatorname{dist}(F, \partial\Omega)$. Write $\tilde{M} = \sup\{|\varphi(w)|: \operatorname{dist}(w, F) \leqslant \delta\}$. By the Schwarz inequality we obtain

$$|\varphi(w) - \varphi(b)| \leqslant \frac{2M}{\delta}|w - b| \quad \text{for } w \in K(b, \delta/2),\ b \in F.$$

Hence, by fixing a k_0 such that $\|f-P_k\|_E < \delta/2$ for $k \geqslant k_0$ and putting $M = 2\tilde{M}/\delta$, we conclude the proof.

Denote by Φ the extremal function of a compact set E in C^n, i.e.

$$\Phi(z, E) = \sup_{\nu\geqslant 1}\{\sup\{|P_\nu(z)|^{1/\nu}: P_\nu \text{ is a polynomial in } z = (z_1, \ldots, z_n) \text{ such that } \deg P_\nu \leqslant \nu \text{ and } \|P_\nu\|_E \leqslant 1\}\}, \quad z \in C^n,$$

introduced by Siciak [5]. We shall often use the following properties of $\Phi(z, E)$ (see [5]):

(1) $$\Phi(z, E) \geqslant 1, \quad z \in C^n \quad \text{and} \quad \Phi(z, E) = 1, \quad z \in E,$$

(2) $$|P(z)| \leqslant \|P\|_E[\Phi(z, E)]^{\deg P}, \quad z \in C^n, \text{ for every polynomial } P,$$

(3) $$\Phi(z, E) \leqslant \Phi(z, F), \quad z \in C^n, \text{ if } F \subset E,$$

(4) $$\Phi(z, E) = \max_{1\leqslant i\leqslant n}\{\Phi(z_i, E_i)\}, \quad z \in C^n, \text{ for } E = E_1 \times \ldots \times E_n.$$

We add that in the case $n = 1$ the function $\Phi(z, E)$ is equivalent to Leja's extremal function $L = L(z, E)$ (see [3]). This note and properties (3), (4) give us some criteria for the continuity of $\Phi(z, E)$, $E \subset C^n$, expressed by properties of L.

The results of the previous lemmas enable us to prove the following

THEOREM 1. *Let E be a compact set in C^n. Let $f \in \mathscr{B}(E, [\{\nu_k\}])$ and let φ be a holomorphic function in an open set Ω in C such that $F = f(E) \subset \Omega$.*

1° *If $E = \hat{E}$ and the following condition is satisfied:*

CONDITION (W). *There exist polynomials $\{P_{\mu_k}\}$, $\deg P_{\mu_k} \leqslant \mu_k$, $\{\mu_k\} \in [\{\nu_k\}]$, a neighbourhood U of E in C^n and constants $A > 0$ and $k_0 > 0$ such that*

(i) $$\|f-P_{\mu_k}\|_E \leqslant M\varrho^{\mu_k} \quad \textit{for } k \geqslant 1,$$

M and ϱ being constants independent of k, $\varrho \in (0, 1)$,

(ii) $$P_{\mu_k}(U) \subset \Omega, \quad k \geqslant k_0,$$

and

(iii) $$\sup_{z\in U}|\varphi(P_{\mu_k}(z))| \leqslant A^{\mu_k} \quad \textit{for } k \geqslant k_0,$$

then $\varphi \circ f \in \mathscr{B}(E, [\{\nu_k\}])$.

2° *If the extremal function $\Phi(z, E)$ is continuous in E and φ is a rational function, then Condition* (W) *is necessary that $\varphi \circ f \in \mathscr{B}(E, [\{\nu_k\}])$.*

Proof. 1° By Condition (W) and Lemma 1, for every $k \geqslant k_0$ there exist polynomials $\{R_\mu^k\}$ such that

(1) $$\|\varphi \circ P_{\mu_k} - R_\mu^k\|_E \leqslant M_1 A^{\mu_k} \cdot \varrho_1^\mu \quad \text{for } \mu = 1, 2, \ldots,$$

M_1 and ϱ_1 being constants independent of k, $\varrho_1 \in (0, 1)$. Take an integer l so large that $A\varrho_1^l \leqslant \varrho_1$. Then, by (1), we obtain

$$\|\varphi \circ P_{\mu_k} - R^k_{l\mu_k}\|_E \leqslant M_1 \varrho_1^{\mu_k} \quad \text{for } k \geqslant k_0. \tag{2}$$

On the other hand, by (i) of Condition (W) and Lemma 5, we have

$$\|\varphi \circ f - \varphi \circ P_{\mu_k}\|_E \leqslant M_2 \varrho^{\mu_k}, \quad k \geqslant k_1, \tag{3}$$

where the constant M_2 does not depend on k. By (2), (3) and the triangle inequality we get

$$\|\varphi \circ f - R^k_{l\mu_k}\|_E \leqslant M \eta^{l\mu_k},$$

where $\eta = [\max(\varrho, \varrho_1)]^{1/l}$ and $M = \max(M_1, M_2)$. This implies that $\varphi \circ f \in \mathscr{B}(E, [\{\nu_k\}])$.

2° Suppose that f, $\varphi \circ f \in \mathscr{B}(E, [\{\nu_k\}])$, where

$$\varphi(w) = \frac{W_p(w)}{Z_q(w)} = \frac{a(w-a_1)\ldots(w-a_p)}{b(w-\beta_1)\ldots(w-\beta_q)},$$

$a_i \neq \beta_j$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, $\beta_j \in C \setminus F$. By our assumptions there exist polynomials $\{P_{\mu_k}\}$, $\{Q_{\omega_k}\}$, where $\{\mu_k\}$, $\{\omega_k\} \in [\{\nu_k\}]$, $\deg P_{\mu_k} \leqslant \mu_k$, $\deg Q_{\omega_k} \leqslant \omega_k$, and constants M, ϱ independent of k, $\varrho \in (0, 1)$, such that

$$\|f - P_{\mu_k}\|_E \leqslant M \varrho^{\mu_k} \quad \text{for } k \geqslant 1, \tag{4}$$

and

$$\|\varphi \circ f - Q_{\omega_k}\|_E \leqslant M \varrho^{\omega_k} \quad \text{for } k \geqslant 1. \tag{5}$$

By the definition of $\mathscr{B}(E, [\{\nu_k\}])$ we may assume that $\mu_k = \omega_k = \nu_k$, $k \geqslant 1$. It follows from (4) and Lemma 5 that

$$\|\varphi \circ f - \varphi \circ P_{\nu_k}\|_E \leqslant M_1 \varrho^{\nu_k}.$$

Hence and by (5), we have

$$\|W_p \circ P_{\nu_k} - (Z_q \circ P_{\nu_k}) Q_{\nu_k}\|_E \leqslant M_1 \|Z_q \circ P_{\nu_k}\|_E \varrho^{\nu_k}. \tag{6}$$

By (3) and Lemma 5 there exists a constant M_2 such that

$$\|Z_q \circ P_{\nu_k}\|_E \leqslant M_2 \quad \text{for } k \geqslant 1.$$

Hence by (6) and property (2) of the extremal function $\Phi(z, E)$ we obtain

$$\left|W_p\big(P_{\nu_k}(z)\big) - Z_q\big(P_{\nu_k}(z)\big) Q_{\nu_k}(z)\right| \leqslant M_3 \varrho^{\nu_k} [\Phi(z, E)]^{r\nu_k}, \tag{7}$$

for $z \in C^n$, where $M_3 = M_1 \cdot M_2$, $r = \max(p, q+1)$. Take a number $\eta \in (\varrho, 1)$. Since $\Phi(z, E)$ is continuous in E and because of property (1) of Φ, there exists an open set U such that $E \subset U$ and $\varrho\Phi^r(z, E) < \eta$ for $z \in U$. Hence by (7) we get

$$(8) \qquad |W_p(P_{\nu_k}(z)) - Z_q(P_{\nu_k}(z))Q_{\nu_k}(z)| \leqslant M_3 \eta^{\nu_k} \quad \text{for } z \in U.$$

Suppose that $Z_q(P_{\nu_k}(\mathring{z})) = 0$ for a point $\mathring{z} \in U$. Then $P_{\nu_k}(\mathring{z}) = \beta_{j_0}$ for a certain $j_0 \in \{1, \ldots, q\}$. Write $\delta = \min_{i,j}\{|a_i - \beta_j|\}$. By (8) we would have

$$0 < |b|\,\delta^p \leqslant |b(\beta_{j_0} - a_1) \ldots (\beta_{j_0} - a_p)| = |W_p(\beta_{j_0})| \leqslant M_3 \eta^{\nu_k}.$$

This is impossible for sufficiently large k. So, we have

$$(9) \qquad Z_q(P_{\nu_k}(z)) \neq 0 \quad \text{for } z \in U,\ k \geqslant k_0 = k_0(\delta, p).$$

This means that (ii) of Condition (W) is satisfied.

It remains to prove (iii). To this end take $\varepsilon = \frac{1}{2}\min_{1 \leqslant j \leqslant q}\{\text{dist}(F, \beta_j)\}$ and write $F_\varepsilon = \{w \in C\colon \text{dist}(w, F) < \varepsilon\}$. By (4),

$$P_{\nu_k}(z) \in F_\varepsilon \quad \text{for } z \in E, \text{ if } k \geqslant k_1 = k_1(\varepsilon).$$

Hence

$$(10) \qquad \min_{z \in E}|Z_q(P_{\nu_k}(z))| \geqslant \inf_{w \in F_\varepsilon}|Z_q(w)| = m > 0 \quad \text{for } k \geqslant k_2,$$

where $k_2 = \max(k_0, k_1)$. By (9), (10) and Lemma 4 there exists an open set V such that $E \subset V \subset U$ and

$$|Z_q(P_{\nu_k}(z))|\,2^{q\nu_k} \geqslant m \quad \text{for } z \in V,\ k \geqslant k_2.$$

Hence, by (4), Lemma 5 and property (2) of Φ, we obtain

$$\sup_{z \in V}|\varphi(P_{\nu_k}(z))| \leqslant 1/m\, \|W_p \circ P_{\nu_k}\|_E [(\eta/\varrho)^{1/r}]^{p\nu_k} 2^{q\nu_k} \leqslant A^{\nu_k}$$

for $k \geqslant k_2$, where A is a suitably chosen constant independent of k. The proof is completed.

Remark. If $E \neq \hat{E}$, then, in general, Condition (W) is not sufficient for $\varphi \circ f \in \mathscr{B}(E, [\{\nu_k\}])$. One can easily come to this conclusion by considering $E = \{z \in C\colon |z| = 1\}$, $f(z) = z$ and $\varphi(w) = 1/w$.

Nevertheless, because of Lemma 3, given $f \in \mathscr{B}(E, [\{\nu_k\}])$, we may always assume that $E = \hat{E}$.

THEOREM 2. *Let E and φ be the same as in 2° of Theorem 1. If Condition (W) is satisfied for polynomials $\{P_{\mu_k}\}$, then it is satisfied for every sequence of polynomials $\{Q_{\varkappa_k}\}$ such that $\{\varkappa_k\} \in [\{\mu_k\}]$, $\deg Q_{\varkappa_k} \leqslant \varkappa_k$ and*

$$(1) \qquad \|f - Q_{\varkappa_k}\|_E \leqslant M_1 \varrho_1^{\varkappa_k}, \quad k \geqslant 1,$$

M_1 and ϱ_1 being constants independent of k, $\varrho_1 \in (0, 1)$.

Proof. We have only to prove (ii) and (iii) of Condition (W) for $\{Q_{\varkappa_k}\}$.

(ii) If $\Omega = C$, the proof is trivial. Assume that $C \setminus \Omega = B \neq \emptyset$. By (1), (i) of Condition (W) for $\{P_{\mu_k}\}$ and the definition of $[\{\mu_k\}]$ we have

$$(2) \qquad \|P_{\mu_k} - Q_{\varkappa_k}\|_E \leqslant M_2 \varrho_2^{\omega_k}, \quad k \geqslant 1,$$

for suitable constants M_2 and ϱ_2, $\varrho_2 \in (0, 1)$, $\omega_k = \max(\mu_k, \varkappa_k)$. Take a number $\eta \in (\varrho_2, 1)$. Since the extremal function $\Phi(z, E)$ is continuous in E, there exists an open set V, $E \subset V$, such that $\varrho_2 \Phi(z, E) \leqslant \eta$ for $z \in \operatorname{cl} V$. Hence, by (2) and property (2) of Φ, we get

$$(3) \qquad |P_{\mu_k}(z) - Q_{\varkappa_k}(z)| \leqslant M_2 \eta^{\omega_k} \quad \text{for } z \in V,\ k \geqslant 1.$$

Write $2m = \operatorname{dist}(F, B)$. By (i) of Condition (W) for every point $w \in B$ we have

$$|P_{\mu_k}(z) - w| \geqslant |f(z) - w| - |f(z) - P_{\mu_k}(z)| \geqslant 2m - M\varrho^{\mu_k} \geqslant m > 0$$

for $z \in E$ and $k \geqslant k_1 = k_1(m)$. Hence, by (ii) of Condition (W), the family $\mathscr{F} = \{P_{\mu_k} - w\colon w \in B,\ k \geqslant k_2 = \max(k_0, k_1)\}$ of polynomials in z satisfies the assumptions of Lemma 4. So, for every $\lambda > 1$ we can find an open set V_λ such that $E \subset V_\lambda \subset U$ and

$$(4) \qquad |P_{\mu_k}(z) - w|\, \lambda^{\mu_k} \geqslant m, \quad z \in V_\lambda,\ w \in B,\ k \geqslant k_2.$$

If we take $\lambda < 1/\eta$, then, by (3) and (4), we get

$$|Q_{\varkappa_k}(z) - w| \geqslant |P_{\mu_k}(z) - w| - |P_{\mu_k}(z) - Q_{\varkappa_k}(z)| \geqslant m\lambda^{-\mu_k} - M_2 \eta^{\omega_k} > 0$$

for $z \in V_\lambda \cap V$, $w \in B$, $k \geqslant k_3$, where k_3 is sufficiently large. Thus, $Q_{\varkappa_k}(V_\lambda \cap V) \subset \Omega$ for $k \geqslant k_3$ as asserted.

(iii) Since the function φ is rational, it is enough to apply property (2) of the extremal function Φ and Lemma 4.

Now let E be a polynomially convex compact set in C (²) and let $f \in \mathscr{B}(E, [\{\nu_k\}])$. Assume that the extremal function $\Phi(z, E)$ is continuous in E. Comparing Lemma 2 and Theorem 1, by the Montel theorem, gives the following

THEOREM 3. *If φ is a rational function with at least two poles lying in $C \setminus f(E)$, then $\varphi \circ f \in \mathscr{B}(E, [\{\nu_k\}])$ if and only if there exists a function $\tilde{f}$ holomorphic in a neighbourhood of E such that $\tilde{f}|_E = f$.*

We shall now illustrate our results by means of some examples.

COUNTER-EXAMPLE 1. Let $E = \{z \in C\colon |z| \leqslant 1\}$. Then $\Phi(z, E) = \max\{|z|, 1\}$ for $z \in C$ (see [5]). Take a sequence $\{\nu_k\}$ of positive integers such that $\nu_{k+1}/\nu_k \to \infty$ as $k \to \infty$ and define

$$f(z) = \sum_{k=1}^{\infty} \frac{z^{\nu_k}}{a^{\nu_{k-1}}} \quad (a > 1).$$

(²) In this case it is well known that $E = \hat{E}$ if and only if the set $C \setminus E$ is connected.

One can easily check that $f \in \mathscr{B}(E, \{\nu_k\})$ and f cannot be analytically continued onto any neighbourhood of E. Let φ be a rational function with at least two poles. Then it follows from Theorem 3 that $\varphi \circ f \notin \mathscr{B}(E, [\{\nu_k\}])$.

COUNTER-EXAMPLE 2. Define E and f as above and set

$$g(z) = f^2(z) - M^2,$$

where $\|f\|_E < M$. Since $\mathscr{B}(E, [\{\nu_k\}])$ is a ring; then $g \in \mathscr{B}(E, [\{\nu_k\}])$. Because of Theorem 3 it is seen that $1/g \notin \mathscr{B}(E, [\{\nu_k\}])$.

COUNTER-EXAMPLE 3. Define the sequence $\{\nu_k\}$ as

$$\nu_0 = 1, \quad \nu_{k+1} = 2^{\nu_k}, \quad k \geqslant 0,$$

and set

$$f(x) = \sum_{l=0}^{\infty} \frac{\cos \nu_k \arccos x}{\nu_k} \quad \text{for } x \in E = [-1, 1].$$

It is known (Bernstein [1], p. 294) that $f \in \mathscr{B}(E, \{\nu_k\})$ and f is not differentiable in E. Define

$$g(x) = f(x) + M,$$

where $\|f\|_E < M$. We will show that $1/g \notin \mathscr{B}(E, [\{\nu_k\}])$. To this end write

$$P_{\nu_k}(z) = M + \sum_{l=0}^{k} \frac{\cos \nu_l \arccos z}{\nu_l} = M + \sum_{l=0}^{k} \frac{(z+\sqrt{z^2-1})^{\nu_l} + (z-\sqrt{z^2-1})^{\nu_l}}{2\nu_l}.$$

It is clear that P_{ν_k} is a polynomial in z of degree ν_k. Moreover,

$$\|g - P_{\nu_k}\|_E < 2(1/2)^{\nu_k}.$$

Fix a number $r > 1$. Since the sequence $\{\nu_k/\nu_{k-1}\}$ is increasing, for $|z+\sqrt{z^2-1}| = R \geqslant r$ we have

$$|P_{\nu_k}(z)| \geqslant \left| \frac{(z+\sqrt{z^2-1})^{\nu_k} + (z-\sqrt{z^2-1})^{\nu_k}}{2\nu_k} \right| -$$

$$- \left| M + \sum_{l=0}^{k-1} \frac{(z+\sqrt{z^2-1})^{\nu_l} + (z-\sqrt{z^2-1})^{\nu_l}}{2\nu_l} \right|$$

$$\geqslant \frac{R^{\nu_k} - R^{-\nu_k}}{2\nu_k} - \sum_{l=0}^{k-1} \frac{R^{\nu_l} + R^{-\nu_l}}{2\nu_l} - M$$

$$> \tfrac{1}{2}\left[\left(\frac{R^{\nu_k/\nu_{k-1}}}{2}\right)^{\nu_{k-1}} - \sum_{l=1}^{k-1}\left(\frac{R^{\nu_k/\nu_{k-1}}}{2}\right)^{\nu_{l-1}}\right] - \left(M + \frac{R}{2} + 1\right) > 0,$$

as $k \geqslant k_0 = k_0(r)$. Thus, all the zeros of the polynomials $\{P_{\nu_k}\}$ $(k \geqslant k_0)$ are contained in the ellipse $\{z \in C\colon |z+\sqrt{z^2-1}| < r\}$ with the foci -1 and 1. So, the sequence $\{P_{\nu_k}\}$ does not satisfy (ii) of Condition (W) for $\varphi(w) = 1/w$. Since $\Phi(z, [-1, 1]) = |z+\sqrt{z^2-1}|$ for $z \in C$ (see [3]), it is continuous in C. Hence, by Theorems 1 and 2, $1/g \notin \mathscr{B}([-1, 1], [\{\nu_k\}])$.

EXAMPLE 4. Let E be a compact set in C^n. If $f \in \mathscr{B}(E, [\{\nu_k\}])$ and φ is a polynomial; then $\varphi \circ f \in \mathscr{B}(E, [\{\nu_k\}])$.

EXAMPLE 5. Let E be a polynomially convex compact set in C^n such that the extremal function $\Phi(z, E)$ is continuous in E. Take an increasing sequence $\{\nu_k\}$ of positive integers such that $\nu_{k+1}/R^{\nu_k} \to \infty$ as $k \to \infty$ for a certain constant $R > 1$. Write $\mu_k = [R^{\nu_k}]$, $k = 1, 2, \ldots$ Given an $\varrho \in (0, 1)$, we put

$$\varepsilon_\nu = \varrho^{\mu_k} \quad \text{for } \nu_k \leqslant \nu < \nu_{k+1},\ k \geqslant 1.$$

By the well-known Bernstein theorem (see [6]) there exists a function $f \in \mathscr{C}(E)$ such that

$$\mathscr{E}_\nu(f, E) = \varepsilon_\nu, \quad \nu = 1, 2, \ldots$$

Since

$$\limsup_{\nu\to\infty} \sqrt[\nu]{\mathscr{E}_\nu(f, E)} \geqslant \lim_{k\to\infty} \varrho^{\mu_k/(\nu_{k+1}-1)} = 1;$$

then, in virtue of Lemma 1 (for the case $f_k = f$), the function f cannot be continued to a holomorphic function onto any neighbourhood of E. Take polynomials P_{ν_k} such that

$$\mathscr{E}_{\nu_k}(f, E) = \|f-P_{\nu_k}\|_E, \quad k \geqslant 1.$$

Since $\mathscr{E}_\mu(f, E) \leqslant \mathscr{E}_\nu(f, E)$ for $\mu \geqslant \nu$, then $f \in \mathscr{B}(E, \{\mu_k\})$. By the assumption of continuity of $\Phi(z, E)$ in E we can choose an open set U, $E \subset U$, such that $\Phi(z, E) < R$ for $z \in U$. Then, by property (2) of Φ, we have

$$|e^{P_{\nu_k}(z)}| \leqslant A^{R^{\nu_k}}, \quad z \in U,$$

and

$$|e^{-P_{\nu_k}(z)}| \leqslant A^{R^{\nu_k}}, \quad z \in U,$$

A being a positive constant. Hence the sequences $P_{\mu_k} \equiv P_{\nu_k}$ and $Q_{\mu_k} \equiv -P_{\nu_k}$ $(k = 1, 2, \ldots)$ satisfy Condition (W) for $\varphi(w) = e^w$. Thus, by Theorem 1, the functions e^f and e^{-f} are members of $\mathscr{B}(E, [\{\mu_k\}])$.

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