EXTENSION OF LOCALLY UNIFORMLY EQUIVALENT METRICS

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Hausdorff showed that if \( A \) is a closed subset of a metric space \((X, d)\) and if \( \rho \) is a metric on \( A \) which is topologically equivalent to \( d|A \times A \), then there is a metric on \( X \) that extends \( \rho \) and is topologically equivalent to \( d \) ([1], II, Theorem 3.2). The word "topologically" can be replaced either by "Lipschitz" [5] or by "locally Lipschitz" [3] or, under some additional assumptions, by "uniformly" [5] (for Lipschitz equivalent and uniformly equivalent metrics the closedness of \( A \) is irrelevant). The purpose of this paper is to show that "locally uniformly" applies as well. The results and the proofs are similar to those for locally Lipschitz equivalent metrics in [3].

A map \( f: S \to T \) between uniform spaces is called \textit{locally uniformly continuous} if each point of \( S \) has a neighborhood on which \( f \) is uniformly continuous. If \( f \) is bijective and both \( f \) and \( f^{-1} \) are locally uniformly continuous, then \( f \) is called a \textit{locally uniform homeomorphism}. Two metrics \( d_1 \) and \( d_2 \) on a set \( D \) are said to be \textit{locally uniformly equivalent} if the identity map \( \text{id}: (D, d_1) \to (D, d_2) \) is a locally uniform homeomorphism. A metric on a subset \( A \) of a metric space \((X, d)\) is said to be \textit{locally uniformly compatible} if it is locally uniformly equivalent to \( d|A \times A \).

**Lemma.** Let \((X, d)\) be a metric space, let \( A \subseteq X \) be closed, and let \( \rho \) be a locally uniformly compatible metric on \( A \) such that \( \rho \leq d|A \times A \). Let \( e(x, y) \) be the minimum of \( d(x, y) \) and \( \inf \{d(x, a) + \rho(a, b) + d(b, y) \mid a, b \in A\} \) for \( x, y \in X \). Then \( e \) is a locally uniformly compatible metric on \( X \) extending \( \rho \).

**Proof.** By [2], p. 517, \( e \) is a metric on \( X \) that extends \( \rho \) and is topologically equivalent to \( d \). To make the proof independent of the last fact, we remark that \( A \) is closed in \((X, e)\) because \( e(x, A) = d(x, A) \) for each \( x \in X \). Observe that \( e \leq d \) and that \( e(x, y) = d(x, y) \) for all \( x, y \in X \) such that \( e(x, A) \geq \varepsilon, e(y, A) \geq \varepsilon, \) and \( e(x, y) < 2\varepsilon \) for some \( \varepsilon > 0 \). Hence it suffices to prove that each \( p \in A \) has a neighborhood \( U \) in \((X, e)\) such that \( \text{id}: (U, e|U \times U) \to (X, d) \) is uniformly continuous. There is an \( r > 0 \) such that if \( V = \{x \in A \mid \rho(x, p) < 2r\} \), then the identity map
id: \((V, \varrho | V \times V) \to (X, \bar{d})\) is uniformly continuous. We show that one can choose
\[ U = \{x \in X \mid e(x, p) < r\}. \]

Let \(\varepsilon > 0\). Choose \(\delta > 0\) with \(\delta < \min(r, \varepsilon/2)\) such that \(a, b \in V\) and \(\varrho(a, b) < \delta\) imply \(d(a, b) < \varepsilon/2\). Suppose that \(x, y \in U\) and \(e(x, y) < \delta\). To prove \(\bar{d}(x, y) < \varepsilon\), we may assume \(e(x, y) \neq \bar{d}(x, y)\). Then there are \(a, b \in A\) such that \(\bar{d}(x, a) + \varrho(a, b) + \bar{d}(b, y) < \delta\). This implies
\[ \varrho(a, p) = e(a, p) \leq e(a, x) + e(x, p) < d(a, x) + r < \delta + r < 2r, \]
whence \(a \in V\); similarly \(b \in V\). Since \(\varrho(a, b) < \delta\), we get
\[ \bar{d}(x, y) \leq \bar{d}(x, a) + \bar{d}(a, b) + \bar{d}(b, y) < \delta + \varepsilon/2 < \varepsilon. \]

**Theorem 1.** Let \(\varrho\) be a locally uniformly compatible metric on a closed subset \(A\) of a metric space \((X, \bar{d})\). Then there is a locally uniformly compatible metric on \(X\) extending \(\varrho\).

**Proof.** Let \(m(A)\) be the space of all bounded real functions on \(A\) (with the sup norm). There is an isometric embedding \(f: (A, \varrho) \to m(A)\) (see [1], II, Proposition 1.1). Since \(f\) is locally uniformly continuous with respect to \(\bar{d}\), by [6], Theorem 1, \(f\) has a locally uniformly continuous extension \(\tilde{f}: X \to m(A)\). Define a metric \(\bar{d}_1\) on \(X\) by
\[ \bar{d}_1(x, y) = \bar{d}(x, y) + \|\tilde{f}(x) - \tilde{f}(y)\|. \]

Then \(\bar{d}_1\) is locally uniformly compatible and \(\varrho \leq \bar{d}_1|A \times A\). Thus an application of the Lemma completes the proof.

Next we study the case of a non-closed \(A\).

**Theorem 2.** Let \(\varrho\) be a locally uniformly compatible metric on a subset \(A\) of a metric space \((X, \bar{d})\). Then \(\varrho\) has an extension to a locally uniformly compatible metric on some neighborhood of \(A\).

**Proof.** From the Lemma in [4] it follows easily that \(\varrho\) has an extension to a locally uniformly compatible metric \(\varrho_1\) on an open neighborhood \(U\) of \(A\) in \(\bar{A}\). Now \(U = V \cap \bar{A}\) for some open neighborhood \(V\) of \(A\) in \(X\). Since \(U\) is closed in \(V\), by Theorem 1 there is a locally uniformly compatible metric \(\varrho_2\) on \(V\) extending \(\varrho_1\), and thus \(\varrho\).

**Remark.** In Theorems 1 and 2 the extension of \(\varrho\) can be chosen to be complete (respectively, totally bounded) if \(\varrho\) is complete (respectively, totally bounded) and \(\bar{d}\) is locally complete (respectively, separable and locally totally bounded). These results can be proved as similar results in [3].
REFERENCES


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