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Finitely generated maximum modulus algebras

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Abstract. In the present note we study finitely generated and finitely rationally generated Fréchet function algebras (A, X) on a locally compact and σ-compact space X, which satisfy a generalized maximum modulus principle, introduced by Rusek [8]. In particular we give a function algebraic characterization of the algebra of all holomorphic functions on a polynomially convex domain $\Omega \subset C^n$.

1. Introduction. In a previous paper [3] the following theorem was proved (for definitions see Section 2).

(1.1) THEOREM. *Let $(A, \sigma A)$ be a singly rationally generated function algebra with locally compact spectrum σA and generating element f. If A satisfies the maximum modulus principle on σA, then $f(\sigma A)$ is an open subset of the complex plane and A is topologically and algebraically isomorphic to $\mathcal{O}(f(\sigma A))$, the algebra of all holomorphic functions on $f(\sigma A)$.*

For finitely rationally generated function algebras such a characterization does not hold. We gave an example of a doubly generated maximum modulus algebra with locally compact spectrum σA, which has no analytic structure at any point of σA (see [3], (4.9)).

In [8] Rusek proposed a generalization of the notion of a maximum modulus algebra in order to obtain a multidimensional analogue of a well-known theorem of Wermer and Aupetit. His results also yield an analogue of Theorem (1.1) in the finitely generated case, our main result Corollary (3.2).

In [4] Heal and Windham gave another characterization of the algebra $\mathcal{O}(\Omega)$, Ω a polynomially convex domain. They showed that a n-generated function algebra A is topologically and algebraically isomorphic to $\mathcal{O}(\Omega)$ iff the spectrum of A is a topological $2n$-dimensional manifold (without boundary).

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2. Preliminaries. For the basic concept of function algebras we refer the reader to the book by Kramm [6].

(2.1) DEFINITION. A pair (A, X) is called a *function algebra* if

(i) $X \neq \emptyset$ is a locally compact and σ-compact Hausdorff space;

(ii) A is a closed subalgebra of the Fréchet algebra $C(X)$ of continuous functions on X;

(iii) A contains the constants and separates the points of X.

$C(X)$ is always equipped with the c.o.-topology. A fundamental system of seminorms is given by $\|f\|_{K_n} = \sup\{|f(x)|: x \in K_n\}$, $n = 1, 2, \ldots$, where $(K_n)_n$ is an admissible exhaustion of X, i.e., $K_n \subset K_{n+1}$, K_n compact for every n and for every compact $K \subset X$ there is n with $K \subset K_n$. The set σA of all non-trivial continuous multiplicative functionals on A equipped with the weak*-topology is called the *spectrum* of A.

For a compact subset $K \subset \sigma A$ we denote by A_K the restriction algebra $A_{|K}$ completed in the $\|\cdot\|_K$ norm. The spectrum σA_K can be identified with the A-convex hull $\hat{K} := \{\psi \in \sigma A: |f(\psi)| \leqslant \|f\|_K$, for all $f \in A\}$ (see [6], (6.6)). As a consequence $\hat{K}$ is compact.

(2.2) We call a function algebra (A, X) *rationally n-generated* provided there exist elements $f_1, \ldots, f_n$ in A such that functions of the form $p \cdot q^{-1}$, p and q polynomials in $f_1, \ldots, f_n$ (q invertible in A), lie dense in A. We call (A, X) *n-generated* if A is the closure of the polynomials in $f_1, \ldots, f_n$. In both cases, $F = (f_1, \ldots, f_n)$: $X \to C^n$ is an injective and continuous map.

A compact set $K \subset C^n$ is rationally convex iff $K = \sigma R(K)$ with $R(K) := \{f \in C(K)$: f can be approximated uniformly on K by rational functions with poles off $K\}$. The spectrum $\sigma R(K)$ can be naturally identified with the compact set $\{x \in C^n$: $p(x) \in p(K)$ for all polynomials $p\}$ (see [2], p. 69).

(2.3) Proposition. *If $(A, \sigma A)$ is rationally n-generated by $f_1, \ldots, f_n$ and $K \subset \sigma A$ is an A-convex compact set, then $F(K) \subset C^n$ is rationally convex.*

Proof. First note that $F_{|K}$: $K \to F(K)$ is a homeomorphism. The map T: $A_K \to R(F(K))$, $g \to g \circ F_{|K}^{-1}$ induces a Banach algebra isomorphism from A_K to a closed subalgebra $\tilde{A}$ of $R(F(K))$. From this follows $\sigma R(F(K)) \subset \sigma \tilde{A}$.

On the other hand, the spectrum map, which is associated to the inverse of T, σT^{-1}: $K \to \sigma \tilde{A}$, $\varphi \to \varphi \circ T^{-1}$ is a homeomorphism (see Kramm [6], (2.1)). It is now easy to show that $\sigma T^{-1}(\varphi) = F(\varphi)$ for all $\varphi \in K$; hence $F(K) = \sigma \tilde{A}$. ∎

We call an open set $\Omega \subset C^n$ *rationally* (resp. *polynomially*) *convex* provided there exists an admissible exhaustion $K_n \subset K_{n+1} \subset \ldots$ of Ω, such that all K_n are rationally (resp. polynomially) convex. Polynomially convex domains are also called *Runge domains*.

(2.4) Let X be a locally compact space. A subalgebra $A \subset C(X)$ (not necessarily closed) is called a *maximum modulus algebra* (*m.m.a*) on X iff

(i) A separates points on X and contains the constants;

(ii) for every $g \in A$ and every compact set $K \subset X$ we have $\|g\|_K \leqslant \|g\|_{\partial K}$ (∂K denotes the topological boundary of K relative to X).

If $\partial K = \emptyset$ we define $\|g\|_{\partial K} = 0$ for all $g \in A$.

We recall Rusek's definition of a structural system.

(2.5) DEFINITION. $\{A, X \xrightarrow{F} \Omega\}$ is a *structural system of order n*, iff

(i) A is a subalgebra of $C(X)$ separating the points of X and containing the constants;

(ii) Ω is a domain in C^n and $F \in A^n$ is a proper mapping of X onto Ω;

(iii) for every affine complex line L in C^n with $\Omega \cap L \neq \emptyset$, $A_{|F^{-1}(\Omega \cap L)}$ is a maximum modulus algebra on $F^{-1}(\Omega \cap L)$.

We shall use the following observation of Järvi [5].

(2.6) THEOREM. *If A is a m.m.a. on X, then every $f \in A$ is a quasiopen mapping from X to C (i.e., for each relatively compact open set U in X we have $\partial f(U) \subset f(\partial U)$).*

In particular, if f is injective, then $f(X)$ is an open set in C and f: $X \to f(X)$ is a homeomorphism.

(2.7) DEFINITION. A σ-compact complex analytic manifold X of dimension n is said to be *Stein* if

(i) $\tilde{K} = \{z \in X\colon |f(z)| \leqslant \|f\|_K$ for every $f \in \mathcal{O}(X)\}$ is a compact subset of X for every compact subset K of X;

(ii) $\mathcal{O}(X)$ separates points of X;

(iii) for every $z \in X$, one can find n functions $f_1, \ldots, f_n \in \mathcal{O}(X)$ which form a coordinate system at z.

Let X be a Stein manifold and $A \subset \mathcal{O}(X)$ a subalgebra such that (i)–(iii) of the above definition are satisfied if we replace $\mathcal{O}(X)$ by A. Then A is dense in $\mathcal{O}(X)$ (see Forster [1], p. 145).

3. Main result. Let (A, X) be a function algebra, $F \in A^n$ and let L be an affine complex line in C^n with $F(X) \cap L \neq \emptyset$; then we define $X_L := F^{-1}(F(X) \cap L)$.

(3.1) THEOREM. (i) *Let (A, X) be a function algebra on a locally compact space X and $A^n \ni F$: $X \to C^n$ a locally injective map (i.e., for every $x \in X$ there exists a neighbourhood of x on which F is injective) such that the restriction algebra $A_{|X_L}$ is a m.m.a. for all complex affine lines $L \subset C^n$ with $L \cap F(X) \neq \emptyset$. Then F is an open mapping and X can be equipped with an uniquely determined structure of a n-dimensional complex manifold such that $A \subset \mathcal{O}(X)$, the algebra of all holomorphic functions on X.*

(ii) *If, moreover, $X = \sigma A$, then X is a Stein manifold and $A = \mathcal{O}(X)$.*

Proof. (i) For $\delta > 0$ denote by B_δ the open ball in C^n centered at $(0, \ldots, 0)$ with radius $\delta > 0$.

Let x be an arbitrary point of X, K a compact neighbourhood of x such that F is injective in an open set U containing K. We may assume $F(x) = (0, \ldots, 0)$. Hence there exists $\delta > 0$ such that $F^{-1}(B_\delta) \cap K \subset \mathring{K}$, where $\mathring{K}$ denotes the interior of K relative to X.

Assume that $(0, \ldots, 0) = F(x)$ is a boundary point of $F(K)$. We choose a point $z \in B_\delta \setminus F(K)$. Let L be the complex line which contains z and Π the orthogonal projection from C^n onto L. Note that $\Pi \circ F \in A$ if we naturally identify L with C, and that $A_{|X_L \cap U}$ is a m.m.a.

It follows from (2.6) that $\Pi \circ F(X_L \cap U)$ is an open set in C. Since X_L is closed in X, $\Pi \circ F(X_L \cap K)$ is a compact set in C and we may choose a boundary point y of $\Pi \circ F(X_L \cap K)$ with $|y| < \delta$ since $\Pi(z) \notin \Pi \circ F(X_L \cap K)$.

For the uniquely determined point $\tilde{y} \in X_L \cap U$ with $\Pi \circ F(\tilde{y}) = y$ we have $\tilde{y} \in F^{-1}(B_\delta) \cap K$ and thus $\tilde{y} \in \mathring{K}$. This is a contradiction since $\Pi \circ F$ is open and y is a boundary point of $\Pi \circ F(X_L \cap K)$. Hence $F(x)$ is not a boundary point of $F(K)$ and so F is an open mapping.

Since F is a local homeomorphism, we get an uniquely determined structure of a n-dimensional complex manifold on X such that F is a holomorphic map.

To prove $A \subset \mathcal{O}(X)$ we now follow exactly the proof of Theorem (3.3) in [8]. We give a short outline.

Let $g \in A$ be an arbitrary element. Let $U \subset X$ be an open set such that $F_{|U}$: $U \to F(U)$ is a homeomorphism and $F(U)$ is an open ball in C^n.

Since $\{A_{|U}, U \xrightarrow{F|U} F(U)\}$ is a structural system of order n, the function $\log|g \circ F_{|U}^{-1} - a|$ is plurisubharmonic in $F(U)$ for every $a \in C$ (Proposition 2.6 in [8]). From this it follows that either $g \circ F_{|U}^{-1}$ or $\overline{g \circ F_{|U}^{-1}}$ is holomorphic in $F(U)$ (Proposition 2.8 in [8]). Now it is easy to deduce that $g \circ F_{|U}^{-1} \in \mathcal{O}(F(U))$.

(ii) It follows from (i) that $A \subset \mathcal{O}(X)$. Since the A-convex hull $\hat{K}$ is already compact (see (2.1)), the holomorph-convex hull $\tilde{K}$ is also compact for every compact set $K \subset X$. $F \in A^n$ (and thus $F \in \mathcal{O}(X)^n$) forms a coordinate system at every point $z \in X$ and $\mathcal{O}(X)$ separates the points of X, since A does. (2.7) implies that X is a Stein manifold and that A is a dense subalgebra of $\mathcal{O}(X)$, hence we have $A = \mathcal{O}(X)$ since A is closed by definition.

(3.2) Corollary. (i) *Let $(A, \sigma A)$ be a rationally n-generated function algebra with locally compact spectrum σA and generating elements $f_1, \ldots, f_n$ such that $A_{|\sigma A_L}$ is a m.m.a. for all complex affine lines $L \subset C^n$ with $L \cap F(\sigma A) \neq \emptyset$ $(F = (f_1, \ldots, f_n))$.*

Then $F(\sigma A)$ is a rationally convex open set in C^n and A is topologically and algebraically isomorphic to $\mathcal{O}(F(\sigma A))$.

(ii) *If $(A, \sigma A)$ is n-generated by $f_1, \ldots, f_n$, then – under the assumptions of part* (i) *– $F(\sigma A)$ is, moreover, a polynomially convex open set.*

Proof. (i) Use (2.3) and (3.1).

(ii) If $(A, \sigma A)$ is n-generated by $f_1, \ldots, f_n$ and $K \subset \sigma A$ is an A-convex compact set, then $F(K)$ is polynomially convex since A_K is n-generated by $f_{1|K}, \ldots, f_{n|K}$ (see [2], p. 68). Note that we identify σA_K with K (see (2.1)) and therefore need not distinguish between $f_{i|K}$ and the Gelfand transforms $\hat{f}_{i|\sigma A_K}$ $(i = 1, \ldots, n)$. ∎

Note added in proof. Theorem (3.1) and Corollary (3.2) remain valid if $A_{|X_L}$ resp. $A_{|\sigma A_L}$ are m.m.a. only for those complex affine lines $L \subset C^n$ which are parallel to a coordinate axis and which satisfy $L \cap F(X) \neq \emptyset$ resp. $L \cap F(\sigma A) \neq \emptyset$. These generalizations can be deduced from Lemma 3 in [7] and from suitable modifications of our proof of Theorem (3.1).

References

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