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Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions

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CONTENTS

INTRODUCTION

Initially this paper (*) was to be an exposition of the following result due to Milutin (cf. Milutin [1], [2]).

If S_1 and S_2 are uncountable compact metric spaces, then the spaces $C(S_1)$ and $C(S_2)$ are linearly homeomorphic (¹).

This result settles a question raised by Banach ([1], p. 185): Are the spaces of all continuous scalar-valued functions on the unit interval and on the unit square linearly homeomorphic?

Milutin's method is based on a very clever construction (cf. Lemma 5.5 in the present paper) of a map Ψ from the Cantor set $\mathscr{C}$ onto the unit interval I such that there exists a projection π of norm one from $C(\mathscr{C})$ onto its subspace consisting of all composed functions $f = g\circ\Psi$ for $g \in C(I)$.

Then the problem can be reduced via the Borsuk-Dugundji theorem on linear extensions to the standard decomposition method.

The projection π is an example of the Birkhoff's averaging operator (cf. Birkhoff [1]).

Subsequently I discovered that averaging operators and extension operators admit a common generalization to the operators which will be called in the sequel "linear exaves". This paper is devoted to a development of the theory of linear exaves acting between spaces of continuous functions on compact Hausdorff spaces, together with some applications including Milutin's result.

We define linear exaves as follows. If $\varphi\colon S \to T$ is a continuous map (S, T — compact), then $\varphi^\circ\colon C(T) \to C(S)$ denotes the induced operator defined by $\varphi^\circ(g) = g\circ\varphi$ for $g \in C(T)$. A linear operator $u\colon C(S) \to C(T)$ is called a *linear exave* if $\varphi^\circ u\varphi^\circ = \varphi^\circ$. That condition is always satisfied whenever u is either left or right inverse for φ°. In the first case u is in fact Birkhoff's averaging operator, in the second u is a linear extension operator.

Indeed if $u\varphi^\circ = \mathrm{id}_{C(T)}$ (where id_X denotes the identity on X), then $C(T)$ may be identified with $\varphi^\circ[C(T)]$, the subalgebra of $C(S)$ consisting of all functions which are constant on each set $\varphi^{-1}(t)$ for $t \in T$. Hence $A = u\varphi^\circ$ is a projection from $C(S)$ onto $\varphi^\circ[C(T)]$. If this projection is positive

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(¹) For the terminology and notation see "Preliminaries".

(i.e. takes non-negative functions into non-negative functions), then u is of the form

$$(uf)(t) = \int_{\varphi^{-1}(t)} \varphi(s)\mu_t(ds) \quad \text{for } t \in T \text{ and for } f \in C(S),$$

where μ_t is a probability measure concentrated on $\varphi^{-1}(t)$ for $t \in T$. Now it is easily seen that A satisfies Birkhoff's averaging condition, $A(f_1 \cdot A(f_2)) = Af_1 \cdot Af_2$ for f_1, f_2 in $C(S)$.

If $\varphi^\circ u = \mathrm{id}_{C(S)}$, then φ° maps $C(T)$ onto $C(S)$ and therefore φ is one-to-one. The compactness of S and T implies then that φ is a homeomorphic embedding of S into T. Identifying S with its homeomorphic image φS in T it is natural to regard φ° as the operator of restriction of function on T to the function on φS. Then the condition $\varphi^\circ u = \mathrm{id}_{C(S)}$ means nothing else but the fact that for each f in $C(S)$, uf is an extension of f.

In the present paper we are mainly concerned with the following three types of exaves (a) linear exaves, (b) regular exaves = linear exaves which are regular operators (cf. § 1 for definition), (c) linear-multiplicative exaves = linear exaves which are linear-multiplicative operators.

Roughly speaking, the existence of an extension operator (resp. averaging operator) from $C(S)$ into $C(T)$ of type either (a) or (b) is closely related to the existence of a bounded linear projection from $C(T)$ onto its subspace isometric to $C(S)$ (resp. from $C(S)$ onto its subspace isometric to $C(T)$), while the existence of linear-multiplicative extension (resp. averaging) is equivalent to the existence of certain retraction from T onto S (resp. from S onto T).

The advantage of regular operators is that they admit some infinite operations (cf. § 1). This enables us to reduce some general theorems on existence of certain exaves to the assertion of the existence of a very particular regular exaves (like Milutin Lemma). On the other hand the non-existence theorems seem to be properly stated for general linear exaves.

Summary of the contents. § 1 is devoted to study regular operators. The formal part of the theory is developed in § 2-§ 4. In § 5 and § 6 we study two classes of compact spaces: Milutin spaces and Dugundji spaces. The first are continuous images of a generalized Cantor set (= product of two point spaces) by an epimorphism admitting linear averaging operator. The second are those which admit an embedding into Tichonov cube such that there exists a linear extension operator. Both classes include compact metric spaces, their products and are, in a certain sense, "closer" to metric spaces than other compact spaces. The proof that each metric space is a Milutin space is rather complicated. A key tool is Lemma 5.5 (the Milutin Lemma).

In § 7 we study linear exaves commuting with certain group of mappings. We conclude this paragraph with a proof that every compact topological group is a Milutin space (an improvement of earlier results of Ivanovskiĭ [1] and Kuzminov [1]).

§ 8 and § 9 are devoted to applications. In § 8 we prove Milutin's result (Theorem 8.3). This, together with some results of C. Bessaga and the author (cf. Bessaga and Pełczyński [1]), enables us to get the complete linear topological classification of spaces $C(S)$ for compact metric S. In the non-metric case we extend Milutin's result to the case of product of compact metric spaces and of arbitrary compact groups as well. It is shown that if G is a compact topological group, then the linear topological type of $C(G)$ is entirely determined by the topological weight of G (Theorem 8.9).

§ 9 is mainly devoted to the study of a class of epimorphisms of order two (i.e. such that the inverse image of each point has at most two points) which have no linear averaging operator. This enables us to obtain some results of Amir [1], [2].

In the Notes and Remarks we discuss some counterexamples and state some open questions. We also discuss the relationships of some of our results and methods to those in literature as well as make some general comments about the references. In the Bibliography we have attempted to include a complete list of references on the subject of extension of continuous functions and related topics.

In this paper we restrict ourselves to spaces $C(S)$ with S compact though extension and averaging operators also appear in different contexts, e.g. in differential equations (cf. Adams, Aronszajn and Smith [1]) and in the group representation theory (if H is a subgroup of a locally compact group G, then $\int_H f(xy)\,dy$ yields an averaging operator from $C(G)$ onto $C(G/H)$ (cf. § 7 for details)). In the Appendix we try to describe a "general concept of an exave" in terms of the theory of categories and we enlist some important models. Also in the Notes and Remarks the reader can find some information concerning various types of exaves.

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PRELIMINARIES

Topological spaces. Throughout this paper by "topological space" we mean "topological Hausdorff space". A *map* $\varphi : S \to T$ means a continuous transformation from S into T. If $\varphi : S \to Q$ and $\psi : Q \to T$ are maps, then the composition $\omega = \psi\varphi : S \to T$ (sometimes we shall write $\psi \circ \varphi$) is the map defined by $\omega(s) = \psi(\varphi(s))$ for $s \in S$. A map $\varphi : S \to T$ is called an *epimorphism* (resp. a *homeomorphic embedding*) if $\varphi S = T$ (if φ regarded as a map from S onto φS is a homeomorphism). The *identity map* on S will be denoted by id_S.

If S is a topological space, then the *topological weight*, in shorter form, the *weight*, of S is the smallest cardinal number $\mathfrak{m}$ such that there exists a base for the topology of S (see Kelley [2], p. 46) of power $\mathfrak{m}$.

If $(S_a)_{a \in A}$ is a family of spaces, then $\boldsymbol{P}_{a \in A} S_a$, in short $\boldsymbol{P} S_a$, denotes the *Cartesian product* of the spaces S_a (i.e. the set of all functions $s = (s_a)_{a \in A}$ on the coordinate set A such that $s_a \in S_a$ for $a \in A$) endowed with the weakest topology in which the *natural projections*

$$p_b : \boldsymbol{P} S_a \to S_b \qquad \text{defined by} \qquad p_b((s_a)) = s_b$$

are continuous functions for all $b \in A$. The cardinality of A will be denoted by $\overline{\overline{A}}$. The Cartesian product of two spaces S and T will be denoted by $S \times T$. If $\mathfrak{m}$ is a cardinal number, then $S^{\mathfrak{m}}$ will denote the Cartesian product of $\mathfrak{m}$ copies of S.

If $(S_a)_{a \in A}$ and $(T_a)_{a \in A}$ are two families of topological spaces and $(\varphi_a : S_a \to T_a)_{a \in A}$ is a family of maps, then the *product map*

$$\boldsymbol{P}\varphi_a : \boldsymbol{P} S_a \to \boldsymbol{P} T_a$$

is the unique map with the property that the diagram

$$\begin{array}{ccc} \boldsymbol{P} S_a & \xrightarrow{\ \boldsymbol{P}\varphi_a\ } & \boldsymbol{P} T_a \\ \big\downarrow{\scriptstyle p_b} & & \big\downarrow{\scriptstyle q_b} \\ S_b & \xrightarrow{\ \varphi_b\ } & T_b \end{array}$$

commutes for each $b \in A$, where p_b and q_b denote the natural projections onto S_b and onto T_b respectively.

In the sequel the capital letters S, T, Q will be reserved for denoting compact spaces. The following symbols will be used for denoting special compact spaces:

$I = [0, 1]$ — the closed unit interval.

$D = \{0\} \cup \{1\}$ — the two-point discrete space.

$[\alpha]$ — the space of all ordinals $\leqslant \alpha$ endowed with the usual order topology (cf. Kelley [2], pp. 57, 266-271).

$I^{\mathfrak{m}}$ — the Cartesian product of $\mathfrak{m}$ copies ($\mathfrak{m}$ is a cardinal number) of I, the *Tichonov cube* of the topological weight $\mathfrak{m}$;

$D^{\mathfrak{m}}$ — the Cartesian product of $\mathfrak{m}$ copies of two-point spaces, the *generalized Cantor set* of power $\mathfrak{m}$. A general point of $D^{\mathfrak{m}}$ will be denoted by $\xi = (\xi_a)_{a \in A}$, where $\xi_a = 0$ or 1, $a \in A$, and A is an arbitrary set of indices of the power $\overline{\overline{A}} = \mathfrak{m}$.

$\mathscr{C} = D^{\aleph_0}$ — the ordinary *Cantor set.* In this case we choose as A the set of all positive integers. A general point of $\mathscr{C}$ is a sequence $\xi = (\xi_i)_{i=1}^{\infty}$ where $\xi_i = 0$ or 1.

Banach spaces. In the sequel X, Y, Z and E will denote Banach spaces unless otherwise specified. If X is a Banach space, then X^* denotes the dual (conjugate) space to X. The elements of X^* are linear functionals on X and will be denoted by $x^*, y^*, z^*, \ldots$ The *weak-star topology* on X^* is the weakest topology on X^* such that for each $x \in X$ the function $x(x^*) = x^*x$ is continuous on X^*. The symbols u, v, w are reserved for denoting linear operators. Linear operators are assumed to be bounded, therefore continuous. If $u: X \to Y$ is a linear operator, then u^* is the *adjoint operator of* u, i.e., u^* is the linear operator from Y^* into X^* defined by $(u^*y^*)(x) = y^*ux$ for $x \in X$ and $y^* \in Y^*$.

A linear operator $u: X \to Y$ is called an *epimorphism* if $uX = Y$, a *linear homeomorphism* if it is an epimorphism and is one-to-one and has a bounded inverse (isomorphism in the terminology of Banach [1]), a *linear homeomorphic embedding* if it is a linear homeomorphism onto a subspace of Y, *isometric embedding* if it is one-to-one and $\|ux\| = \|x\|$ for $x \in X$, an *isometry* if it is an isometric embedding and an epimorphism.

A Banach space X is said to be *linearly homeomorphic* (*isometric*) to a Banach space Y if there is a linear homeomorphism (an isometry) $u: X \to Y$.

The *Cartesian product* $X \times Y$ of *Banach spaces* X and Y is the space of all pairs (x, y), $x \in X$, $y \in Y$, with the usual operations of addition and multiplication by scalars; we admit $\|(x, y)\| = \max(\|x\|, \|y\|)$.

If E is a subspace of X, then X/E denotes the *quotient space* X by E; the elements of X/E are cosets modulo E, i.e. the sets $[x] = \{z \in X: z = x + e$

for some $e \in E\}$. The norm in X/E is defined by $\|[x]\| = \inf_{e \in E} \|x+e\|$. The epimorphism $x \to [x]$ is called the *quotient map*. A subspace E of a Banach space X is said to be *complemented* in X if there exists a *projection* u ($=$ a bounded linear idempotent, i.e. $u^2 = u$) from X onto E. Let us recall that E is complemented in X if and only if there exists a linear homeomorphism v from X onto the Cartesian product $X/E \times E$ such that $ve = (0, e)$ for $e \in E$.

Any unexplained terminology and notation will be that of Dunford-Schwartz [1].

Spaces of continuous functions and measure spaces. If S is a compact space, then $C(S)$ (respectively $C_R(S)$) denotes the Banach space of all continuous complex (respectively real) valued functions on S with the norm $\|f\| = \sup_{s \in S} |f(s)|$, and $M(S)$ denotes the space of all complex finite regular Borel measures on S with the norm $\|\mu\| =$ the total variation of μ on S. According to the Riesz representation theorem (Dunford-Schwartz [1], p. 265) we identify $M(S)$ with the space dual to $C(S)$. We shall employ the notation $\mu(f) = \int_S f(s)\mu(ds)$ for μ in $M(S)$ and $f \in C(S)$. A μ in $M(S)$ is said to be *non-negative* provided $\mu(f) \geqslant 0$ whenever f is *non-negative*, i.e. $f(s) \geqslant 0$ for $s \in S$. The set of all non-negative measures in $M(S)$ is called the *positive cone* of $M(S)$. If $\|\mu\| = 1$, then μ is called *normalized*. A measure μ in $M(S)$ is said to be *concentrated* on a (closed) subset F of S provided $\int_F f(s)\mu(ds) = \int_S f(s)\mu(ds)$ for every f in $C(S)$. For s in S we denote by δ_s the unit point mass at the point s, i.e. $\delta_s(f) = f(s)$ for $f \in C(S)$. By 1_S we denote the function on S which is identically one. If f and g are in $C(S)$, then $f \geqslant g$ means that $f(s) \geqslant g(s)$ for all $s \in S$.

If $\varphi\colon S \to T$ is a map, then $\varphi^\circ\colon C(T) \to C(S)$ denotes the *induced operator* defined by $\varphi^\circ(f) = f \circ \varphi$ for $f \in C(T)$. The induced operator φ° is linear and multiplicative; it is an isometric embedding (an epimorphism) if and only if φ is an epimorphism (a homeomorphic embedding) (cf. Gillman-Jerison [1], p. 141).

§ 1. REGULAR OPERATORS AND THEIR PRODUCTS

1.1. Definition. A linear operator $u\colon C(S) \to C(T)$ is said to be *regular* provided $\|u\| = 1$ and $u 1_S = 1_T$.

1.2. Proposition. *For every linear operator $u\colon C(S) \to C(T)$ the following conditions are equivalent*:

(1.2.1) *u is regular.*

(1.2.2) $u1_S = 1_T$ *and* $uf \geqslant 0$ *whenever* $f \geqslant 0$.

(1.2.3) *The adjoint operator* u^* *maps isometrically the positive cone of* $M(T)$ *into the positive cone of* $M(S)$.

(1.2.4) *For* t *in* T *and for* f *in* $C(S)$

$$(uf)(t) = \int_S f(s)\mu_t(ds),$$

where $t \to \mu_t = u^*\delta_t$ *is a continuous function from* T *into* $M(S)$ *endowed with the weak-star topology, and for each* t *in* T, μ_t *is a positive normalized measure.*

Proof. (1.2.1) $\Rightarrow$ (1.2.2). Let $0 \neq f \in C(S)$ and let $f \geqslant 0$. Then for every ε with $\|f\|^{-1} \geqslant \varepsilon > 0$ we have $\|1_S - \varepsilon f\| \leqslant 1$. The regularity of u implies that $|(u1_S - uf)(t)| = |1 - \varepsilon(uf)(t)| \leqslant 1$ for t in T and for $\|f\|^{-1} \geqslant \varepsilon > 0$. Clearly the last inequality implies that $(uf)(t) \geqslant 0$ for $t \in T$. Hence $uf \geqslant 0$ whenever $f \geqslant 0$.

(1.2.2) $\Rightarrow$ (1.2.3). Let $\nu \in M(T)$ and let $\nu \geqslant 0$. Then since the condition $f \geqslant 0$ implies $uf \geqslant 0$, we have

$$(u^*\nu)(f) = \nu(uf) \geqslant 0 \quad \text{for} \quad 0 \leqslant f \in C(S).$$

Hence $u^*\nu \geqslant 0$. Finally, for the non-negative measures ν and $u^*\nu$ we have

$$\|u^*\nu\| = (u^*\nu)(1_S) = \nu(u1_S) = \nu(1_T) = \|\nu\|.$$

This shows that u maps isometrically the positive cone of $M(T)$ into the positive cone of $M(S)$.

(1.2.3) $\Rightarrow$ (1.2.4). Let us put $\mu_t = u^*\delta_t$ for $t \in T$. Since $\delta_t \geqslant 0$ and $\|\delta_t\| = 1$, condition (1.2.3) implies that μ_t is a positive normalized measure in $M(S)$. Clearly for $f \in C(S)$ and $t \in T$ we have $(uf)(t) = \delta_t(uf) = (u^*\delta_t)(f) = \mu_t(f) = \int_S f(s)\mu_t(ds)$. Finally, since $t \to \delta_t$ is a continuous function from T into $M(T)$ equipped with the weak-star topology and since the adjoint operator $u^*\colon M(T) \to M(S)$ is continuous if both spaces $M(T)$ and $M(S)$ carry their weak-star topologies, the composed function $t \to \mu_t$ is also weak-star continuous.

(1.2.4) $\Rightarrow$ (1.2.1). Since μ_t is a continuous function on t in the weak-star topology of $M(S)$, for every $f \in C(S)$ the function $\mu_t(f)$ is continuous on T. Hence the formula $(uf)(t) = \int_S f(s)\mu_t(ds)$ for $f \in C(S)$ and $t \in T$ defines an operator from $C(S)$ into $C(T)$. Clearly u is linear and $\|u\| \leqslant \sup_t \|\mu_t\| = 1$. Since $\mu_t(1_S) = \mu_t(S) = 1$ for $t \in T$, we obtain $u1_S = 1_T$ and therefore $\|u\| \geqslant \|u1_S\| = 1$. Hence $\|u\| = 1$. That completes the proof.

1.3. PROPOSITION. *Let $(S_a)_{a\in A}$ and $(T_a)_{a\in A}$ be families of compact spaces. Let $u_a\colon C(S_a)\to C(T_a)$ be regular linear operators $(a\in A)$. Then there is the unique regular linear operator*

$$u = \otimes u_a\colon C(\boldsymbol{P}S_a) \to C(\boldsymbol{P}T_a)$$

such that for arbitrary finite subset $B\subset A$

$$u\Big(\prod_{a\in B} p_a^{\circ} f_a\Big) = \prod_{a\in B} q_a^{\circ}(u_a f_a) \qquad (f_a\in C(S_a);\ a\in B) \tag{1.3.1}$$

where p_a and q_a denote natural projections from $\boldsymbol{P}S_a$ onto S_a and from $\boldsymbol{P}T_a$ onto T_a respectively ($\prod_{a\in B} g_a$ denotes the ordinary product of functions g_a $(a\in B)$).

Proof. By (1.2.4) for every t_a in T_a $(a\in A)$ there is a positive normalized measure μ_{t_a} in $M(S_a)$ such that

$$(u_a f)(t_a) = \int_{S_a} f(s_a)\,\mu_{t_a}(ds_a) \qquad \text{for} \qquad f\in C(S_a).$$

Let us set $\mu_{(t_a)} = \bigotimes_{a\in A} \mu_{t_a}$ for $(t_a)\in\boldsymbol{P}T_a$. Clearly the product measure $\mu_{(t_a)}$ s a positive normalized measure in $M(\boldsymbol{P}S_a)$.

i Let us set

$$uf\big((t_a)\big) = \int_{\boldsymbol{P}S_a} f(s)\,\mu_{(t_a)}(ds) \qquad \text{for } f\in C(\boldsymbol{P}S_a) \text{ and for } (t_a)\in\boldsymbol{P}T_a. \tag{1.3.2}$$

Clearly u (defined by (1.3.2)) is a linear operator from $C(\boldsymbol{P}S_a)$ into the space $B(\boldsymbol{P}T_a)$ of all bounded complex valued functions on $\boldsymbol{P}T_a$ with the usual sup-norm. Moreover, $\|u\| = \sup_{(t_a)\in\boldsymbol{P}T_a} \|\mu_{(t_a)}\| = 1$. It follows from Fubini's Theorem and from the well-known properties of product measures that the operator u satisfies (1.3.1). Hence in particular $uf\in C(\boldsymbol{P}T_a)$ for each function $f\in C(\boldsymbol{P}S_a)$ of the form

$$f = \prod_{a\in B} p_a^{\circ} f_a \qquad (f_a\in C(S_a);\ a\in B;\ B \text{ is a finite subset of } A). \tag{1.3.3}$$

Let $C_0(\boldsymbol{P}S_a)$ denote the smallest linear manifold spanned by the functions of the form (1.3.3). By the linearity of u, uf is a continuous function on $\boldsymbol{P}T_a$ for every $f\in C_0(\boldsymbol{P}S_a)$. Since (by the Stone-Weierstrass Theorem) $C_0(\boldsymbol{P}S_a)$ is dense in $C(\boldsymbol{P}S_a)$ and since $C(\boldsymbol{P}T_a)$ can be regarded as a closed linear subspace of $B(\boldsymbol{P}T_a)$, the continuity of u implies that $u[C(\boldsymbol{P}S_a)]\subset C(\boldsymbol{P}T_a)$. Thus u can be treated as a linear operator from $C(\boldsymbol{P}S_a)$ into $C(\boldsymbol{P}T_a)$. Clearly u is regular and, as it has been observed, u satisfies (1.3.1).

Finally observe that (1.3.1) determines uniquely the linear operator from $C(\boldsymbol{P}S_a)$ into $C(\boldsymbol{P}T_a)$, because the functions of the form (1.3.3) span a dense linear manifold in $C(\boldsymbol{P}S_a)$.

Proposition 1.3 can be generalized to the case of inverse systems of compact spaces (cf. Eilenberg-Steenrod [1], pp. 213-220). We employ in the next proposition the terminology and notation of that treatise.

Remark. Let us observe that the Cartesian product of a family $(S_a)_{a\in A}$ of compact spaces can be regarded as the inverse limit of the system $\{S, \pi\}$ over the set M of all finite subsets of A directed by inclusion. If $B \subset M$, then $S_B = \mathop{\boldsymbol{P}}\limits_{a\in B} S_a$, and if $B' \supset B$, then $\pi_B^{B'}$ is the natural projection of $S_{B'}$ onto S_B. Therefore the proof of Proposition 1.3 can be reduced verifying the hypothesis of Proposition 1.4 that the diagrams (1.4.1) are commutative, which verification amounts to proving Proposition 1.3 for finite products.

1.4. Proposition. *Let $\{S, \pi\}$ and $\{T, \sigma\}$ be an inverse systems of compact spaces over a directed set M with inverse limits S_∞ and T_∞ respectively. Let us suppose that the limit maps, $\pi_a\colon S_\infty \to S_a$ and $\sigma_a\colon T_\infty \to T_a$, are epimorphisms ($a \in M$). Let further $u_a\colon C(S_a) \to C(T_a)$ be regular operators such that the diagram*

(1.4.1)
$$\begin{array}{ccc} C(S_a) & \xrightarrow{(\pi_a^\beta)^\circ} & C(S_\beta) \\ \downarrow{\scriptstyle u_a} & & \downarrow{\scriptstyle u_\beta} \\ C(T_a) & \xrightarrow{(\sigma_a^\beta)^\circ} & C(T_\beta) \end{array}$$

is commutative for $a < \beta$ ($a, \beta \in M$).

Then there is the unique regular linear operator

$$u = \varinjlim u_a\colon C(S_\infty) \to C(T_\infty)$$

such that the diagram

(1.4.2)
$$\begin{array}{ccc} C(S_a) & \xrightarrow{(\pi_a)^\circ} & C(S_\infty) \\ \downarrow{\scriptstyle u_a} & & \downarrow{\scriptstyle u} \\ C(T_a) & \xrightarrow{(\sigma_a)^\circ} & C(T_\infty) \end{array}$$

commutes for each $a \in M$.

Proof. Let us set

$$C_0(S_\infty) = \bigcup_{a\in M} (\pi_a)^\circ[C(S_a)].$$

Since the subalgebra $C_0(S_\infty)$ separates the points of S_∞ and with each function f contains the adjoint function $\bar{f}$, the Stone-Weierstrass Theorem implies that $C_0(S_\infty)$ is dense in $C(S_\infty)$. Since π_a are epimorphisms, the relation $\pi_a = \pi_a^\beta \pi_\beta$ implies that π_a^β are epimorphisms for $\beta > a$ $(a \in M)$. Therefore $(\pi_a)^\circ$ and $(\pi_a^\beta)^\circ$ are invertible. Moreover, if $a < \beta$, then $(\pi_a)^\circ[C(S_a)] \subset (\pi_\beta)^\circ[C(S_\beta)]$. Let $f_1 \in C(S_a)$ and let $f = (\pi_a)^\circ f_1 = (\pi_\beta)^\circ(\pi_a^\beta)^\circ f_1$. Then (1.4.1) implies

$$\begin{aligned}(\sigma_\beta)^\circ u_\beta[(\pi_\beta)^\circ]^{-1} f &= (\sigma_\beta)^\circ u_\beta[(\pi_\beta)^\circ]^{-1}(\pi_\beta)^\circ(\pi_a^\beta)^\circ f_1\\ &= (\sigma_\beta)^\circ u_\beta(\pi_a^\beta)^\circ f_1 = (\sigma_\beta)^\circ(\sigma_a^\beta)^\circ u_a f_1\\ &= (\sigma_a)^\circ u_a f_1 = (\sigma_a)^\circ u_a[(\pi_a)^\circ]^{-1} f.\end{aligned}$$

This shows that the formula

$$(1.4.3) \qquad uf = (\sigma_a)^\circ u_a[(\pi_a)^\circ]^{-1} f \quad \text{for} \quad f \in (\pi_a)^\circ[C(S_a)];\ a \in M$$

well defines a linear operator from $C_0(S_\infty)$ into $C(T_\infty)$. Since $C_0(S_\infty)$ is dense in $C(S_\infty)$ and since $\|uf\| \leqslant \|f\|$ for f in $C_0(S_\infty)$, the operator u can be uniquely extended to a linear operator from $C(S_\infty)$ into $C(T_\infty)$. Clearly such defined u is regular. Let $f_1 \in C(S_a)$. Then (1.4.3) implies

$$u(\pi_a)^\circ f_1 = (\sigma_a)^\circ u_a[(\pi_a)^\circ]^{-1}(\pi_a)^\circ f_1 = (\sigma_a)^\circ u_a f_1 \qquad (a \in M).$$

Hence u satisfies (1.4.2).

Conversely, if u satisfies (1.4.2), then for each $f \in (\pi_a)^\circ[C(S_a)]$ we have (1.4.3). Therefore u is uniquely defined on the dense subset $C_0(S_\infty)$ of $C(S_\infty)$. That completes the proof.

§ 2. EXAVES. EXTENSION AND AVERAGING OPERATORS

2.1. DEFINITION. Let $\varphi\colon S \to T$ be a map (S, T are compact spaces). A linear operator $u\colon C(S) \to C(T)$ is said to be a *linear exave* for φ provided $\varphi^\circ u$ is the identity on $\varphi^\circ[C(T)]$ or equivalently $\varphi^\circ u \varphi^\circ = \varphi^\circ$. A *regular exave* is a linear exave which is a regular operator. If φ is a homeomorphic embedding, then a linear exave (regular exave) for φ is called *linear extension operator* (*regular extension operator*). If φ is an epimorphism, then a (regular) linear exave for φ is called *linear averaging operator* (*regular averaging operator*).

A linear exave u for φ is said to be *normal* if $u\varphi^\circ u = u$. Let us observe that if u_1 is an arbitrary (regular) linear exave for φ, then $u = u_1\varphi^\circ u_1$ is a normal (regular) linear exave for φ. Linear extension and averaging operators are always normal.

We leave to the reader the simple proofs of the next proposition and corollaries.

2.2. PROPOSITION. *Let $\varphi: S \to T$ be a map. Then a linear operator $u: C(S) \to C(T)$ is a linear extension operator (averaging operator) if and only if $\varphi^\circ u = \mathrm{id}_{C(S)}$ (respectively $u\varphi^\circ = \mathrm{id}_{C(T)}$).*

2.3. COROLLARY. *An epimorphism $\varphi: S \to T$ has a linear (regular) averaging operator of norm $\leqslant \lambda$ if and only if there is a projection of norm $\leqslant \lambda$ (of norm one) from $C(S)$ onto its subspace $\varphi^\circ[C(T)]$ isometric to $C(T)$.*

2.4. COROLLARY. *If $u: C(S) \to C(T)$ is a linear (regular) extension operator for $\varphi: S \to T$, then u is a linear homeomorphism (linear isometry) from $C(S)$ onto a complemented subspace of $C(T)$.*

2.5. PROPOSITION. *Let u be a linear exave for a map $\varphi: S \to T$. Let T_1 be a closed subset of T such that $\varphi S \subset T_1$. Then $\overset{\circ}{\iota}_1 u$ is a linear exave for the map $\varphi_1: S \to T_1$, where $\varphi_1 s = \varphi s$ for $s \in S$ and ι_1 denotes the natural (identical) embedding of T_1 into T.*

Proof. Since $\iota_1\varphi_1 = \varphi$, we have $\overset{\circ}{\varphi}_1(\overset{\circ}{\iota}_1 u)\overset{\circ}{\varphi}_1 = (\iota_1\varphi_1)^\circ u\overset{\circ}{\varphi}_1 = \varphi^\circ u\overset{\circ}{\varphi}_1$. Now, let us choose g_1 in $C(T_1)$. By the Tietze-Urysohn extension theorem there is g in $C(T)$ such that $\overset{\circ}{\iota}_1 g = g_1$. Hence $\overset{\circ}{\varphi}_1 g_1 = \overset{\circ}{\varphi}_1\overset{\circ}{\iota}_1 g = (\iota_1\varphi_1)^\circ g = \varphi^\circ g$. Therefore, since $\varphi^\circ u\varphi^\circ = \varphi^\circ$, we get

$$\overset{\circ}{\varphi}_1(\overset{\circ}{\iota}_1 u)\overset{\circ}{\varphi}_1 g_1 = \varphi^\circ u\overset{\circ}{\varphi}_1 g_1 = \varphi^\circ u\varphi^\circ g = \varphi^\circ g = \overset{\circ}{\varphi}_1\overset{\circ}{\iota}_1 g = \overset{\circ}{\varphi}_1 g_1.$$

Hence $\overset{\circ}{\varphi}_1(\overset{\circ}{\iota}_1 u)\overset{\circ}{\varphi}_1 = \overset{\circ}{\varphi}_1$. That completes the proof.

If Q is a subset of T, then T/Q denotes the compact space obtained from T by identification of all points of Q. The natural epimorphism from T onto T/Q will be denoted ι/Q. Clearly $(\iota/Q)^\circ$ is an isomorphism from the algebra $C(T/Q)$ onto the subalgebra of $C(T)$ consisting of all continuous functions on T which are constant of Q. In the sequel we shall identify this subalgebra with $C(T/Q)$.

2.6. PROPOSITION. *A map $\varphi: S \to T$ admits a linear extension operator if and only if $\iota/\varphi S: T \to T/\varphi S$ admits a linear averaging operator.*

Proof. Let u be a linear extension operator for φ. Let us fix t_0 in φS and set

$$Pf = f - u\varphi^\circ f + f(t_0)u1_S \quad \text{for} \quad f \in C(T).$$

Clearly P is a projection from $C(T)$ onto $C(T/\varphi S)$. Thus, by Corollary 2.4, $\iota/\varphi S$ admits a linear averaging operator.

Conversely, if $\iota/\varphi S$ admits a linear averaging operator, then again by Corollary 2.4 there is a projection P from $C(T)$ onto $C(T/\varphi S)$. Let $\tilde{u}$ be an operator (not necessarily linear or continuous) from $C(S)$ into $C(T)$

such that $\|\tilde{u}f\| = \|f\|$ and $\varphi^{\circ}\tilde{u}f = f$ for $f \in C(S)$. The existence of $\tilde{u}$ follows from the Tietze-Urysohn extension theorem. Let us fix t_0 in φS and let us set

$$uf = \tilde{u}f - P\tilde{u}f + (P\tilde{u}f)(t_0)1_T \quad \text{for} \quad f \in C(S).$$

Since $P\tilde{u}f \in C(T/\varphi S)$, $(P\tilde{u}f)(\varphi_s) = (P\tilde{u}f)(t_0)$ for each s in S. Hence uf is an extension of f, because $\varphi^{\circ}\tilde{u}f = f$ for $f \in C(S)$. Since $\|\tilde{u}f\| = \|f\|$, the definition of u implies that

$$\|uf\| \leqslant (2\|P\|+1)\|f\| \quad \text{for} \quad f \text{ in } C(S).$$

Therefore to complete the proof that u is a linear extension operator for φ it is enough to establish the linearity of u. Since u is bounded, it is enough to show that if f_1 and f_2 are in $C(S)$, then $u(f_1+f_2) = uf_1 + uf_2$. To do this, first we note that for arbitrary g_1 in $C(T)$ if

(2.6.1) $$\varphi^{\circ}g_1 = \varphi^{\circ}g_2$$

then

(2.6.2) $$g_1 - Pg_1 + (Pg_1)(t_0)1_T = g_2 - Pg_2 + (Pg_2)(t_0)1_T.$$

Indeed, (2.6.1) implies that $(g_1 - g_2)(t) = 0$ for $t \in \varphi S$. Thus $g_1 - g_2 \in C(T/\varphi S)$, and consequently

(2.6.3) $$g_1 - g_2 = P(g_1 - g_2) = Pg_1 - Pg_2.$$

In particular

(2.6.4) $$0 = (g_1 - g_2)(t_0) = Pg_1(t_0) - Pg_2(t_0).$$

Clearly (2.6.3) and (2.6.4) imply (2.6.2). Now if $g_1 = \tilde{u}f_1 + \tilde{u}f_2$ and $g_2 = \tilde{u}(f_1+f_2)$ for f_1 and f_2 in $C(S)$, then such defined g_1 and g_2 satisfy (2.6.1) and therefore (2.6.2). But for this particular choice of g_1 and g_2, (2.6.2) together with the definition of u imply $u(f_1+f_2) = uf_1 + uf_2$. That completes the proof.

Remark. If Q is a two-point subset of T, then $C(T/Q)$ is subspace of $C(T)$ of codimension one. Hence for every $\varepsilon > 0$ there exists a projection of norm $< 2+\varepsilon$ (cf. Bohnenblust [1], Grünbaum [1], Isbell and Semadeni [1]). Thus if Q is a two-point subset of a compact space T and if $\varepsilon > 0$, then ι/Q admits a linear averaging operator of norm $< 2+\varepsilon$.

2.7. Definition. By $C(S, E)$ (S compact, E — a Banach space) we denote the space of all continuous functions from S into E with the norm $\|f\| = \sup_{s \in S} \|f(s)\|$.

A linear operator $u: C(S, E) \to C(T, E)$ is said to be an *E-valued linear exave* for a map $\varphi: S \to T$ provided

$$\overset{\circ}{\varphi}_E u \overset{\circ}{\varphi}_E = \overset{\circ}{\varphi}_E, \quad \text{where} \quad \overset{\circ}{\varphi}_E(f) = f \circ \varphi \text{ for } f \in C(T, E).$$

The next proposition shows that for a given φ an E-valued linear exave exists if and only if there exists a (scalar valued) linear exave.

2.8. Proposition. *For arbitrary map $\varphi: S \to T$ the following conditions are equivalent:*

(2.8.1) *For some Banach space E_0, φ has an E_0-valued linear exave.*

(2.8.2) *φ has a (scalar valued) linear exave.*

(2.8.3) *For every Banach space E, φ has an E-valued linear exave.*

Proof. (2.8.1) $\Rightarrow$ (2.8.2) Let $\boldsymbol{u}$ be an E_0-valued linear exave for φ. Let us choose e in E and e^* in E^* such that $\|e\| = \|e^*\| = e^*e = 1$. Let us put

$$uf = e^*\boldsymbol{u}(f\cdot e) \quad \text{for} \quad f \in C(S),$$

where $(f\cdot e)(s) = f(s)e$ for $s \in S$. Clearly u is a linear exave for φ.

(2.8.2) $\Rightarrow$ (2.8.3). Let $u: C(S) \to C(T)$ be a linear exave for φ and let E be an arbitrary Banach space. Let $C_0(S, E)$ denote the linear manifold in $C(S, E)$ consisting of all functions of the form

$$(2.8.4) \qquad \boldsymbol{g} = \sum_{i=1}^{n} f_i\cdot e_i, \qquad f_i \in C(S);\ e_i \in E \ (i = 1, 2, \ldots, n;\ n = 1, 2, \ldots).$$

We set for $\boldsymbol{g}$ defined by (2.8.4),

$$(2.8.5) \qquad \boldsymbol{ug} = \sum_{i=1}^{n} uf_i\cdot e_i.$$

It is easily seen that $\boldsymbol{ug}$ does not depend on the representation (2.8.4) of $\boldsymbol{g}$, i.e., if $\boldsymbol{g} = \sum_{i=1}^{n} f_i\cdot e_i = \sum_{j=1}^{m} f'_j\cdot e'_j$, then $\sum_{i=1}^{n} uf_i\cdot e_i = \sum_{j=1}^{m} uf'_j\cdot e'_j$.

Further we have

$$\begin{aligned}\|\boldsymbol{ug}\| &= \sup_{t\in T}\Big\|\sum_{i=1}^{n}(uf_i)(t)e_i\Big\| = \sup_{t\in T}\sup_{\|e^*\|=1}\Big|\sum_{i=1}^{n}(uf_i)(t)e^*e_i\Big| \\ &= \sup_{\|e^*\|=1}\Big\|u\Big(\sum_{i=1}^{n}e^*e_if_i\Big)\Big\| \leq \|u\|\sup_{\|e^*\|=1}\sup_{s\in S}\Big|e^*\Big(\sum_{i=1}^{n}f_i(s)e_i\Big)\Big| \\ &= \|u\|\Big\|\sum_{i=1}^{n}f_i\cdot e_i\Big\| = \|u\|\|\boldsymbol{g}\|.\end{aligned}$$

Hence (2.8.5) defines a bounded linear operator from $C_0(S, E)$ into $C(T, E)$ (actually into $C_0(T, E)$). Since $C_0(S, E)$ is dense in $C(S, E)$, $\boldsymbol{u}$ has the unique extension to a bounded linear operator from $C(S, E)$ into $C(T, E)$. It follows from the assumption that u is a linear exave for φ and from (2.8.4) that this extension is the E-valued linear exave for φ.

(2.8.3) $\Rightarrow$ (2.8.1). This implication is trivial.

Remark. In terms of tensor product the construction used in the proof of the implication (2.8.2) $\Rightarrow$ (2.8.3) can be described as follows. If E is a Banach space and S is compact, then $C(S, E)$ can be identified with the weak tensor product $C(S)\hat{\hat{\otimes}} E$ (cf. Grothendieck [1], Chap. I, p. 90). Thus, if u is a linear exave for $\varphi\colon S \to T$, then $\mathbf{u} = u\hat{\hat{\otimes}}\,\mathrm{id}_E$, where id_E denotes the identity on E, is an E-valued linear exave for φ (for the definition of $v_1\hat{\hat{\otimes}}\, v_2$ — the weak tensor product of linear operators v_1 and v_2, cf. Grothendieck [1], Chap. I, p. 93).

The next proposition shows that there is no essential difference between the "real" and "complex theory" of linear exaves. By $C_R(S)$ we denote the real Banach space of all real valued continuous functions on S.

2.9. Proposition. *Let $c \geqslant 1$. Then for arbitrary map $\varphi\colon S \to T$ the following conditions are equivalent:*

(2.9.1) *There exists a linear exave $u\colon C(S) \to C(T)$ for φ with $\|u\| \leqslant c$.*

(2.9.2) *There exists a real linear exave $u_R\colon C_R(S) \to C_R(T)$ for φ with $\|u_R\| \leqslant c$.*

Moreover, u is regular if and only if u_R is regular.

Proof. (2.9.1) $\Rightarrow$ (2.9.2). We put

$$(u_Rf)(t) = \mathrm{re}\,[(uf)(t)] \quad \text{for} \quad f\in C_R(S).$$

Clearly u_R is a linear exave for φ and $\|u_R\| \leqslant \|u\|$.

(2.9.2) $\Rightarrow$ (2.9.1). We put

$$uf = u_R(\mathrm{re} f) - iu_R(\mathrm{re}(if)) \quad \text{for} \quad f\in C(S).$$

Using literally the same arguments as in the Bohnenblust-Sobczyk proof of the "complex" Hahn-Banach extension theorem (cf. Dunford and Schwartz [1], p. 63) we verify that u is a complex linear operator from $C(S)$ into $C(T)$ with $\|u\| = \|u_R\|$. Moreover, if $f\in C_R(S)$, then $uf = u_Rf$. Finally since u_R is a real exave for φ, i.e. $\varphi^\circ u_R\varphi^\circ = \varphi^\circ$, we get $\varphi^\circ u\varphi^\circ = \varphi^\circ$. Hence u is a complex linear exave for φ.

For regular exaves we have a statement slightly stronger than Proposition 2.9.

2.10. Definition. Let $C_+(S)$ denote the cone of all non-negative functions in $C(S)$. An operator $v\colon C_+(S) \to C_+(T)$ is said to be an *affine exave* for a map $\varphi\colon S \to T$ if v is continuous, and $v(a_1f_1 + a_2f_2) = a_1vf_1 + a_2vf_2$ for $a_1, a_2 \geqslant 0$ and for f_1, f_2 in $C_+(S)$, and $(\varphi^\circ v\varphi^\circ)(f) = \varphi^\circ f$ for $f\in C_+(S)$.

2.11. Proposition. *A map $\varphi\colon S \to T$ has a regular exave if and only if there is an affine exave for φ.*

Proof. Clearly the restriction of a regular exave to the cone of non-negative functions is an affine exave (cf. Proposition 1.2).

Conversely, let $v: C_+(S) \to C_+(T)$ be an affine exave for φ. Let us fix s_0 in S and let $U = \{t \in T: (v1_s)(t) > 2^{-1}\}$. Then U is open and $U \supset \varphi S$. Choose λ in $C_+(T)$ with $0 \leqslant \lambda \leqslant 1$ such that $\lambda^{-1}(1) \supset \varphi S$ and $\lambda^{-1}(0) \supset T \setminus U$. Clearly $\lambda \circ \varphi = 1_S$ and $(1_T - \lambda) \circ \varphi = 0$. Thus $v_1: C_+(S) \to C_+(T)$ defined by

$$v_1 f = vf + (1_T - \lambda) f(s_0) \quad \text{for} \quad f \in C_+(S)$$

is an affine exave for φ. We have $(v_1 1_S)(t) \geqslant 4^{-1}$ for $t \in T$. Indeed, if $\lambda(t) > 2^{-1}$, then $t \in U$ and $(v1_S)(t) > 2^{-1}$. Therefore $(v_1 1_S)(t) \geqslant \lambda(t)(v1_S)(t) \geqslant 4^{-1}$. If $\lambda(t) \leqslant 2^{-1}$, then $(v_1 1_S)(t) \geqslant (1_T - \lambda)(t) \cdot 1_S(s_0) \geqslant 2^{-1}$. Let us set

$$v_2 f = \frac{v_1 f}{v_1 1_S} \quad \text{for} \quad f \in C_+(S).$$

Since $v_1 1_S$ is a positive function on T and since v_1 is an affine exave for φ, it is easily seen that v_2: $C_+(S) \to C_+(T)$ is a well defined affine exave for φ. Clearly $v_2 1_S = 1_T$. We extend v_2 on $C_R(S)$ in the standard way. We set

$$ug = v_2 g^+ - v_2 g^- \quad \text{for} \quad g \in C_R(S)$$

where $g^+ = \max(g, 0)$ and $g^- = (-g)^+$.

Clearly $uf = v_2 f$ for $f \in C_+(S)$ and $uag = aug$ for every real a and for $g \in C_R(S)$. To prove the additivity of u first observe that if $g = f_1 - f_2$ for some f_1 and f_2 in $C_+(S)$, then $ug = uf_1 - uf_2 = v_2 f_1 - v_2 f_2$. Indeed, since $g = g^+ - g^- = f_1 - f_2$, we get $g^+ + f_2 = g^- + f_1$. Thus, since v_2 is an affine operator,

$$v_2(g^+ + f_2) = v_2 g^+ + v_2 f_2 = v_2 g^- + v_2 f_1 = v_2(g^- + f_1).$$

Hence $ug = v_2 g^+ - v_2 g^- = v_2 f_1 - v_2 f_2$. Now, let g_1 and g_2 be in $C_R(S)$. We have

$$g_1 + g_2 = g_1^+ - g_1^- + g_2^+ - g_2^- = (g_1^+ + g_2^+) - (g_1^- + g_2^-).$$

Since u is an extension of v_2 and since v_2 is additive on the cone of non-negative functions, the preceding remark implies

$$\begin{aligned} u(g_1 + g_2) &= u(g_1^+ + g_2^+) - u(g_1^- + g_2^-) = ug_1^+ + ug_2^+ - ug_1^- - ug_2^- \\ &= ug_1^+ - ug_1^- + ug_2^+ - ug_2^- = ug_1 + ug_2. \end{aligned}$$

This shows that u is additive. Finally, since u takes the positive cone of $C_+(S)$ into the positive cone of $C_+(T)$, for each $g \in C_R(S)$ with $\|g\| \leqslant 1$ we have $1_S \pm g \in C_+(S)$. Thus $u(1_S \pm g) = u1_S \pm ug = v_2 1_S \pm ug = 1_T \pm \pm ug \in C_+(T)$. Hence $\|ug\| \leqslant 1$. This shows that $\|u\| \leqslant 1$. Clearly, by the respective property of v_2, we have $\varphi^\circ u \varphi^\circ = \varphi^\circ$. Thus u is a regular real exave for φ from $C_R(S)$ into $C_R(T)$. To complete the proof we apply Proposition 2.9.

§ 3. LINEAR MULTIPLICATIVE EXAVES AND RETRACTIONS. LOCALIZATION PRINCIPLE

3.1. Definition. A map $\varphi: S \to T$ is a *coretraction* corresponding to a *retraction* $r: T \to S$ provided $r\varphi = \mathrm{id}_S$. Clearly coretractions are homeomorphic embeddings and retractions are epimorphisms.

A *neighbourhood coretraction* is a map $\varphi: S \to T$ such that for each s in S there is a closed neighbourhood of s, say V, such that $\varphi_V: V \to T$ is a coretraction, where φ_V denotes the restriction of φ to V.

We recall that a compact space S is said to be an *absolute retract* (*a neighbourhood absolute retract*) if $\varphi: S \to T$ is a coretraction (a neighbourhood corectraction) for every homeomorphic embedding φ of S into an arbitrary compact space T.

3.2. Definition. A *regular linear-multiplicative exave* for a map $\varphi: S \to T$ is a regular linear exave for φ, say $u: C(S) \to C(T)$, such that $u(f_1 f_2) = uf_1 \cdot uf_2$ for f_1, f_2 in $C(S)$.

3.3. Proposition. *The following conditions are equivalent:*

(3.3.1) *$\varphi: S \to T$ is a coretraction.*

(3.3.2) *There is a regular linear-multiplicative extension operator for φ.*

Proof. (3.3.1) $\Rightarrow$ (3.3.2). Let $r: T \to S$ be a retraction corresponding to φ. Then $u = r^\circ$ is a regular linear-multiplicative extension operator for φ.

(3.3.2) $\Rightarrow$ (3.3.1). Let u be a regular linear-multiplicative extension operator for $\varphi: S \to T$. Then u is an isomorphism of the algebra $C(S)$ into $C(T)$ such that $u1_S = 1_T$. Hence (cf. Gillman and Jerison [1], p. 141) there is a map $r: T \to S$ such that $u = r^\circ$. Since u is a linear extension operator, $\varphi^\circ u = \mathrm{id}_{C(S)}$. Therefore $\varphi^\circ r^\circ = (r\varphi)^\circ = (\mathrm{id}_S)^\circ$. Thus $r\varphi = \mathrm{id}_S$. That completes the proof.

3.4. Proposition. *The following conditions are equivalent:*

(3.4.1) *r is a retraction.*

(3.4.2) *There is a linear-multiplicative averaging operator for r.*

The proof is analogous to that of Proposition 3.3.

3.5. Proposition. *If φ is a neighbourhood coretraction, then there is a regular extension operator for φ.*

This proposition is an immediate consequence of Proposition 3.3 and the following lemma.

3.6. Lemma (Localization Principle). *Let $\varphi: S \to T$. Let $\{T_a\}_{a\in A}$ be a family of closed subsets of T the interiors of which cover φS. Let $S_a = \varphi^{-1}(T_a \cap \varphi S)$ and let $\varphi_a: S_a \to T_a$ denote the restriction of φ to $S_a (a \in A)$.*

Let us assume that for each $a \in A$ there exists a (regular) linear exave u_a for φ_a. Then there is a (regular) linear exave for φ.

Proof. Since T is compact, there is a finite subfamily $\{T_{a_i}\}_{1 \leq i \leq N}$ of $\{T_a\}_{a \in A}$ such that $\bigcup_{i=1}^{N} U_i \supset \varphi S$ where U_i denotes the interior of $T_{a_i} = T_i$. Let us set $S_i = S_{a_i}$, $\varphi_i = \varphi_{a_i}$ and $u_i = u_{a_i}$ $(i = 1, 2, \ldots, N)$. Let U_0 be an open subset of T such that $T \setminus \varphi S \supset U_0 \supset T \setminus \bigcup_{i=1}^{N} U_i$. Then the family $\{U_i\}_{0 \leq i \leq N}$ is an open cover for T. Let $\{\lambda_i\}_{0 \leq i \leq N}$ be a partition of unity such that λ_i vanishes outside U_i for $i = 1, 2, \ldots, N$. Let s_0 be a fixed point in S. Let us set

$$uf = f(s_0)\lambda_0 + \sum_{i=1}^{N} \lambda_i u_i f \quad \text{for} \quad f \in C(S) \tag{3.6.1}$$

where $(u_i f)(t) = 0$ for $t \in T \setminus T_i$ and $(u_i f)(t) = u_i f_i(t)$ for $t \in T_i$ (f_i denotes the restriction of f to S_i). Clearly $\lambda_i u_i f \in C(T)$ for every f in $C(S)$ and for $i = 1, 2, \ldots, N$. Furthermore $\varphi^0 \lambda_0 = 0$, because $\{t \in T: \lambda_0(t) \neq 0\} \subset U_0 \subset T \setminus \varphi S$. Thus

$$\varphi^\circ\Big(\sum_{i=0}^{N} \lambda_i\Big) = \sum_{i=1}^{N} \varphi^\circ \lambda_i = 1_S. \tag{3.6.2}$$

Let $f \in \varphi^\circ[C(T)]$. Then $f_i \in \varphi_i^\circ[C(T_i)]$. Hence $(\varphi_i^\circ u_i f_i)(s) = f_i(s) = f(s)$ for every s in S_i, because u_i is a linear exave for φ_i $(i = 1, 2, \ldots, N)$. Since $\{t \in T: \lambda_i(t) \neq 0\} \subset U_i \subset T_i$, we get $\varphi^\circ(\lambda_i u_i f)(s) = \lambda_i(\varphi s) \cdot f(s)$ for every s in S and for $i = 1, 2, \ldots, N$. Hence, (3.6.1) and (3.6.2) imply, for each s in S,

$$(\varphi^\circ uf)(s) = \big[\varphi^\circ\big(f(s_0)\lambda_0\big)\big](s) + \Big[\varphi^\circ\Big(\sum_{i=1}^{N} \lambda_i u_i f\Big)\Big](s) = \sum_{i=1}^{N} \lambda_i(\varphi s) \cdot f(s) = f(s).$$

Therefore u is a linear exave for φ.

If all u_i are regular, then $|(\lambda_i u_i f)(t)| \leq \lambda_i(t)\|u_i f_i\| \leq \lambda_i(t)\|f\|$ (for $t \in T$ and for $f \in C(S)$) and $\lambda_i u_i 1_S = \lambda_i$ $(i = 0, 1, \ldots, N)$. Thus $|(uf)(t)| \leq \sum_{i=0}^{N} \lambda_i(t)\|f\| = \|f\|$ for $t \in T$ and for $f \in C(S)$, and $u 1_S = \sum_{i=0}^{N} \lambda_i = 1_T$. Hence u is regular.

§ 4. INTEGRAL REPRESENTATIONS AND COMPOSITIONS OF LINEAR EXAVES

4.1. Proposition. *Let $u: C(S) \to C(T)$ be a regular exave for a map $\varphi: S \to T$. Let $u^*: M(T) \to M(S)$ be the adjoint operator to u and let δ_t denote the unit point-mass at t $(t \in T)$. Then*

$$(uf)(t) = \int_S f(s)\mu_t(ds) \quad \textit{for } t \in T \textit{ and for } f \in C(S), \tag{4.1.1}$$

where

$$\mu_t = u^* \delta_t \quad \textit{for} \quad t \in T, \tag{4.1.2}$$

and the following conditions are satisfied:

(i) $t \to \mu_t$ *is a continuous function from* T *into* $M(S)$ *endowed with the weak-star topology and* μ_t *is a non-negative normalized measure in* $M(S)$ *for each* t *in* T.

(ii) *if* $t \in \varphi S$, *then* μ_t *is concentrated on the set* $\varphi^{-1}(t)$, *i.e.*

$$(uf)(t) = \int_{\varphi^{-1}(t)} f(s)\mu_t(ds) \quad \textit{for } t \in \varphi S \textit{ and } f \in C(S). \tag{4.1.3}$$

Conversely any function $t \to \mu_t$ *satisfying* (i) *and* (ii) *uniquely determines by* (4.1.1) *a regular exave for* φ *such that* (4.1.2) *holds.*

Proof. It $u : C(S) \to C(T)$ is a regular operator, then, by Proposition 1.2, it has the representation (4.1.1) with μ_t defined by (4.1.2) and satisfying (i). To verify (ii) let us fix t in φS. Let F be a closed subset of S disjoint with $\varphi^{-1}(t)$. Then $t \notin \varphi(F) = F_1$ and, by Urysohn's Lemma, there exists a non-negative function g in $C(T)$ such that $g^{-1}(1) \supset \varphi F$ and $g(t) = 0$. Let $f = \varphi^\circ g$. Since $\varphi^\circ u \varphi^\circ = \varphi^\circ$, for $s \in \varphi^{-1}(t)$ we obtain

$$(uf)(t) = (uf)(\varphi s) = (u\varphi^\circ g)(\varphi s) = (\varphi^\circ u \varphi^\circ g)(s) = (\varphi^\circ g)(s) = g(\varphi s) = g(t) = 0.$$

Clearly $f \geqslant 0$ and $f(s) = 1$ for $s \in F \subset \varphi^{-1}(F_1)$. Thus since μ_t is a non-negative measure, representation (4.1.1) implies

$$0 \leqslant \mu_t(F) \leqslant \int_S f(s)\mu_t(ds) = (uf)(t) = 0.$$

Hence $\mu_t(F) = 0$ for arbitrary closed subset F of S disjoint with $\varphi^{-1}(t)$. Therefore μ_t is concentrated on $\varphi^{-1}(t)$.

Conversely, if a function $t \to \mu_t$ satisfies (i) and if u is defined by (4.1.1), then (by Proposition 1.2) u is a regular operator from $C(S)$ into $C(T)$ and $\mu_t = u^* \delta_t$ for $t \in T$.

Finally, let μ_t satisfy (ii). Let us fix $s \in S$ and let $t = \varphi s$. Then, by (i), (ii) and (4.1.3), for arbitrary g in $C(T)$ we have:

$$\begin{aligned}(\varphi^\circ u \varphi^\circ g)(s) &= (u\varphi^\circ g)(\varphi s) = \int_{\varphi^{-1}(t)} g(\varphi s)\mu_t(ds) = \int_{\varphi^{-1}(t)} g(t)\mu_t(ds) \\ &= g(t) \int_{\varphi^{-1}(t)} \mu_t(ds) = g(t) = (\varphi^\circ g)(s).\end{aligned}$$

Hence $\varphi^\circ u \varphi^\circ = \varphi^\circ$. This completes the proof that u is a regular exave for φ.

Remark. One can easily show that every linear operator $u : C(S) \to C(T)$ has the representation (4.1.1) where $\mu_t = u^* \delta_t$. If u is a linear exave for a map $\varphi : S \to T$, then the carrier of μ_t contains $\varphi^{-1}(t)$, but in general these two sets do not coincide (cf. Example 2 in Notes and Remarks). However, in the case of linear extension operators we have

4.2. PROPOSITION. *Let $\varphi: S \to T$ be a homeomorphic embedding. Let $u : C(S) \to C(T)$ be a linear extension operator for φ. Let $\mu_t = u^* \delta_t$ for $t \in T$. Then u has the representation* (4.1.1) *and the function $t \to \mu_t$ from T into $M(S)$ endowed with the weak-star topology is continuous and*

(iii) $\mu_{\varphi s} = \delta_s$ *for* $s \in S$.

Conversely, if $t \to \mu_t$ is a continuous function from T into $M(S)$ endowed with the weak-star topology and satisfying (iii), *then there is the unique linear extension operator $u : C(S) \to C(T)$ for φ such that the formulas* (4.1.1) *and* (4.1.2) *hold.*

Proof. For arbitrary linear operator $u : C(S) \to C(T)$ the function $t \to \mu_t = u^* \delta_t$ from T into $M(S)$ endowed with the weak-star topology is continuous (cf. the proof of Proposition 1.2, the implication (1.2.3) $\Rightarrow$ (1.2.4)). Clearly u has the representation (4.1.1). Now, if u is a linear extension operator for φ, then by (4.1.1) for s in S and for f in $C(S)$

$$\mu_{\varphi s}(f) = (u^* \delta_{\varphi s})(f) = \delta_{\varphi s}(uf) = uf(\varphi s) = f(s).$$

This proves (iii).

The second part of the proposition is obvious.

We are now ready to prove the basic facts on compositions and product of linear exaves. The results are much more complete for regular exaves than for arbitrary exaves.

4.3. PROPOSITION. *Let v and u be regular exaves for maps $\varphi_1 : Q \to S$ and $\varphi : S \to T$ respectively. Let $\varphi_1 Q = \varphi^{-1} T_0$ for some subset T_0 of T. Then $w = u \circ v$ is a regular exave for $\psi = \varphi \circ \varphi_1$.*

Proof. Let us set $\mu_t = u^* \delta_t$ for $t \in T$ and $\nu_s = v^* \delta_s$ for $s \in S$. Let $f = \psi^\circ g$ for some $g \in C(T)$. Then for any fixed t in ψQ

$$f(q) = g(t) = c \quad \text{for every} \quad q \in \psi^{-1}(t).$$

Clearly the assumption $\varphi_1 Q = \varphi^{-1} T_0$ implies that $T_0 = \psi Q$, and implies that if $s \in \varphi^{-1}(t)$, then $\varphi_1^{-1}(s) \subset \psi^{-1}(t)$. Hence, by Proposition 4.1,

$$(vf)(s) = \int_{\varphi_1^{-1}(s)} f(q) \nu_s(dq) = c \quad \text{for} \quad s \in \varphi^{-1}(t).$$

Again applying Proposition 4.1 we obtain

$$(wf)(t) = \int_{\varphi^{-1}(t)} vf(s) \mu_t(ds) = c.$$

Since $f = \psi^\circ g$ and since $c = g(t)$, the last formula means that $(w\psi^\circ g)(t) = g(t)$ for $t \in \psi Q$ and for $g \in C(T)$. Thus if $q \in Q$, then

$$(\psi^\circ g)(q) = g(\psi q) = (w\psi^\circ g)(\psi q) = (\psi^\circ w\psi^\circ)(q) \quad \text{for} \quad g \in C(T).$$

Hence $\psi^\circ w \psi^\circ = \psi^\circ$. Thus w is a linear exave for ψ. The regularity of w is trivial.

Examples 3 and 4 in Notes and Remarks show that the assumptions of Proposition 4.3 (the regularity of u and v and the condition $\varphi_1 Q = \varphi^{-1} T_0$) are in general essential. The next proposition shows however that in certain cases these assumptions can be dropped.

4.4. PROPOSITION. *Let v and u be linear exaves for maps $\varphi_1 : Q \to S$ and $\varphi : S \to T$ respectively. Let either* (a) *v be a linear averaging operator, or* (b) *u be a linear extension operator, then $w = u \circ v$ is a linear exave for $\psi = \varphi\varphi_1$.*

Proof. (a) implies that $v\varphi_1^\circ = \mathrm{id}_{C(S)}$. Thus, since $\psi^\circ = \varphi_1^\circ \varphi^\circ$, we get $w\psi^\circ = uv\varphi_1^\circ\varphi^\circ = u\varphi^\circ$. Since $\varphi^\circ u \varphi^\circ = \varphi^\circ$, we obtain:

$$\psi^\circ w \psi^\circ = \varphi_1^\circ \varphi^\circ u \varphi^\circ = \varphi_1^\circ \varphi^\circ = \psi^\circ.$$

(b) implies $\varphi^\circ u = \mathrm{id}_{C(S)}$. Hence $\psi^\circ w = \varphi_1^\circ \varphi^\circ u v = \varphi_1^\circ v$. Thus, since $\varphi_1^\circ v \varphi_1^\circ = \varphi_1^\circ$,

$$\psi^\circ w \psi^\circ = \varphi_1^\circ v \varphi_1^\circ \varphi^\circ = \varphi_1^\circ \varphi^\circ = \psi^\circ.$$

Hence in both cases w is a linear exave for ψ.

Example 3 in Notes and Remarks shows that the composition of a linear exave with a linear extension operator, as well as with a linear averaging operator, taken in the opposite order as in Proposition 4.4, is not in general a linear exave. However, we have:

4.5. PROPOSITION. *Let u be a (regular) linear exave for a map $\varphi : S \to T$ and let $\varphi_1 : S_1 \to S$ be a coretraction corresponding to a retraction $r : S \to S_1$. If $r^\circ[C(S_1)] \supset \varphi^\circ[C(T)]$, then ur° is a (regular) linear exave for $\varphi\varphi_1$.*

Proof. By the assumption for every g in $C(T)$ there is an $f \in C(S_1)$ such that $\varphi^\circ g = r^\circ f$. Since $\varphi_1^\circ r^\circ = \mathrm{id}_{C(S_1)}$ and since $\varphi^\circ u \varphi^\circ = \varphi^\circ$, for arbitrary g in $C(T)$ we have

$$(\varphi\varphi_1)^\circ u r^\circ (\varphi\varphi_1)^\circ g = \varphi_1^\circ \varphi^\circ u r^\circ \varphi_1^\circ \varphi^\circ g = \varphi_1^\circ \varphi^\circ u r^\circ \varphi_1^\circ r^\circ f = \varphi_1^\circ \varphi^\circ u r^\circ f$$
$$= \varphi_1^\circ \varphi^\circ u \varphi^\circ g = \varphi_1^\circ \varphi^\circ g = (\varphi\varphi_1)^\circ g.$$

Hence ur° is a linear exave for $\varphi\varphi_1$. Clearly, if u is regular, then ur° has the same property.

4.6. Proposition. *Let*

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ \uparrow{\scriptstyle\psi_1} & & \uparrow{\scriptstyle\psi_2} \\ S_0 & \xrightarrow{\varphi_0} & T_0 \end{array}$$

be a commutative diagram, where ψ_1 and ψ_2 are homeomorphic embeddings, φ is an epimorphism and $\psi_1 S_0 = \varphi^{-1}(\psi_2 T_0)$. If there exist a regular extension operator u_1 for ψ_1 and a regular averaging operator v for φ, then there exist a regular averaging operator v_0 for φ_0 and a regular extension operator u_2 for ψ_2.

Proof. Let us set

$$v_0 = \psi_2^\circ v u_1, \qquad u_2 = v u_1 \varphi_0^\circ.$$

Clearly v_0 and u_2 are regular operators. Since v and u_1 are regular operators and since $\psi_1 S_0 = \varphi^{-1}(\psi_2 T_0)$, if follows from Proposition 4.3 that vu_1 is a regular exave for $\varphi\psi_1$. Since ψ_2 is a homeomorphic embedding and since $(\varphi\psi_1)S_0 = \psi_2 T_0$ and $\varphi_0\psi_2 = \varphi\psi_1$, Proposition 2.5 implies that $v_0 = \psi_2^\circ v u_1$ is a regular exave for φ_0. Since $\varphi_0 S_0 = T_0$, v_0 is a regular averaging operator for φ_0. Hence $v_0\varphi_0^\circ = \mathrm{id}_{C(S_0)}$. Finally we have

$$\psi_2^\circ u_2 \psi_2^\circ = \psi_2^\circ v u_1 \varphi_0^\circ \psi_2^\circ = v_0 \varphi_0^\circ \psi_2^\circ = \psi_2^\circ.$$

Hence u_2 is a regular exave for ψ_2. Since ψ_2 is a homeomorphic embedding, u_2 is a regular extension operator. That completes the proof.

Remark. In general $vu_1 \neq u_2 v_0$, but we have the identity

$$u_2 v_0 = v u_1 (\psi_2\varphi_0)^\circ v u_1.$$

Thus, if the exave vu_1 is normal (cf. Definition 2.1), then $vu_1 = u_2 v_0$.

4.7. Proposition. *Let $\{\varphi_a : S_a \to T_a\}_{a\in A}$ be a family of maps. Let $u_a : C(S_a) \to C(T_a)$ be a regular exave for φ_a $(a \in A)$.*

Then $u = \otimes u_a$ is a regular exave for the map $\boldsymbol{P}\varphi_a : \boldsymbol{P}S_a \to \boldsymbol{P}T_a$.

Moreover, if all u_a are regular extension operators resp. regular averaging operators, then u has the same property.

Proof. Let $\varphi = \boldsymbol{P}\varphi_a$. By the definition of φ, if $p_a : \boldsymbol{P}S_a \to S_a$ and $q_a : \boldsymbol{P}T_a \to T_a$ denote the natural projections, then

$$\varphi_a p_a = q_a \varphi \quad \text{for} \quad a \in A. \tag{4.7.1}$$

Since u and φ° are linear operators, to verify the identity $\varphi^\circ u \varphi^\circ = \varphi^\circ$ it is enough to show that $\varphi^\circ u \varphi^\circ g = \varphi^\circ g$ for g in a linearly dense subset G of $C(T)$. Let G be the set of all functions g of the form

$$g = \prod_{a\in B} q_a^\circ g_a \qquad (g_a \in C(T_a);\ a \in B,\ B \text{ is a finite subset of } A).$$

Using the fact that φ° is a linear-multiplicative operator and applying formulas (1.3.1) of Proposition 1.3 and (4.7.1), we obtain

$$\varphi^\circ u\varphi^\circ g = \varphi^\circ u\prod_{a\in B}\varphi^\circ \mathring{q}_a g_a = \varphi^\circ u\prod_{a\in B}\mathring{p}_a\mathring{\varphi}_a g_a = \varphi^\circ \prod_{a\in B}\mathring{q}_a u_a\mathring{\varphi}_a g_a = \prod_{a\in B}\varphi^\circ \mathring{q}_a u_a\mathring{\varphi}_a g_a$$
$$= \prod_{a\in B}\mathring{p}_a\mathring{\varphi}_a u_a\mathring{\varphi}_a g_a = \prod_{a\in A}\mathring{p}_a\mathring{\varphi}_a g_a = \prod_{a\in B}\varphi^\circ \mathring{q}_a g_a = \varphi^\circ\prod_{a\in B}\mathring{q}_a g_a = \varphi^\circ g.$$

This completes the proof that u is a regular exave for φ.

The second part of the Proposition follows from the fact that if φ_a are homeomorphic embeddings (resp. epimorphisms) for all $a\in A$, then φ has the same property.

§ 5. MILUTIN SPACES

We recall that a compact space is called *dyadic* if it is a continuous image of a generalized Cantor set $D^{\mathfrak{m}}$.

This section is devoted to study some subclass of the class of all dyadic spaces.

5.1. Definition. A *Milutin space* (resp. an *almost Milutin space*) is a compact space T such that there exists an epimorphism $\varphi: D^{\mathfrak{m}} \to T$ which has a regular averaging operator (a linear averaging operator).

The next proposition is a modification of a result of Šanin [1].

5.2. Proposition. *If T is a Milutin space (almost Milutin space) of weight $\mathfrak{n}$, then there exists a regular averaging operator (a linear averaging operator) for an epimorphism $\varphi: D^{\mathfrak{n}} \to T$.*

Proof. The case where $\mathfrak{n}$ is finite is trivial. Let us suppose that $\mathfrak{n}$ is infinite. Let $\varphi: D^{\mathfrak{m}} \to T$ be an epimorphism. Clearly $\mathfrak{m} \geqslant \mathfrak{n}$. Then (cf. Engelking and Pełczyński [1], Theorem 1) there is a coretraction $\varphi_1: D^{\mathfrak{n}} \to D^{\mathfrak{m}}$ such that $r^\circ[C(D^{\mathfrak{n}})] \supset \varphi^\circ[C(T)]$, where $r^\circ: D^{\mathfrak{m}} \to D^{\mathfrak{n}}$ is a retraction corresponded to φ_1. Hence, by Proposition 4.5, if u is a regular averaging operator (a linear averaging operator) for φ, then ur° is a regular averaging operator (a linear averaging operator) for $\psi = \varphi\varphi_1$.

For sake of completness we describe the construction of the coretraction $\varphi_1: D^{\mathfrak{n}} \to D^{\mathfrak{m}}$. Let $D^{\mathfrak{m}} = \underset{a\in A}{\boldsymbol{P}} D_a$ where $\bar{\bar{A}} = \mathfrak{m}$ and D_a is a two point space $(a\in A)$. If B is a subset of A, then $p_B: \underset{a\in A}{\boldsymbol{P}} D_a \to \underset{a\in B}{\boldsymbol{P}} D_a$ denotes the natural projection. Clearly p_B is a retraction which corresponds to a coretraction $\varphi_B: \underset{a\in B}{\boldsymbol{P}} D_a \to \underset{a\in A}{\boldsymbol{P}} D_a$ such that for $\eta = (\eta_a)\in \underset{a\in B}{\boldsymbol{P}} D_a$, $\varphi_B\eta = (\xi_a)_{a\in A}$ where $\xi_a = \eta_a$ for $a\in B$ and $\xi_a = 0$ for $a\in A\setminus B$. If $f\in C(D^{\mathfrak{m}})$, then $B_f = \bigcap \{B \mid f\in p_B^\circ(C(\underset{a\in B}{\boldsymbol{P}} D_a))\}$ is the smallest set of coordinates on which f

"essentially depends", i.e. if $\xi_a = \xi'_a$ for $a \in B_f$, then $f(\xi) = f(\xi')$ for $\xi = (\xi_a)$ and $\xi' = (\xi'_a)$ in $D^{\mathfrak{m}}$. Since (by the Stone-Weierstrass theorem) the set of all functions depending on a finite number of coordinates is dense in $C(D^{\mathfrak{m}})$, B_f is at most countable for each f in $C(D^{\mathfrak{m}})$. Since the weight of T is equal to the density character of $C(T)$ and since $\varphi^\circ[C(T)]$ is isometric to $C(T)$ (because φ is an epimorphism), there is in $\varphi^\circ[C(T)]$ a dense subset, say W, such that $\overline{\overline{W}} = \mathfrak{n}$. Let $A_0 = \bigcup_{f \in W} B_f$. Clearly $\overline{\overline{A}} = \mathfrak{n}$ and $B_f \subset A_0$ for each $f \in \varphi^\circ[C(T)]$. Hence

$$(*) \qquad \varphi^\circ[C(T)] \subset p^\circ_{A_0}[C(D^{\mathfrak{n}})] \quad \text{where} \quad D^{\mathfrak{n}} = \mathop{\boldsymbol{P}}_{a \in A_0} D_a.$$

We put $\varphi_1 = \varphi_{A_0}$. Since φ_1° restricted to $p^\circ_{A_0}[C(D^{\mathfrak{n}})]$ is an algebraic isomorphism, it follows from (*) that $(\varphi\varphi_1)^\circ = \varphi_1^\circ\varphi^\circ$ is an algebraic isomorphism from $C(T)$ into $C(D^{\mathfrak{n}})$, equivalently $\varphi\varphi_1(D^{\mathfrak{n}}) = T$.

5.3. PROPOSITION. *The Cartesian product of an arbitrary family of Milutin spaces is a Milutin space. More precisely, if $\{\varphi_a : D^{\mathfrak{m}_a} \to T_a\}_{a \in A}$ is a family of epimorphisms and if u_a is a regular averaging operator for φ_a $(a \in A)$, then $\otimes u_a$ is a regular averaging operator for the epimorphism $\boldsymbol{P}\varphi_a : \boldsymbol{P}D^{\mathfrak{m}_a} \to \boldsymbol{P}T_a$.*

Proof. This is an immediate consequence of Proposition 4.7 and of the fact that $\mathop{\boldsymbol{P}}_{a \in A} D^{\mathfrak{m}_a}$ is homeomorphic to $D^{\mathfrak{m}}$ where $\mathfrak{m} = \max(\overline{\overline{A}}, \sup_{a \in A} \mathfrak{m}_a)$.

We shall say that a compact space T is *locally Milutin* (*locally almost Milutin*) if each t in T has a closed neighbourhood which is a Milutin space (an almost Milutin space).

5.4. PROPOSITION. *Every locally Milutin space (locally almost Milutin space) is a Milutin space (an almost Milutin space).*

Proof. By the standard arguments the proof reduces to verify that if T is a compact space with the property that there is a finite open cover $\{U_i\}_{i=1}^N$ of T such that the closure T_i of each U_i is a Milutin space (an almost Milutin space) for $i = 1, 2, \ldots, N$, then T is a Milutin space (an almost Milutin space).

Let the epimorphisms $\varphi_i : D^{\mathfrak{m}_i} \to T_i$ admit regular averaging operators (linear averaging operators). Let S denote the discrete sum of spaces $D^{\mathfrak{m}_i}$ and let $\varphi : S \to T$ be the map induced by φ_i, i.e.

$$\varphi(x) = \varphi_i(x) \quad \text{for} \quad x \in D^{\mathfrak{m}_i} \subset S \ (i = 1, 2, \ldots, N).$$

Since $\bigcup_{i=1}^N T_i = T$ and since $\varphi_i(D^{\mathfrak{m}_i}) = T_i$ for $i = 1, 2, \ldots, N$, φ is an epimorphism. Clearly $\{D^{\mathfrak{m}_i}\}_{i=1}^N$ is an open and closed covering of S. The restriction of φ to $D^{\mathfrak{m}_i}$ being φ_i admits a regular averaging operator (a linear averaging operator). Hence, by Lemma 3.6, φ has a regular averaging operator (a linear averaging operator). Let $\Sigma_N^{(\mathfrak{m})}$ denote the discrete sum of N copies of $D^{\mathfrak{m}}$, where $\mathfrak{m} = \max(\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_N, \aleph_0)$. Since $\mathfrak{m} \geqslant \mathfrak{m}_i$, each of the copies of $D^{\mathfrak{m}}$ admits a retraction onto $D^{\mathfrak{m}_i}$ $(i = 1, 2, \ldots, N)$. These retractions induce the retraction r from $\Sigma_N^{(\mathfrak{m})}$ onto S. Hence, by Propositions 3.4 and 4.4, there exists a regular averaging operator (a linear averaging operator) for φr. To complete the proof it is enough to observe that if $\mathfrak{m} \geqslant \aleph_0$, then $\Sigma_N^{(\mathfrak{m})}$ is homeomorphic to $D^{\mathfrak{m}}$.

Let "$\sim$" denote the relation "is homeomorphic to". Then clearly $\Sigma_N^{(\mathfrak{m})} \sim [N] \times D^{\mathfrak{m}}$, where $[N]$ denotes the discrete space consisting of N points. Since $D^{\aleph_0} \times [N]$ is a zero-dimensional compact perfect metrizable space, it is homeomorphic to $D^{\aleph_0}$ (cf. Kuratowski [2], p. 58). If $\mathfrak{m} \geqslant \aleph_0$, then $\mathfrak{m} + \aleph_0 = \mathfrak{m}$. Hence

$$D^{\mathfrak{m}} \sim D^{\mathfrak{m}+\aleph_0} \sim D^{\mathfrak{m}} \times D^{\aleph_0} \sim D^{\mathfrak{m}} \times D^{\aleph_0} \times [N] \sim D^{\mathfrak{m}} \times [N].$$

Therefore if $\mathfrak{m} \geqslant \aleph_0$, then $\Sigma_N^{(\mathfrak{m})} \sim D^{\mathfrak{m}} \times [N] \sim D^{\mathfrak{m}}$. That completes the proof.

We recall that $\mathscr{C} = D^{\aleph_0}$ denotes the *Cantor set*, that is the countable product of two-point spaces $D_i = D = \{0\} \cup \{1\}$ $(i = 1, 2, \ldots)$. A general point of $\mathscr{C}$ is denoted by $\xi = (\xi_i)$ where $\xi_i = 0$ or 1 $(i = 1, 2, \ldots)$. If ξ and η are in $\mathscr{C}$, then $\xi \leqslant \eta$ means that either $\xi = \eta$, or there is an index i_0 such that $\xi_i = \eta_i$ for $i < i_0$ and $\xi_{i_0} < \eta_{i_0}$ (that is the lexicographical order).

By the product measure on $\mathscr{C}$ we mean the product measure $\otimes m_i$ where $m = m_i$ $(i = 1, 2, \ldots)$ is the measure on D such that $m(\{0\}) = m(\{1\}) = 2^{-1}$. We shall write $\int_{\mathscr{C}} f(\xi)\,d\xi$ instead of $\int_{\mathscr{C}} f(\xi) \otimes m_i(d\xi)$.

5.5. LEMMA (Milutin [1], [2]). *The closed interval $I = [0, 1]$ is a Milutin space, i.e. there exists an epimorphism Ψ from the Cantor discontinuum $\mathscr{C}$ onto I which admits a regular averaging operator.*

Proof. Since $\mathscr{C} \times \mathscr{C}$ is homeomorphic to $\mathscr{C}$, we can use $\mathscr{C} \times \mathscr{C}$ instead of $\mathscr{C}$. Let us set

$$h(\xi) = \sum_{i=1} 2^{-1} \xi_i \quad \text{for} \quad \xi = (\xi_i) \in \mathscr{C}.$$

It is well known that the "*Cantor map*" h is continuous and maps $\mathscr{C}$ onto I. Let us put $h_+^{-1}(s) = \max_{h(\xi)=s} \xi$ for $s \in I$. Then $h_+^{-1}: I \to \mathscr{C}$ is a measurable map which has a countable set of points of discontinuity (the set of all dyadic points in $(0, 1)$). Moreover,

$$\int_{\mathscr{C}} g(\xi)\,d\xi = \int_0^1 g\big(h_+^{-1}(s)\big)\,ds \tag{5.5.1}$$

for every complex valued function g on $\mathscr{C}$ integrable with respect to the product measure on $\mathscr{C}$.

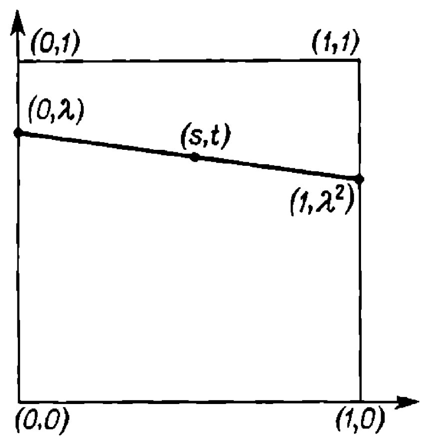
Let us define the epimorphism $\Psi : \mathscr{C}\times\mathscr{C} \to I$ and the regular averaging operator $u : C(\mathscr{C}\times\mathscr{C}) \to C(I)$ for Ψ by:

$$(5.5.2)\quad \Psi(\xi,\zeta) = \begin{cases} \dfrac{h(\xi)-1+\sqrt{(h(\xi)-1)^2+4h(\xi)h(\zeta)}}{2h(\xi)} & \text{for} \quad \xi \neq 0, \\ h(\zeta) & \text{for} \quad \xi = 0, \end{cases}$$

$$(5.5.3)\qquad (uf)(\lambda) = \int_{\mathscr{C}} f\Big(\xi,\, h_+^{-1}\big[\lambda^2 h(\xi)+\lambda\big(1-h(\xi)\big)\big]\Big)\, d\xi$$

$$\text{for} \quad 0 \leqslant \lambda \leqslant 1 \text{ and for } f \in C(\mathscr{C}\times\mathscr{C}).$$

The map Ψ can be described as follows. It is the composition of the map $h\times h : \mathscr{C}\times\mathscr{C} \to I\times I$ and the epimorphism $\Psi_1 : I\times I \to I$ which assings to each point (s,t) of the square $I\times I$ the unique $\lambda \in I$ such that the point (s,t) lies in the interval joining the points $(0,\lambda)$ and $(1,\lambda^2)$. Hence $\Psi(\xi,\zeta)$ is the non-negative root of the quadratic equation $h(\zeta) = (\lambda^2-\lambda)h(\xi)+\lambda$. Clearly Ψ is continuous and maps $\mathscr{C}\times\mathscr{C}$ onto I.



We shall show that u is a regular averaging operator for Ψ. First, let us consider the integrand in the right hand of (5.5.3). If $y(\xi,\lambda) = \lambda^2 h(\xi)+\lambda\big(1-h(\xi)\big)$, then $0 \leqslant \lambda \leqslant 1$ and $0 \leqslant h(\xi) \leqslant 1$ implies that $\lambda \geqslant y(\xi,\lambda) \geqslant \lambda^2$ for ξ in $\mathscr{C}$. Clearly $y(\xi,0) = 0$ and $y(\xi,1) = 1$ for $\xi \in \mathscr{C}$. If $0 < \lambda < 1$, then for every fixed λ, $y(\xi,\lambda)$ as a function of ξ, is continuous, decreasing, and strictly decreasing except a countable set of ξ. Thus $h_+^{-1}\big(y(\xi,\lambda)\big)$ and $f\Big(\xi,\, h_+^{-1}\big(y(\xi,\lambda)\big)\Big)$ as functions of ξ have at most a countable set of points of discontinuity. For $\lambda = 0, 1$, we have

$$f\Big(\xi,\, h_+^{-1}\big(y(\xi,0)\big)\Big) = f(\xi,0) \qquad \text{and} \qquad f\Big(\xi,\, h^{-1}\big(y(\xi,1)\big)\Big) = f(\xi,1).$$

Therefore the integrand in (5.5.3) is a bounded measurable function on $\mathscr{C}$ for every λ in $I = [0,1]$ and the integral in (5.5.3) exists for each $f \in C(\mathscr{C}\times\mathscr{C})$ and for each $\lambda \in I$. Therefore u can be regarded as a linear operator from $C(\mathscr{C}\times\mathscr{C})$ into $B(I)$, where $B(I)$ denotes the space of all bounded complex-valued functions on I with the usual sup-norm. Clearly $\|u\| = 1$ and $u1_{\mathscr{C}\times\mathscr{C}} = 1_I$. We shall show that the range of u is contained in the space $C(I)$, which may be regarded as a closed linear subspace of $B(I)$. Since u is a bounded linear operator, it is enough to establish that $uf\in C(I)$ for every f in a certain linearly dense subset W of $C(\mathscr{C}\times\mathscr{C})$.

For $\xi\in\mathscr{C}$ let χ_ξ denote the characteristic function of the set $\{\xi'\in\mathscr{C}: 0\leqslant\xi'\leqslant\xi\}$. For $(\xi,\zeta)\in\mathscr{C}\times\mathscr{C}$ let $\chi_\xi\otimes\chi_\zeta$ be the function defined by

$$(\chi_\xi\otimes\chi_\zeta)(\xi',\zeta') = \chi_\xi(\xi')\cdot\chi_\zeta(\zeta') \qquad \big((\xi',\zeta')\in\mathscr{C}\times\mathscr{C}\big).$$

Let W be the set of all functions $f=\chi_\xi\otimes\chi_\zeta$ for $(\xi,\zeta)\in\Lambda\times\Lambda$, where

$$\Lambda = \{\xi = (\xi_i)\in\mathscr{C}: \lim_i \xi_i = 1\}.$$

Observe that if $\xi\in\Lambda$, then $\{\xi'\in\mathscr{C}: 0\leqslant\xi'\leqslant\xi\}$ is a closed and open subset of $\mathscr{C}$. Therefore $\chi_\xi\in C(\mathscr{C})$ for $\xi\in\Lambda$. Hence $W\subset C(\mathscr{C}\times\mathscr{C})$. Furthermore if ξ,ζ,η,σ are in $\mathscr{C}$, then

$$\chi_\xi\otimes\chi_\zeta\cdot\chi_\eta\otimes\chi_\sigma = \chi_{\min(\xi,\eta)}\otimes\chi_{\min(\zeta,\sigma)}.$$

Hence the set of all linear combinations of functions which belong to W is a subalgebra of $C(\mathscr{C}\times\mathscr{C})$ (because if ζ and η are in Λ, then $\min(\zeta,\eta)\in\Lambda$). This subalgebra separates the points of $\mathscr{C}\times\mathscr{C}$ and contains the constant functions. Therefore, by the Stone-Weierstrass theorem, W is linearly dense in $C(\mathscr{C}\times\mathscr{C})$.

If $f=\chi_\xi\otimes\chi_\zeta$ belongs to W, then, by (5.5.3) and (5.5.1),

$$\begin{aligned}(uf)(\lambda) &= \int_{\mathscr{C}} \chi_\xi(\eta)\chi_\zeta\Big(h_+^{-1}\big[\lambda^2 h(\eta)+\lambda\big(1-h(\eta)\big)\big]\Big)d\eta\\ &= \int_0^1 \chi_\xi(h_+^{-1}s)\chi_\zeta\big(h_+^{-1}[\lambda^2 s+\lambda(1-s)]\big)ds.\end{aligned}$$

Since $\chi_\xi(h_+^{-1}s) = \chi_{[0,h(\xi)]}$ for almost all s in I (where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$), we get

$$(uf)(\lambda) = \int_0^1 \chi_{[0,h(\xi)]}(s)\chi_{[0,h(\zeta)]}\big(\lambda^2 s+\lambda(1-s)\big)ds \qquad \text{for } \lambda\in I.$$

The last integral expresses the lenght of the orthogonal projection onto s-axis of the intersection of the interval $[(s,t): t=(\lambda^2-\lambda)s+\lambda;\ 0\leqslant s\leqslant 1]$ with the rectangle $[0\leqslant s\leqslant h(\xi);\ 0\leqslant t\leqslant h(\zeta)]$. An easy computation shows that for $(\xi,\zeta)\in\Lambda\times\Lambda$

(5.5.4) $$(uf)(\lambda) = u(\chi_\xi\otimes\chi_\zeta)(\lambda) = \begin{cases} h(\xi) & \text{for } 0\leqslant\lambda\leqslant h(\zeta),\\ h(\xi)-\dfrac{\lambda-h(\zeta)}{\lambda-\lambda^2} & \text{for } h(\zeta)<\lambda\leqslant\Psi(\xi,\zeta),\\ 0 & \text{for } \Psi(\xi,\zeta)<\lambda\leqslant 1.\end{cases}$$

Since $0 \notin A$, (5.5.2) implies that $\Psi(\xi, \zeta) > 0$ for $(\xi, \zeta) \in A \times A$. Therefore

$$\lim_{\lambda \to \Psi(\xi,\zeta)-0} \left[h(\xi) - \frac{\lambda - h(\zeta)}{\lambda - \lambda^2} \right] = 0$$

because $\Psi(\xi, \zeta)$ is the positive root of the equation $(\lambda^2 - \lambda) h(\xi) + \lambda - h(\zeta) = 0$. This together with (5.5.4) show that $uf \in C(I)$ for every $f \in W$.

Hence u is a regular operator from $C(\mathscr{C} \times \mathscr{C})$ into $C(I)$. Now, let $f \in \Psi^{\circ}[C(I)]$, i.e. $f = g \circ \Psi$ for some $g \in C(I)$. Then $(\xi, \zeta) \in \varphi^{-1}(\lambda)$ (equivalently $h(\zeta) = (\lambda - \lambda^2) h(\xi) + \lambda$) implies that $f(\xi, \zeta) = g(\lambda)$ and for almost all ξ

$$\zeta = h_{+}^{-1}[(\lambda - \lambda^2) h(\xi) + \lambda].$$

Hence the integrand in (5.5.3) is equal to $g(\lambda)$ for almost all ξ. Thus $(uf)(\lambda) = g(\lambda)$. This shows that u is a regular averaging operator for Ψ. That completes the proof.

5.6. THEOREM. *The Cartesian product of an arbitrary family of compact metric spaces is a Milutin space.*

Proof. According to Proposition 5.3 it is enough to show that every compact metric space is a Milutin space. First observe that Milutin's Lemma 5.5 and Proposition 5.3 imply that $I^{\aleph_0}$ is a Milutin space. Hence, by Proposition 5.2 there exists a map $\varphi: \mathscr{C} \to I^{\aleph_0}$ which admits a regular averaging operator, say v. Let T_0 be an arbitrary compact metric space. Then there exists a homeomorphic embedding $\psi_2: T_0 \to I^{\aleph_0}$. Let $S_0 = \varphi^{-1}(\psi_2 T_0)$ and let ψ_1 be the identical embedding of S_0 into $\mathscr{C}$. Since there exists a retraction of $\mathscr{C}$ onto each of its non-empty closed subset (cf. Kuratowski [1], p. 169), ψ_1 is a coretraction. Therefore, by Proposition 3.3, there exists a regular extension operator for ψ_1, say u_1. Finally, let φ_0 denote the restriction of φ to S_0. Then we are in the position of Proposition 4.6. Hence there exist a regular averaging operator v_0 for φ_0 and a regular extension operator u_2 for ψ_2. Since ψ_1 is a coretraction, Propositions 3.4 and 4.4 imply that $v_0 \psi_1^{\circ}$ is a regular averaging operator for $\varphi_0 \psi_1$. That completes the proof.

The foregoing proof implies immediately

5.7. COROLLARY. *Every homeomorphic embedding of an arbitrary compact metric space into the Hilbert cube has a regular extension operator.*

5.8. PROPOSITION. *Each compact absolute neighbourhood retract is a Milutin space.*

Proof. According to Proposition 5.4 it is enough to show that each absolute retract is a Milutin space. Let T be an absolute retract and let φ be a homeomorphic embedding of T into some Tichonov cube $I^{\mathfrak{m}}$. Then φ is a coretraction. Therefore, by Proposition 3.4, φ° is a regular averaging operator for a retraction r corresponding to φ. Since $I^{\mathfrak{m}}$ is (by Theorem 5.6) a Milutin space, there is a regular averaging operator, say v, for an epimorphism $\psi : D^{\mathfrak{m}} \to I^{\mathfrak{m}}$. Hence, by Proposition 4.4, $\varphi^\circ v$ is a regular averaging operator for the epimorphism $r\psi : D^{\mathfrak{m}} \to T$. This completes the proof.

We complete this section by showing that Milutin spaces have a special separation property. This allows us to prove that there are dyadic spaces which are not Milutin spaces (cf. Notes and Remarks, Example 5).

5.9. Definition. A compact space S is said to have *Bockstein Separation Property* — [B.S.P.] if every pair of disjoint open subsets of S can be separated by open F_σ sets.

Let us observe that [B.S.P.] is equivalent to the following property

(5.9.1). *For every pair* (U_0, U_1) *of open disjoint subsets of* S *there is a non-negative* f *in* $C(S)$ *such that* $f^{-1}(0) \supset U_0$ *and* $f^{-1}(0) \cap U_1 = \emptyset$.

Indeed if S has [B.S.P.] and (U_0, U_1) is a pair of (non empty) open disjoint sets in S, then there are disjoint open F_σ-sets, say V_0 and V_1, separating U_0 and U_1. Since V_1 is an open F_σ-set, there is a non-negative function f in $C(S)$ such that $f^{-1}(0) = S \setminus V_1 \supset U_0$. Clearly f has the property required in (5.9.1).

Conversely suppose that a compact space S satisfies (5.9.1). Let (U_0, U_1) be any pair of non-empty disjoint open sets in S. Let f be as in (5.9.1). Let us put $V_1 = S \setminus f^{-1}(0)$. Then V_1 is an open F_σ-set such that $V_1 \supset U_1$ and $V_1 \cap U_0 = \emptyset$. Now let us consider the pair (V_1, U_0) and again by (5.9.1) let g be a positive function in $C(S)$ such that $g^{-1}(0) \supset V_1$ and $g^{-1}(0) \cap U_0 = \emptyset$. We put $V_0 = S \setminus g^{-1}(0)$.

5.10. Proposition. *Let a map* $\varphi : S \to T$ *admit a regular linear exave, say* u. *Let either* S *or* T *have* [B.S.P.]. *Then* φS *has* [B.S.P.].

Proof. Let S have [B.S.P.]. Let (U_0, U_1) be an arbitrary pair of non-empty open (in the relative topology) disjoint sets in φS. Let us put $V_j = \varphi^{-1} U_j$ $(j = 0, 1)$. Then $V_0 \cap V_1 = \emptyset$. Therefore, by (5.9.1), there exists a non-negative f in $C(S)$ such that $f^{-1}(0) \supset V_0$ and $f^{-1}(0) \cap V_1 = \emptyset$. Since u is regular, Proposition 4.1 implies that

$$(uf)(t) = \int_{\varphi^{-1}(t)} f(s)\mu_t(ds) \quad \text{for} \quad t \in \varphi(S)$$

where μ_t is a positive normalized non-negative measure concentrated on $\varphi^{-1}(t)$ $(t \in \varphi S)$. Hence, if $t \in U_0$, then $\varphi^{-1}(t) \subset V_0 \subset f^{-1}(0)$ and therefore $(uf)(t) = 0$. If $t \in U_1$, then $\varphi^{-1}(t) \subset V_1$. Therefore $f(s) > 0$ for every $s \in \varphi^{-1}(t)$. Hence the integral representation implies that if $t \in U_i$, then $(uf)(t) > 0$. Thus if g is the restriction of uf to φS, then $g^{-1}(0) \supset U_0$ and $g^{-1}(0) \cap U_1 = \emptyset$. Hence φS satisfies (5.9.1) and therefore it has [B.S.P.].

Now, let T have [B.S.P.] and let (U_0, U_1) be an arbitrary pair of open disjoint subsets of φS. Clearly to show that φS has [B.S.P.] it is enough to find open disjoint subsets of T, say V_0 and V_1, such that $V_j \supset U_j$ for $j = 0, 1$.

Let us fix δ, with $0 < \delta < 2^{-1}$. Let us set for $j = 0, 1$

$$A_j = \{g \in C(\varphi S): -2+j \leqslant g \leqslant j; \emptyset \neq g^{-1}(-2+j) \subset U_j; g^{-1}(-j) \supset \varphi S \setminus U_j\},$$

$$B_j = \{f \in C(S): f = g \circ \varphi; g \in A_j\}, \quad V_j = \bigcup_{f \in B_j} \{t \in T: (uf)(t) < -2+j+\delta\}.$$

Clearly V_j are open. Furthermore $U_j \subset V_j$ for $j = 0, 1$. Indeed if $t \in U_j$, then there is $g \in A_j$ such that $-g(t) = \|g\| = 2-j$. Therefore, since u is a linear exave, $u(g \circ \varphi)$ is an extension of g on T (this is nothing else but $\varphi^\circ u \varphi^\circ = \varphi^\circ$). Therefore

$$u(g \circ \varphi)(t) = g(t) = 2+j < -2+j+\delta.$$

Hence, if $t \in U_j$, then $t \in V_j$ (because $g \circ \varphi \in B_j$).

Finally, let us suppose that the sets V_0 and V_1 are not disjoint. Let $t_0 \in V_0 \cap V_1$. Then there exists $f_j \in B_j$ such that

$$(uf_j)(t_0) < -2+j+\delta \quad \text{for} \quad j = 0, 1.$$

Thus

$$\|u(f_0+f_1)\| \geqslant |(uf_0)(t_0) + uf_1(t_0)| > 3-2\delta.$$

On other hand, if $f_j = g_j \circ \varphi$ for $j = 0, 1$, then

$$\|f_0+f_1\| = \|g_0+g_1\| = \sup_{t \in \varphi S} |g_0(t)+g_1(t)|.$$

It follows from the definitions of A_j that if $g_j \in A_j$ for $j = 0, 1$, then $|g_0(t)+g_1(t)| \leqslant 1$ for $t \in \varphi S$. Hence $\|f_0+f_1\| \leqslant 1$.

This shows that if the sets V_0 and V_1 are not disjoint, then $\|u\| > 3-2\delta$. Hence, in particular, if $\|u\| = 1$, then V_0 and V_1 are disjoint. That completes the proof.

Remark. Actually we proved that if a compact space T has [B.S.P.], then, for any map $\varphi: S \to T$ such that φS does not have [B.S.P.], the norm of every exave for φ is $\geqslant 3$.

5.11. COROLLARY. *Every Milutin space has* [B.S.P.].

Proof. This is an immediate consequence of Proposition 5.10 and the fact that $D^{\mathfrak{m}}$ has [B.S.P.] (cf. Bockstein [1], K. A. Ross and A. H. Stone [1], R. Engelking [1]).

§ 6. DUGUNDJI SPACES

6.1. DEFINITION. A compact space S is a *Dugundji space* (resp. *almost Dugundji space*) if for every compact space T every homeomorphic embedding $\varphi: S \to T$ has a regular extension operator (resp. linear extension operator).

6.2. PROPOSITION. *For every compact space S the following conditions are equivalent*:

(6.2.1) *S is a Dugundji space (resp. an almost Dugundji space).*

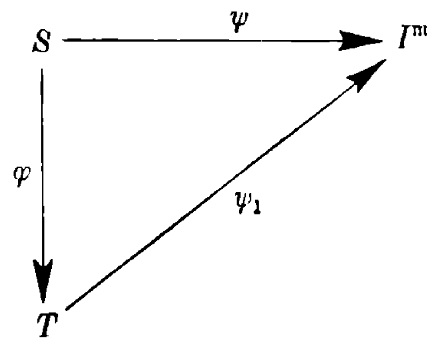
(6.2.2) *There exists a Dugundji space T (resp. an almost Dugundji space T) and a homeomorphic embedding $\varphi: S \to T$ which admits a regular extension operator (resp. linear extension operator).*

(6.2.3) *There is a homeomorphic embedding $\psi: S \to I^{\mathfrak{m}}$ which admits a regular extension operator (resp. a linear extension operator).*

Proof. (6.2.1) $\Rightarrow$ (6.2.2). Put $S = T$ and $\varphi = \mathrm{id}_S$.

(6.2.2) $\Rightarrow$ (6.2.3). This is an immediate consequence of the fact that every compact space can be homeomorphically embedded into $I^{\mathfrak{m}}$ for some cardinal $\mathfrak{m}$ and of the fact that the composition of two regular extension operators (resp. linear extension operators) is a regular extension operator (resp. linear extension operator) (cf. Proposition 4.4).

(6.2.3) $\Rightarrow$ (6.2.1). Let $v: C(S) \to C(I^{\mathfrak{m}})$ be a regular extension operator (resp. linear extension operator) for $\psi: S \to I^{\mathfrak{m}}$. Let $\varphi: S \to T$ be a homeomorphic embedding of S into an arbitrary compact space T. Since $I^{\mathfrak{m}}$ is an absolute retract, there exists an extension $\psi_1: T \to I^{\mathfrak{m}}$ of ψ such that the diagram



is commutative, i.e. $\psi_1\varphi = \psi$. Let us set $u = \psi_1^\circ v$. Since v is a linear extension operator for ψ, $\psi^\circ v = \mathrm{id}_{C(S)}$. Hence

$$\varphi^\circ u \varphi^\circ = \varphi^\circ \psi_1^\circ v \varphi^\circ = \psi^\circ v \varphi^\circ = \mathrm{id}_{C(S)}\varphi^\circ = \varphi^\circ.$$

This shows that u is a linear exave for the homeomorphic embedding φ. Thus u is a linear extension operator. Clearly u is regular, whenever v has the same property. That completes the proof.

Remark. Actually the proof of the implication (6.2.3) $\Rightarrow$ (6.2.1) shows that if S is a compact space with the property that there is a homeomorphic embedding $\psi: S \to I^{\mathfrak{m}}$ which admits a linear extension operator of norm $\leqslant a$, then for every compact space T, every homeomorphic embedding of S into T admits a linear extension operator of norm $\leqslant a$.

A compact space S is said to be a *locally Dugundji space* (*locally almost Dugundji space*) if each s in S has a closed neighbourhood which is a a Dugundji space (almost Dugundji space).

6.3. PROPOSITION. *Every locally Dugundji space* (*resp. locally almost Dugundji space*) *is a Dugundji space* (*resp. an almost Dugundji space*).

Proof. Let $\varphi: S \to I^{\mathfrak{m}}$ be a homeomorphic embedding of a locally Dugundji space S (resp. almost locally Dugundji space) into the Tichonov cube $I^{\mathfrak{m}}$. For each s in S let S_s denote the closed neighbourhood of s which is a Dugundji space (an almost Dugundji space). Let us set $T_s = I^{\mathfrak{m}}$ and let $\varphi_s: S_s \to T_s$ denote the restriction of φ to S_s. Clearly there are regular extension operators (resp. linear extension operator) for φ_s because S_s are Dugundji spaces (resp. almost Dugundji spaces). Hence, by Localization Principle 3.6, there is a regular extension operator (resp. linear extension operator) for φ. Therefore S satisfies (6.2.3). That completes the proof.

6.4. COROLLARY. *Every compact absolute neighbourhood retract is a Dugundji space.*

This is an immediate consequence of Proposition 6.3 and the fact that every absolute retract is a Dugundji space.

6.5. PROPOSITION. *The Cartesian product of an arbitrary family of Dugundji spaces is a Dugundji space.*

Proof. Let $(S_a)_{a\in A}$ be a family of Dugundji spaces. Let $\varphi_a: S_a \to I^{\mathfrak{m}_a}$ be homeomorphic embeddings, where cardinal numbers $\mathfrak{m}_a$ depend on S_a. Then each φ_a has a regular extension operator, say u_a $(a \in A)$. Hence, by Proposition 4.7, $\otimes u_a$ is a regular extension operator for the map $\boldsymbol{P}\varphi_a: \boldsymbol{P}S_a \to \boldsymbol{P}I^{\mathfrak{m}_a}$. Since the product $\boldsymbol{P}I^{\mathfrak{m}_a}$ is homeomorphic to the Tichonov cube $I^{\mathfrak{m}}$ for $\mathfrak{m} = \max(\overline{\overline{A}}, \sup_{a\in A} \mathfrak{m}_a)$, the space $\boldsymbol{P}S_a$ satisfies (6.2.3). That completes the proof.

6.6. THEOREM. *The Cartesian product of an arbitrary family of compact metric spaces is a Dugundji space.*

Proof. Every compact metric space is a Dugundji space. This is a particular case of Borsuk-Dugundji theorem (cf. Borsuk [1], Dugundji [1], Michael [1]). Alternatively it follows from Corollary 5.7 and Proposition 6.2. The assertion of the theorem follows immediately from the previous remark and Proposition 6.5.

The next corollary is an analogue of Corollary 5.11.

6.7. COROLLARY. *Every Dugundji space has* [B.S.P.].

This follows immediately from Propositions 6.2, 5.10, and the fact that the Tichonov cube $I^{\mathfrak{m}}$ has [B.S.P.] (cf. Bockstein [1], K. A. Ross and A. H. Stone [1]).

§ 7. EXAVES AND TOPOLOGICAL GROUPS

7.1. Definition. A topological group G *acts on a space* S provided to each pair (g, s) in $G\times S$ there corresponds a point $\gamma(g, s)$ in S such that the following conditions are satisfied:

(7.1.1) The transformation function $\gamma : G\times S \to S$ is continuous;

(7.1.2) $\gamma(g, \cdot) = \gamma_g : S \to S$ is a homeomorphism of S onto itself $(g\in G)$,

(7.1.3) $\gamma_{gg_1} = \gamma_g\circ\gamma_{g_1}$; $\gamma_{g^{-1}} = (\gamma_g)^{-1}$ for g, g_1 in G.

The operator $\sigma_g = (\gamma_g)^\circ : C(S)\to C(S)$, with $\sigma_g(f) = f\circ\gamma_g$ for $f\in C(S)$, will be called the *shift operator*.

A *map* $\varphi : S\to T$ is said to be *G-invariant* if

(7.1.4) G acts on S and on T with transformation functions $\gamma' : G\times S\to S$ and $\gamma'' : G\times T\to T$ respectively;

(7.1.5) $\varphi\circ\gamma'_g = \gamma''_g\circ\varphi$ for each g in G.

A *linear exave* u for a G-invariant map $\varphi : S\to T$ is said to be *G-invariant* if

(7.1.6) $u\sigma'_g = \sigma''_g u$ for $g\in G$.

7.2. Proposition. *Let G be a compact topological group and let v be a linear exave (a regular exave) for a G-invariant map $\varphi : S\to T$. Then there is a G-invariant linear exave (regular exave) for φ.*

Proof. Let us set

$$u_g = \sigma''_g v\sigma'_{g^{-1}} \quad \text{for} \quad g\in G, \tag{7.2.1}$$

$$uf = \int_G u_g f\,dg \quad \text{for} \quad f\in C(S), \tag{7.2.2}$$

where the integral in (7.2.2) is taken with respect to the normalized Haar measure of G.

It follows from (7.1.1) that $g\to u_g f$ is a continuous function from G into $C(T)$ for every fixed f in $C(S)$. Thus the compactness of G implies that the integral in (7.2.2) exists. Therefore u is a linear operator from $C(S)$ into $C(T)$ with $\|u\|\leqslant \sup_{g\in G}\|u_g\|\leqslant\|v\|$. Furthermore for every g in G (7.1.3), (7.1.5), and the identity $\varphi^\circ v\varphi^\circ = \varphi^\circ$ imply

$$\varphi^\circ u_g\varphi^\circ = \varphi^\circ\sigma''_g v\sigma'_{g^{-1}}\varphi^\circ = \sigma'_g\varphi^\circ v\varphi^\circ\sigma''_{g^{-1}} = \sigma'_g\varphi^\circ\sigma''_{g^{-1}} = \sigma'_g\sigma'_{g^{-1}}\varphi^\circ = \varphi^\circ.$$

Thus for every f in $C(T)$ we get

$$(\varphi^\circ u\varphi^\circ)(f) = \varphi^\circ\int_G (u_g\varphi^\circ)(f)\,dg = \int_G (\varphi^\circ u_g\varphi^\circ)(f)\,dg$$

$$= \int_G \varphi^\circ f\,dg = \varphi^\circ f\int_G dg = \varphi^\circ f.$$

Hence u is a linear exave for φ. Moreover, if v is regular, then u has the same property.

Finally, let us fix h in G. Then (7.1.3) and (7.2.1) imply that for every g in G,

$$u_g\sigma'_h = \sigma''_g v\sigma'_{hg^{-1}} = \sigma''_h \sigma''_{gh^{-1}} v\sigma'_{(gh^{-1})^{-1}} = \sigma''_h u_{gh^{-1}},$$

because for any shift operator σ, $\sigma_{gg_1} = \sigma_{g_1}\circ\sigma_g$ for g, g_1 in G. Hence the invariantness under translations of the Haar measure implies that

$$(u\sigma'_h)(f) = \int_G u_g\sigma'_h f dg = \int_G \sigma''_h u_{gh^{-1}} f dg$$

$$= \sigma''_h \int_G u_{gh^{-1}} f dg = \sigma''_h \int_G u_g f dg = (\sigma''_h u)(f)$$

for every f in $C(S)$ and for every h in G. Thus $u\sigma'_h = \sigma''_h u$ for every $h \in G$. That completes the proof.

7.3. DEFINITION. Let G be a compact topological group and let H be a closed subgroup of G. We shall denote by G/H the *left coset-space of G modulo H* (cf. Montgomery and Zippin [1], p. 26) and by ψ_H the natural map from G onto G/H, i.e. $\psi_H g = [gH]$ for $g \in G$ where $[gH] = \{g_1 \in G : g^{-1}g_1 \in H\}$. Clearly the map $\psi_H : G \to G/H$ is G-invariant (the transformation functions $\gamma' : G\times G \to G$ and $\gamma'' : G\times G/H \to G/H$ are defined by $\gamma'(g, g_1) = g\cdot g_1$ and $\gamma''(g, [g_1H]) = [g\cdot g_1 H]$ for g, g_1 in G).

7.4. PROPOSITION. *There exists the unique G-invariant regular operator for ψ_H. This operator is defined by*

(7.4.1) $$(uf)([gH]) = \int_H f(gh)dh \quad \text{for } f \in C(G) \text{ and for } g \in G,$$

where the integral in (7.4.1) is taken with respect to the normalized Haar measure of H.

Proof. Let us suppose that u is a regular G-invariant averaging operator for ψ_H. Then, by Proposition 4.1, there is a normalized non-negative measure μ_H in $M(G)$ such that

$$(uf)([H]) = \int_H f(g)\mu_H(dg) = \int_G f(g)\mu_H(dg) \quad \text{for} \quad f \in C(G).$$

(We regard here $H = \psi_H^{-1}([H])$ as a subset of G). Since every f' in $C(H)$ can be extended to an f in $C(G)$ and since u is G-invariant, we have

$$\int_G f(h_1h)\mu_H(dh) = \int_G f(h)\mu_H(dh) = \int_H f'(h)\mu_H(dh)$$

$$\text{for } f' \in C(H) \text{ and } h_1 \in H.$$

Hence regarding μ_H as an element of $M(H)$ we infer that μ_H is a non-negative normalized measure in $M(H)$ which is invariant under translations by elements of H. Thus μ_H is the Haar measure of H. Therefore

$$(uf)([H]) = \int_H f(h)\,dh \quad \text{for} \quad f\in C(G).$$

Finally, since u is G-invariant, (7.1.6) implies that

$$(uf)([gH]) = (\sigma_g'' uf)([H]) = (u\sigma_g' f)([H]) = \int_H (\sigma_g' f)(h)\,dh$$

$$= \int_H f(gh)\,dh \quad (g\in G).$$

This completes the proof of the uniqueness part of the Proposition. We leave to the reader the simple checking that (7.4.1) defines a G-invariant regular averaging operator for ψ_H.

7.5. THEOREM. *Every coset-space of a compact topological group (less generally, every compact topological group) is a Milutin space.*

The proof of this theorem is based on the next proposition.

7.6. PROPOSITION. *Every compact topological group is homeomorphic to the quotient group of a Cartesian product of a family of compact metric groups.*

Proof. We repeat with a slight modification the arguments of Kuzminov [1].

Let G be a compact topological group and let G_0 be the component of unit. Then G_0 is a connected invariant subgroup of G such that the quotient group G/G_0 is zero-dimensional. Hence, by a result of Mostert ([1], Theorem 8), G is homeomorphic to $G_0\times G/G_0$. Since every infinite zero-dimensional compact topological group is homeomorphic to $D^{\mathfrak{n}}$ for some cardinal number $\mathfrak{n}$ (cf. Hulanicki [1], Kuzminov [1], Hewitt [1], Hewitt and Ross [1], pp. 95-98), G is homeomorphic either to $G_0\times[N]$ (where $[N]$ denotes the finite discrete space consisting of exactly N points), or to $G_0\times D^{\mathfrak{n}}$.

Now, G_0, as a compact connected topological group, is isomorphic(1) to the quotient group $(A\times\Sigma^*)/Z$, where A is a compact Abelian group, Σ^* is a Cartesian product of a family of simple and simply connected compact Lie groups and Z is a closed subgroup of $A\times\Sigma^*$ which is contained in the centrum of $A\times\Sigma^*$ (Weil [1], pp. 89-93). To complete the proof it is enough to apply the following facts:

(α) Every compact Abelian group is isomorphic to a Cartesian product of a family of compact metric groups which are either finite, or are iso-

(1) For topological groups an isomorphism means always an algebraic isomorphism which is a homeomorphism,

morphic to the group of all rotations of the unit circle (cf. Weil [1], p. 89-93).

(β) Every compact Lie group is metrizable.

(γ) If H_1 and H_2 are (topological) groups, Z_2 is an invariant subgroup of H_2 and $\{e\}$ is the subgroup of H_1 consisting of the unit of H_1, then the group $H_1\times H_2/Z_2$ is isomorphic to $(H_1\times H_2)/Z$, where $Z=\{e\}\times Z_2$.

Proof of Theorem 7.5. It follows immediately from Propositions 7.4 and 4.4 that if a compact group is a Milutin space, then every of its coset-spaces has the same property.

Now, by Proposition 7.6 and Theorem 5.6, every compact group is homeomorphic to a coset-space of a topological group which is a Milutin space. Hence every topological group, and therefore every of its coset-spaces, is a Milutin space.

§ 8. APPLICATION TO LINEAR TOPOLOGICAL CLASSIFICATION OF SPACES OF CONTINUOUS FUNCTIONS

8.1. Definition. A Banach space Y is said to be a *factor* of a Banach space X (in symbols $Y|X$) if there is a Banach space Z such that X is linearly homeomorphic to $Y\times Z$, or equivalently if Y is linearly homeomorphic to a complemented subspace of X.

The next proposition is an immediate consequence of Corollaries 2.3 and 2.4.

8.2. Proposition. *If an epimorphism $\varphi: S\to T$ has a linear averaging operator, then $C(T)|C(S)$.*

If a homeomorphic embedding $\varphi: S\to T$ has a linear extension operator, then $C(S)|C(T)$.

8.3. Proposition. *Let a Banach space Y be a factor of $C(D^{\mathfrak{n}})$ and let $C(D^{\mathfrak{n}})$ be a factor of Y. Then Y is linearly homeomorphic to $C(D^{\mathfrak{n}})$.*

Proof. First we need some notation. Let "$\sim$" denote the relation "is linearly homeomorphic to". If E is a Banach space and S a compact space, then $C(S,E)$ denote the space of all continuous functions on S with values on E. The symbol $(E\times E\times\ldots)_{c_0}$ denotes the Banach space of all sequences (x_n) such that $x_n\in E$ $(n=1,2,\ldots)$ and $\lim\limits_n\|x_n\|=0$. We admit $\|(x_n)\|=\sup\limits_n\|x_n\|$. By c_0 we denote the space of all scalar valued sequences convergent to zero.

Let us observe that

$$(8.3.1)\qquad (C(D^{\mathfrak{n}})\times C(D^{\mathfrak{n}})\times\ldots)_{c_0}\sim C(D^{\mathfrak{n}})\qquad\text{for}\qquad \mathfrak{n}\geqslant\aleph_0.$$

Indeed, let $[\omega]$ denote the one-point compactification of a countable discrete space and let $\mathfrak{n}\geqslant\aleph_0$. Then $D^{\mathfrak{n}}$ is homeomorphic to $D^{\mathfrak{n}}\times[\omega]$, because $D^{\mathfrak{n}}$ is homeomorphic to $D^{\mathfrak{n}}\times\mathscr{C}$ and $\mathscr{C}$ is homeomorphic to $\mathscr{C}\times[\omega]$ (as a zero-dimensional compact perfect space). Hence $C(D^{\mathfrak{n}})\sim C(D^{\mathfrak{n}}\times[\omega])$. The space $C(D^{\mathfrak{n}}\times[\omega])$ can be identified with $C\big(D^{\mathfrak{n}}, C([\omega])\big)$. Since $C([\omega])$ can be identified with the space c of all convergent sequences of scalars, we obtain $C(D^{\mathfrak{n}})\sim C(D^{\mathfrak{n}}, c)$. Since $c\sim c_0$ (cf. Banach [1], p. 182-184), the definition of $C(S,E)$ implies that $C(D^{\mathfrak{n}},c)\sim C(D^{\mathfrak{n}},c_0)$. Finally the space $C(D^{\mathfrak{n}},c_0)$ can be identified with $\big(C(D^{\mathfrak{n}})\times C(D^{\mathfrak{n}})\times\ldots\big)_{c_0}$ by assigning to every $f(\cdot)\in C(D^{\mathfrak{n}},c_0)$ the sequence of its coordinates $\big(f_n(\cdot)\big)$. This proves (8.3.1).

Now, the assumptions of the proposition imply

$$Y\sim C(D^{\mathfrak{n}})\times Z_1 \quad\text{and}\quad C(D^{\mathfrak{n}})\sim Y\times Z_2 \tag{8.3.2}$$

for some Banach spaces Z_1 and Z_2.

Thus if $\mathfrak{n}\geqslant\aleph_0$ then (cf. Pełczyński [3], Bessaga [1]) we get

$$\begin{aligned}
Y &\sim C(D^{\mathfrak{n}})\times Z_1\sim\big(C(D^{\mathfrak{n}})\times C(D^{\mathfrak{n}})\times\ldots\big)_{c_0}\times Z_1\\
&\sim Z_1\times C(D^{\mathfrak{n}})\times\big(C(D^{\mathfrak{n}})\times C(D^{\mathfrak{n}})\times\ldots\big)_{c_0}\\
&\sim Y\times\big(C(D^{\mathfrak{n}})\times C(D^{\mathfrak{n}})\times\ldots\big)_{c_0}\\
&\sim Y\times\big((Y\times Z_2)\times(Y\times Z_2)\times\ldots\big)_{c_0}\\
&\sim Y\times(Y\times Y\times\ldots)_{c_0}\times(Z_2\times Z_2\times\ldots)_{c_0}\\
&\sim(Y\times Y\times\ldots)_{c_0}\times(Z_1\times Z_2\times\ldots)_{c_0}\sim\big((Y\times Z_2)\times(Y\times Z_2)\times\ldots\big)_{c_0}\\
&\sim\big(C(D^{\mathfrak{n}})\times C(D^{\mathfrak{n}})\times\ldots\big)_{c_0}\sim C(D^{\mathfrak{n}}).
\end{aligned}$$

Finally if $\mathfrak{n}<\aleph_0$, then (8.3.2) implies that $Y\sim C(D^{\mathfrak{n}})$ (because the spaces $C(D^{\mathfrak{n}})$ and Y, in this case, are of the same finite dimension).

8.4. PROPOSITION. *Let S be an infinite compact space satisfying the following conditions:*

(8.4.1) *S is either an almost Milutin space, or an almost Dugundji space,*

(8.4.2) *S contains a subset homeomorphic to $D^{\mathfrak{n}}$, where $\mathfrak{n}$ is the topological weight of S.*

Then $C(S)$ is linearly homeomorphic to $C(D^{\mathfrak{n}})$.

Proof. According to Proposition 8.3 it is enough to prove the implications

$$(8.4.1)\Rightarrow C(S)\,|\,C(D^{\mathfrak{n}}),\qquad (8.4.2)\Rightarrow C(D^{\mathfrak{n}})\,|\,C(S).$$

If S is an almost Milutin space, then, by Proposition 8.2, $C(S)\,|\,C(D^{\mathfrak{n}})$.

If S is an almost Dugundji space, then there is a map $\varphi: S\to I^{\mathfrak{n}}$ which admits a linear extension operator. Thus, by Proposition 8.2, $C(S)\,|\,C(I^{\mathfrak{n}})$. Since $I^{\mathfrak{n}}$ is a Milutin space (by Theorem 5.6), $C(I^{\mathfrak{n}})\,|\,C(D^{\mathfrak{n}})$. Hence the transitivity of the relation "to be a factor" implies that $C(S)\,|\,C(D^{\mathfrak{n}})$. This completes the proof of the first implication.

The second implication is an immediate consequence of Proposition 8.2 and the fact that $D^{\mathfrak{n}}$ is a Dugundji space (by Theorem 6.6).

8.5. THEOREM (Milutin). *Let S be an uncountable compact metric space. Then $C(S)$ is linearly homeomorphic to $C(\mathscr{C})$.*

Proof. It follows immediately from Theorem 5.6 that every compact metric space satisfies (8.4.1).

Since S is an uncountable compact metric space, it contains a subset homeomorphic to $\mathscr{C} = D^{\aleph_0}$ (cf. Hausdorff [1], p. 136-138). Since every compact metric space is separable, its topological weight is $\aleph_0$. Thus S satisfies (8.4.2). To complete the proof we apply Proposition 8.4.

Let us recall (cf. Kelley [2] p. 42) that the first derived set $S^{(1)}$ of a topological space S is the set of all non-isolated points of S. For ordinals $a > 1$ the a-th derived set of S, denoted by $S^{(a)}$, is defined inductively. If $a = \beta+1$, then $S^{(a)} = (S^{(\beta)})^{(1)}$; if a is a limit ordinal number, then $S^{(a)} = \bigcap_{\beta<a} S^{(\beta)}$.

8.6. DEFINITION. Let us assign to every compact space S the ordinal number $\chi(S)$ as follows:

if $S^{(a)}$ is non-empty for all ordinals $a \geqslant 1$, then $\chi(S) = 0$,

if S is finite, equivalently if $S^{(1)} = \emptyset$, then $\chi(S) =$ the number of elements of S,

if $S^{(1)} \neq \emptyset$, but $S^{(a)} = \emptyset$ for some $a \geqslant 1$, then $\chi(S) = \beta^{\omega}$, where β is the smallest ordinal such that $S^{(\beta)} = \emptyset$ and ω denotes the first infinite ordinal number.

The next corollary is an immediate consequence of Theorem 8.5, Theorem 2 of Bessaga and Pełczyński [1], and the fact that if S is metric and countable, then $\chi(S) \neq 0$ (cf. Bessaga and Pełczyński [1], the proof of Theorem 3).

8.7. COROLLARY. *Let S and S_1 be compact metric spaces. Then $C(S)$ is linearly homeomorphic to $C(S_1)$ if and only if $\chi(S) = \chi(S_1)$.*

The problem of linear topological classification of spaces of continuous functions on non-metrizable compact spaces seems to be much more complicated than the metric case. There is rather narrow class of those compact spaces S for which $C(S)$ is linearly homeomorphic to $C(D^{\mathfrak{n}})$ (cf. Propositions 8.11 and 8.13). However, products of compact metric spaces and compact topological groups belong to this class.

8.8. THEOREM. *Let $(S_a)_{a\in A}$ be an infinite family of compact metric spaces each of which contains at least two points. Then $C(\boldsymbol{P}S_a)$ is linearly homeomorphic to $C(D^{\mathfrak{n}})$ where $\mathfrak{n} = \overline{\overline{A}}$.*

Proof. According to Theorem 5.6 the space $C(\boldsymbol{P}S_a)$ satisfies (8.4.1). Let D_a be a fixed two-point subset of S_a $(a \in A)$. Then $\boldsymbol{P}_{a\in A} D_a = D^{\overline{\overline{A}}}$ can be in a natural way homeomorphically embedded in $\mathop{\boldsymbol{P}}_{a\in A} S_a$. Thus if $\mathfrak{n}$ is the topological weight of $\boldsymbol{P} S_a$, then $\mathfrak{n} \geqslant \bar{\bar{A}}$ because $\bar{\bar{A}}$ is the topological weight of $D^{\bar{\bar{A}}}$. Since every S_a is a continuous image of $D^{\aleph_0}$ $(a \in A)$, the product $\mathop{\boldsymbol{P}}_{a\in A} S_a$ is a continuous image $D^{\aleph_0 \bar{\bar{A}}}$. Therefore $\mathfrak{n} \leqslant \aleph_0 \bar{\bar{A}} = \bar{\bar{A}}$ and $\mathfrak{n} = \bar{\bar{A}}$. This shows that $\mathop{\boldsymbol{P}}_{a\in A} S_a$ satisfies (8.4.2). To complete the proof we apply Proposition 8.4.

8.9. THEOREM. *Let G be an infinite compact group. Then $C(G)$ is linearly homeomorphic to $C(D^{\mathfrak{n}})$, where $\mathfrak{n}$ is the topological weight of G.*

Proof. According to Proposition 8.4 and Theorem 7.5 it is enough to prove that G satisfies (8.4.2). This is shown in the next Proposition.

8.10. PROPOSITION. *If G is an infinite compact group of topological weight $\mathfrak{n}$, then G contains a subset homeomorphic to $D^{\mathfrak{n}}$.*

Proof. Our arguments are similar to those of Ivanovskiĭ [1]. We use the following result of Pontryagin ([1], p. 327).

(P) Let G be an infinite compact group of topological weight $\mathfrak{n}$. Then there exist a transfinite sequence of compact groups $(G_a)_{a<\vartheta}$ and group-epimorphisms $\varphi_a^\beta : G_\beta \to G_a$ such that

(8.10.1) G_1 is a Lie group which has at least two different points, and $G_\vartheta = G$.

(8.10.2) $\varphi_a^\beta \varphi_\beta^\gamma = \varphi_a^\gamma$ for $0 < a < \beta < \gamma \leqslant \vartheta$.

(8.10.3) If $\delta \leqslant \vartheta$ is a limit ordinal number, then $\bigcap_{a<\delta} K_a^\delta = \{e\}$, where $K_a^\delta = \ker \varphi_a^\delta$ and $\{e\}$ is the neutral element of G_δ.

(8.10.4) $\ker \varphi_a^{a+1} = K_a^{a+1}$ is a Lie group which has at least two different points.

(8.10.5) ϑ is the smallest ordinal number of the power $\mathfrak{n}$.

Let $(D_\xi)_{0<\xi<\vartheta}$ be a family of two-point spaces. We shall construct by transfinite induction homeomorphic embeddings $\psi_a : \mathop{\boldsymbol{P}}_{\xi<a} D_\xi \to G_a$ $(0 < a \leqslant \vartheta)$ such that

$$\varphi_a^\beta \psi_\beta = \psi_a p_a^\beta \qquad (0 < a < \beta \leqslant \vartheta), \tag{8.10.6}$$

where $p_a^\beta : \mathop{\boldsymbol{P}}_{\xi<\beta} D_\xi \to \mathop{\boldsymbol{P}}_{\xi<a} D_\xi$ is the natural projection defined by

$$p_a^\beta x = (x_\xi)_{\xi<a} \in \mathop{\boldsymbol{P}}_{\xi<a} D_\xi \quad \text{for} \quad x = (x_\xi)_{\xi<\beta} \in \mathop{\boldsymbol{P}}_{\xi<\beta} D_\xi.$$

By (8.10.1) there is a homeomorphic embedding $\psi_1 : D_0 \to G_1$. Let us suppose that for $1 \leqslant \xi < \delta \leqslant \vartheta$ the homeomorphisms ψ_ξ have been defined in such a way that (8.10.6) is satisfied for $0 < a < \beta < \delta$. We shall define ψ_δ. Let us consider two cases.

1° δ is a limit ordinal number. First, let us observe that in this case the intersection $\bigcap_{\xi<\delta}(\varphi_\xi^\delta)^{-1}\{\psi_\xi p_\xi^\delta x\}$ is a one-point set for each $x \in \mathbf{P}_{\xi<\delta} D_\xi$. Indeed, using (8.10.2) and (8.10.6) we get for $0 < \alpha < \beta < \delta$,

$$\begin{aligned}(\varphi_\alpha^\delta)^{-1}\{\psi_\alpha p_\alpha^\delta x\} &= (\varphi_\alpha^\beta \circ \varphi_\beta^\delta)^{-1}\{\psi_\alpha p_\alpha^\delta x\} = (\varphi_\beta^\delta)^{-1}[(\varphi_\alpha^\beta)^{-1}\{\psi_\alpha p_\alpha^\beta p_\beta^\delta x\}] \\ &= (\varphi_\beta^\delta)^{-1}[(\varphi_\alpha^\beta)^{-1}\{\varphi_\alpha^\beta \psi_\beta p_\beta^\delta x\}] \supset (\varphi_\beta^\delta)^{-1}\{\psi_\beta p_\beta^\delta x\}.\end{aligned}$$

Thus since $(\varphi_\xi^\delta)^{-1}\{\psi_\xi p_\xi^\delta x\}$ are closed subsets of a compact group G_δ, the intersection $\bigcap_{\xi<\delta}(\varphi_\xi^\delta)^{-1}\{\psi_\xi p_\xi^\delta x\}$ is non-empty. Now, if y_1 and y_2 are in $\bigcap_{\xi<\delta}(\varphi_\xi^\delta)^{-1}\{\psi_\xi p_\xi^\delta x\}$, then $\varphi_\xi^\delta y_1 = \varphi_\xi^\delta y_2 = \psi_\xi p_\xi^\delta x$ for $\xi < \delta$. Hence $y_1 y_2^{-1} \in K_\xi^\delta = \ker\varphi_\xi^\delta$ for $\xi < \delta$. Thus (8.10.3) implies that $y_1 = y_2$.

We define $\psi_\delta x$ as the unique element of the intersection $\bigcap_{\xi<\delta}(\varphi_\xi^\delta)^{-1}\{\psi_\xi p_\xi^\delta x\}$ for $x \in \mathbf{P}_{\xi<\delta} D_\xi$. Clearly $\varphi_\alpha^\delta \psi_\delta x = \psi_\alpha p_\alpha^\delta x$ for $x \in \mathbf{P}_{\xi<\delta} D_\xi$ and for $0 < \alpha < \delta$. Let x' and x'' be in $\mathbf{P}_{\xi<\alpha} D_\xi$. If $x' \neq x''$, then for some $\alpha < \delta$, $p_\alpha^\delta x' \neq p_\alpha^\delta x''$ and therefore $\psi_\alpha p_\alpha^\delta x' \neq \psi_\alpha p_\alpha^\delta x''$ (because ψ_α is a homeomorphic embedding). Thus $\psi_\delta x' \neq \psi_\delta x''$. Finally, let $y = \psi_\delta x \in G_\delta$. Let V be an open neighbourhood of y. It follows from (8.10.3) that $\bigcap_{\xi<\delta} yK_\xi^\delta = \{y\}$. Since (8.10.2) implies $K_\alpha^\delta \supset K_\beta^\delta$ for $0 < \alpha < \beta < \delta$, there is an α_0 such that $yK_{\alpha_0}^\delta \subset V$. Thus, since the operation of multiplication in a compact group is a continuous function of two variables, there is a neighbourhood V_1 of y such that $V_1 K_{\alpha_0}^\delta \subset V$. Let us set

$$U = (\psi_{\alpha_0} p_{\alpha_0}^\delta)^{-1}\{\varphi_{\alpha_0}^\delta V_1 K_{\alpha_0}^\delta\},$$

Since $\varphi_{\alpha_0}^\delta$ is a group epimorphism acting between two compact groups, $\varphi_{\alpha_0}^\delta$ is an open map (Hewitt and Ross [1]). Thus U is open. Since $\psi_{\alpha_0} \circ p_{\alpha_0}^\delta = \varphi_{\alpha_0}^\delta \psi_\delta$ and since $V_1 K_{\alpha_0}^\delta = (\varphi_{\alpha_0}^\delta)^{-1}\{\varphi_{\alpha_0}^\delta(V_1 \cdot K_{\alpha_0}^\delta)\}$, we get

$$U = (\varphi_{\alpha_0}^\delta \psi_\delta)^{-1}\{\varphi_{\alpha_0}^\delta V_1 K_{\alpha_0}^\delta\} = \psi_\delta^{-1}\{V_1 K_{\alpha_0}^\delta\}.$$

This proves the continuity of ψ_δ. Therefore ψ_δ is a continuous one-to-one map from one compact space into another. Hence ψ_δ is a homeomorphic embedding.

2° $\delta = \alpha + 1$. Since $K_\alpha^{\alpha+1}$ is a Lie group, it follows from a result of Gleason (cf. e.g. Montgomery and Zippin [1], p. 221) that the epimorphism $\varphi_\alpha^{\alpha+1}: G_{\alpha+1} \to G_\alpha$ is a local projection, i.e. for every $y \in G_\alpha$ there is a neighbourhood V_y of y and a homeomorphic embedding $\tau_y: V_y \times K_\alpha^{\alpha+1} \to G_{\alpha+1}$ such that $\varphi_\alpha^{\alpha+1}\tau_y(y', k) = y'$ for $k \in K_\alpha^{\alpha+1}$ and $y' \in V_y$. Since $\psi_\alpha: \mathbf{P}_{\xi<\alpha} D_\xi \to G_\alpha$ is a homeomorphic embedding, $\psi_\alpha(\mathbf{P}_{\xi<\alpha} D_\xi)$ is a zero-dimensional compact subset of G_α. Therefore there exist in G_α a finite open cover $\{V_i\}_{i=1}^N$ of mutually disjoint sets, and homeomorphic embeddings

$\tau_i : V_i \times K_\alpha^{\alpha+1} \to G_{\alpha+1}$, with $\varphi_\alpha^{\alpha+1}\tau_i(y, k) = y$ for $k \in K_\alpha^{\alpha+1}$ and $y \in V_i$ $(i = 1, 2, \ldots, N)$. Let us set

$$\tau(y, k) = \tau_i(y, k) \quad \text{for} \quad k \in K_\alpha^{\alpha+1} \text{ and } y \in V_i \ (i = 1, 2, \ldots, N).$$

Clearly τ is a homeomorphic embedding from $\bigcup_{i=1}^{N} V_i \times K_\alpha^{\alpha+1}$ into $G_{\alpha+1}$.

By (8.10.4) $K_\alpha^{\alpha+1}$ has at least two different points, say k_1 and k_2. We shall identify the set D_α with the set $\{k_1, k_2\}$. Let us set

$$\psi_{\alpha+1}x = \tau(\psi_\alpha p_\alpha^{\alpha+1}x, k_\nu) \quad \text{for } x = (x_\xi)_{\xi \leqslant \alpha} \in \mathop{\boldsymbol{P}}_{\xi<\alpha+1} D_\xi \text{ with } x_\alpha = k_\nu \ (\nu = 1, 2).$$

It is easily seen that $\psi_{\alpha+1}$ is a homeomorphic embedding from $\mathop{\boldsymbol{P}}_{\xi<\alpha+1} D_\xi$ into $G_{\alpha+1}$ such that $\varphi_\alpha^{\alpha+1}\psi_{\alpha+1} = \psi_\alpha p_\alpha^{\alpha+1}$. By the inductive hypothesis $\varphi_\xi^\alpha \psi_\alpha = \psi_\xi p_\alpha^\xi$ for $0 < \xi < \alpha$. Therefore (8.10.2) implies $\varphi_\xi^{\alpha+1}\psi_{\alpha+1} = \varphi_\xi^\alpha \varphi_\alpha^{\alpha+1}\psi_{\alpha+1} = \varphi_\xi^\alpha \psi_\alpha p_\alpha^{\alpha+1} = \psi_\xi p_\xi^\alpha p_\alpha^{\alpha+1} = \psi_\xi p_\xi^{\alpha+1}$ for $0 < \xi < \alpha$.

To complete the proof of the Proposition let us observe that ψ_ϑ is, by (8.10.5) and (8.10.1), a homeomorphic embedding of $\mathop{\boldsymbol{P}}_{\xi<\vartheta} D_\xi = D^{\mathfrak{n}}$ into $G_\vartheta = G$.

Remark. Proposition 8.10 and Theorem 8.9 remain valid for arbitrary coset space of a compact group. The proofs are analogous to those for the group. We use a modified version of (P) in which the groups G_α are replaced by coset spaces and group epimorphisms by coset-space-epimorphisms which are defined as follows. Let G' and G'' be compact groups, H', H'' its closed subgroups. Let $\varphi : G' \to G''$ be a group epimorphism and let $\varphi(H') = H''$. Then the coset-space-epimorphism $\Phi : G'/H' \to G''/H$ induced by φ is the unique map such that $\psi''\varphi = \Phi\psi'$, where $\psi' : G' \to G'/H'$ and $\psi'' : G'' \to G''/H''$ are natural maps (cf. Definition 7.3). We modify (P) as follows. Let G be a compact group and let H be its closed subgroup. Let $(G_\alpha)_{1\leqslant\alpha<\vartheta}$ and $(\varphi_\alpha^\beta)_{1\leqslant\alpha<\beta\leqslant\vartheta}$ be as in (P). Let us set $H_\alpha = \varphi_\alpha^\vartheta(H)$ for $1 \leqslant \alpha < \vartheta$, and $H_\vartheta = H$. It follows from (8.10.2) that $H_\alpha = \varphi_\alpha^\beta(H_\beta)$ for $1 \leqslant \alpha < \beta \leqslant \vartheta$. Therefore the coset maps $\Phi_\alpha^\beta : G_\beta/H_\beta \to G_\alpha/H_\alpha$ induced by φ_α^β are well defined for $0 < \alpha < \beta \leqslant \vartheta$. We withdraw from the sequence $(G_\alpha/H_\alpha)_{1\leqslant\alpha\leqslant\vartheta}$ the intervals $\alpha < \gamma \leqslant \beta$ such that $\bar{\Phi}_\alpha^\beta$ are one-to-one. The remaining coset spaces we enumerate by succesive ordinal numbers from 1 to some ϑ' in such a way that we preserve the initial order. Similarly we enumerate coset-space-epimorphisms. For the new sequences of coset spaces and coset-space-epimorphisms we repeat the same transfinite construction as in the proof of Proposition 8.10.

The next results indicate that for a non-metrizable compact space S, in general, $C(S)$ is not linearly homeomorphic to $C(D^{\mathfrak{m}})$, where $\mathfrak{m}$ is the topological weight of S.

Let X be a Banach space. Let us consider the following two properties.

(a) If a linear subspace Y of X is linearly homeomorphic to c_0, then Y is complemented in X,

(b) if W is a set in X with the property that every sequence in W contains a weak Cauchy subsequence, then W is separable.

8.11. PROPOSITION. *If a Banach space X is linearly homeomorphic to a closed linear subspace of $C(D^{\mathfrak{m}})$, then X possesses both properties* (a) *and* (b).

Proof. First remark that the properties (a) and (b) are linear homeomorphism invariants and are hereditary in the sense that if a space possesses one of these properties, then each of its closed linear subspaces possesses the same property. Therefore it is enough to show that for every cardinal number $\mathfrak{m}$ the space $C(D^{\mathfrak{m}})$ possesses the properties (a) and (b). For the property (a) this is shown in Engelking and Pełczyński [1], Lemma 6. We shall show here that $C(D^{\mathfrak{m}})$ possesses the property (b). Let $L_2(D^{\mathfrak{m}})$ denote the Hilbert space of all square integrable functions with respect to the usual product measure λ on $D^{\mathfrak{m}}$. Let $I: C(D^{\mathfrak{m}}) \to L_2(D^{\mathfrak{m}})$ denote the natural embedding (i.e. $I(f)$ is the λ-equivalence class of f). It follows immediately from the characterization of the weak convergence in $C(S)$ (cf. Dunford and Schwartz [1], p. 265) and from the Lebesgue dominated convergence principle that I takes weak Cauchy sequences in $C(D^{\mathfrak{m}})$ into Cauchy sequences in the norm topology of $L_2(D^{\mathfrak{m}})$. Hence, if W is a subset of $C(D^{\mathfrak{m}})$ with the property described in (b), then IW is compact in $L_2(D^{\mathfrak{m}})$ and therefore IW is separable. To complete the proof it is enough to use the following consequence of the Peter-Weil theorem applied to the group $D^{\mathfrak{m}}$ (cf. Pontryagin [1], p. 23, Weil [1], p. 74-76). If G is a compact topological group, then every separable set B in $L_2(G)$ belongs to the smallest closed subspace of $L_2(G)$ generated by a sequence of finite dimensional subspaces (E_n); each E_n is spanned by entiers $a_{ij}^{(n)}(\cdot)$ of some irreducible representation of G $(n = 1, 2, \ldots)$. Moreover, if f is continuous and $f \in B$, then f is in the uniform closure of the linear subspace spanned by the sequence (E_n).

8.12. COROLLARY. *If S is either a dyadic space (less generally, if S is an almost Milutin space), or if S is an almost Dugundji space, then $C(S)$ in linearly homeomorphic to a subspace of $C(D^{\mathfrak{m}})$, where $\mathfrak{m}$ is the topological weight of S. Hence $C(S)$ has both properties* (a) *and* (b).

Proof. It follows from a result of Šanin [1] (cf. Engelking and Pełczyński [1], Theorem 1) that for every dyadic space S of topological weight $\mathfrak{m}$ there is an epimorphism φ from $D^{\mathfrak{m}}$ onto S. Hence $\varphi^\circ: C(S) \to C(D^{\mathfrak{m}})$ is an isometric embedding.

If S is an almost Dugundji space of topological weight $\mathfrak{m}$, then there is a homeomorphic embedding $\psi: S \to I^{\mathfrak{m}}$ which has a linear extension operator. Hence, by Proposition 8.2 and Theorem 8.8, $C(S)$ is linearly homeomorphic to a complemented subspace of $C(D^{\mathfrak{m}})$.

8.13. Corollary. *Let S be an non-metrizable compact space having one of the following properties*

(8.13.1) *S is extremally disconnected (cf. Kelley [2], p. 216),*

(8.13.2) *S is scattered (clairsemé in French, Kuratowski [1], p. 95),*

(8.13.3) *S does not satisfy the σ-chain condition, i.e. there is in S an uncountable family of open pairwise disjoint non-empty sets.*

Then $C(S)$ is not linearly homeomorphic to any subspace of $C(D^{\mathfrak{m}})$.

Proof. If S is non-metrizable and extremally disconnected, then $C(S)$ does not have property (a). (For the proof see Engelking and Pełczyński [1], p. 61.)

If S is a non-metrizable scattered space, then the unit ball of $C(S)$ is not separable but every sequence in the unit ball contains a weak Cauchy subsequence (cf. Pełczyński and Semadeni [1], p. 214). Hence $C(S)$ does not have property (b).

If S contains an uncountably family $(U_a)_{a\in A}$ of open non-empty and pairwise disjoint subsets, then one can construct a family $(f_a)_{a\in A}$ of continuous functions on S such that $f_a(s) = 0$ for $s\in S\setminus U_a$ and $\|f_a\| = 1$ $(a\in A)$. Let $W = \bigcup_{a\in A}\{f_a\}$. Then W is an non-separable subset of $C(S)$, because if $a \neq b$, then $\|f_a - f_b\| = 1$. On the other hand every sequence of different elements of W weakly converges to zero. Thus $C(S)$ does not have property (b).

8.14. Corollary. *If S is a non-metrizable compact space having one of the properties* (8.13.1)-(8.13.3), *then S is neither dyadic, nor an almost Dugundji space.*

§ 9. LINEAR AVERAGING OPERATORS AND PROJECTIONS ONTO SPACES OF CONTINUOUS FUNCTIONS

9.1. Definition. Let $\lambda \geqslant 1$. A Banach space X (a separable Banach space X) is called a $\mathfrak{P}_\lambda$ *space* (resp. a $\mathfrak{P}'_\lambda$ *space*) if for every Banach space Y (separable Banach space Y resp.) and every isometric embedding $u: X \to Y$ there is a projection from Y onto uX of norm $\leqslant \lambda$.

It is well known (cf. Day [1], p. 95, Kelley [3], Cohen [1], Hasumi [1]) that $C(S)$ is a $\mathfrak{P}_1$ space if and only if the compact space S is extremally disconnected.

9.2. Definition. A map $\varphi: A \to B$ is called *irreducible* if $\varphi(A) = B$ and $\varphi(F) \neq B$ for every proper closed subset F of A. We recall (cf. Gleason [1]) that for every compact space T there is an irreducible map $\varphi_T: G_T \to T$ from an extremally disconnected compact space G_T onto T. The map φ_T

will be called a *Gleason epimorphism onto T*. It is unique up to a homeomorphism of G_T. Precisely, if $\varphi'_T : G'_T \to T$ and $\varphi''_T : G''_T \to T''$ are Gleason epimorphisms onto T, then there is a homeomorphism ψ from G'_T onto G''_T such that $\varphi''_T = \psi\varphi'_T\psi^{-1}$.

9.3. PROPOSITION. *Let $\lambda \geqslant 1$. Then for every compact space T the following conditions are equivalent:*

(9.3.1) *Every epimorphism φ from an arbitrary compact space S onto T has a linear averaging operator of norm $\leqslant \lambda$.*

(9.3.2) *The Gleason epimorphism $\varphi_T : G_T \to T$ has a linear averaging operator of norm $\leqslant \lambda$.*

(9.3.3) *$C(T)$ is a $\mathfrak{P}_\lambda$ space.*

Proof. (9.3.1) $\Rightarrow$ (9.3.2). This implication is trivial.

(9.3.2) $\Rightarrow$ (9.3.3). Combining (9.3.2) with Corollary 2.3 we infer that there exists a projection of norm $\leqslant \lambda$ from $C(G_T)$ onto its subspace $\varphi_T^\circ[C(T)]$ isometric to $C(T)$. Since G_T is extremally disconnected, $C(G_T)$ is a $\mathfrak{P}_1$ space. Therefore $C(T)$ is a $\mathfrak{P}_\lambda$ space, because it is isometric to the range of a projection of norm $\leqslant \lambda$ from a $\mathfrak{P}_1$ space (cf. Goodner [1], Day [1], p. 99).

(9.3.3) $\Rightarrow$ (9.3.1). Let $\varphi : S \to T$ be an epimorphism. Then $C(T)$ is isometric to the subspace $\varphi^\circ[C(T)]$ of $C(S)$. Therefore (9.3.3) implies that there is a projection from $C(S)$ onto $\varphi^\circ[C(T)]$ of norm $\leqslant \lambda$. To complete the proof we use Corollary 2.3.

9.4. COROLLARY. *A compact space T is extremally disconnected if and only if every epimorphism from an arbitrary compact space onto T admits a regular averaging operator.*

It is well known (cf. Grothendieck [2], Amir [1], [2], Pełczyński [3], Pełczyński and Sudakov [1], Semadeni [1], Lindenstrauss [1], Nachbin [1]) that in general $C(T)$ is not a $\mathfrak{P}_\lambda$ space for any λ ($1 \leqslant \lambda < +\infty$). For such a compact space T the Gleason epimorphism $\varphi_T : G_T \to T$ does not possesses linear averaging operators.

9.5. DEFINITION. A map $\varphi: S \to T$ is said to be of *order n*, in symbols $o(\varphi) = n$, if n is the least integer (if such an integer exists) such that for every t in T the inverse image $\varphi^{-1}(t)$ consists of at most n points.

Isbell and Semadeni [1] (cf. also Amir [1]) examined the relationships between the order of the Gleason epimorphism onto T and the *projection constant*,

$$P_{C(T)} = \inf\{\lambda \geqslant 1 : C(T) \text{ is a } \mathfrak{P}_\lambda \text{ space}\}.$$

Combining their Theorem 1 with Proposition 9.3 and Corollary 9.4 we get(²)

(²) Isbell and Semadeni considered the spaces of real-valued functions. Combining their result with Proposition 2.9 we get the same result for complex-valued functions.

9.6. *Let the Gleason epimorphism $\varphi_T\colon G_T \to T$ have a linear averaging operator, say u. Then*

(9.6.1) *if* $\|u\| < 3$, *then* $o(\varphi_T) < 2(3-\|u\|)^{-1}$;

(9.6.2) *if* $\|u\| < 2$, *then there exists a regular averaging operator for* φ_T; *equivalently T is extremally disconnected.*

Our next step will be Proposition 9.8 which enables us to give an alternative proof of Amir's characterization of those compact metric spaces T for which $C(T)$ are $\mathfrak{P}'_\lambda$ spaces for some $\lambda \geqslant 1$ (cf. Amir [1], [2]). We shall consider a class of epimorphisms of order 2 which do not have linear averaging operators. The typical example of such an epimorphism is the Cantor map $h: \mathscr{C} \to I$ (see Lemma 5.5 for the definition).

9.7. Definition. Let S and T be compact metric spaces and let $d(\cdot,\cdot)$ denote the metric of S. A *map* $\varphi: S \to T$ is said to be *of Cantor type* if

(9.7.1) φ is an epimorphism;

(9.7.2) $o(\varphi) = 2$;

(9.7.3) *if* $\delta > 0$, then the set $F_\varphi(\delta)$ is finite, where

$$F_\varphi(\delta) = \{s \in S : d(s, s') > \delta \text{ and } \varphi(s) = \varphi(s') \text{ for some } s' \text{ in } S\}.$$

Remark. One can restate (9.7.3) in purely topological terms (by eliminating the metric $d(\cdot,\cdot)$ from the definition of $F_\varphi(\delta)$) as follows:

(9.7.4) if $\mathscr{U} = \{U_a\}_{a\in A}$ is an open cover of a compact space S, then the set $F_\delta(\mathscr{U})$ is finite, where

$$F_\delta(\mathscr{U}) = \{s \in S : \text{there is } s' \in S \text{ such that } \varphi(s) = \varphi(s') \text{ but both } s \text{ and } s' \text{ do not belong to the same } U_a \text{ for any } a \in A\}.$$

Replacing (9.7.3) by (9.7.4) one can extend Definition 9.7 to the case of an arbitrary compact spaces.

9.8. Proposition. *Let S and T be metric spaces and let $\varphi: S \to T$ be of Cantor type. Let*

$$F_\varphi^{[0]} = F_\varphi = \bigcup_{\delta>0} F_\varphi(\delta),$$

$$F^{[k]} = \{s \in F_\varphi : \textit{there is } s' \in F_\varphi \textit{ such that } s \neq s',\ \varphi(s) = \varphi(s'),\ \textit{both } s \textit{ and } s' \textit{ are limit points of } F^{[k-1]}\} \quad (k = 1, 2, \ldots).$$

If for some positive integer m the set $F^{[m]}$ is non-empty and if $u: C(S) \to C(T)$ is a linear averaging operator for φ, then $\|u\| \geqslant m$.

The proof of Proposition 9.8 requires some preliminary results. We begin with a generalization of the Stone-Weierstrass theorem (Proposition 9.9) which may be of interest for other purposes. In fact we shall need only Corollary 9.10 which is a very particular case of Proposition 9.9 and which can be proved directly, more simply than the general case.

9.9. PROPOSITION. *Let S and T be compact spaces and let $\varphi\colon S\to T$ be an epimorphism. Then for every f in $C(S)$*

(9.9.1) $$\varrho\big(f,\varphi^\circ[C(T)]\big) = \inf_{g\in C(T)}\|f-g\circ\varphi\| = \sup_{t\in T} r(t),$$

where $r(t)$ is the radius of the smallest circle in the complex plane circumscribed on the set $f\big(\varphi^{-1}(t)\big)$.

Proof. Throughout this proof f is a fixed function in $C(S)$. Let

$$B = \{\mu\in M(S) : \|\mu\|\leqslant 1,\ \mu(\varphi^\circ g) = 0 \text{ for } g\in C(T)\}.$$

Then B is a weak-star compact and convex subset of $M(S)$. Therefore, by the Krein-Milman theorem (cf. Dunford and Schwartz [1], p. 440), B is the weak-star closure of the convex hull of the set B^e of all extreme points of B. Hence $\sup_{\mu\in B}|\mu(f)| = \sup_{\mu\in B^e}|\mu(f)|$. Combining this identity with the Hahn-Banach extension principle (cf. Dunford and Schwartz [1], pp. 62-65) we get

(9.9.2) $$\varrho\big(f,\varphi^\circ[C(T)]\big) = \sup_{\mu\in B^e}|\mu(f)|.$$

Next we shall show that if $\mu\in B^e$, then μ is concentrated on $\varphi^{-1}(t)$ for some t in T. Indeed, let $\mu\in B$ and let there exist in T two different points, say t_0 and t_1, such that the carrier of μ meets both sets $\varphi^{-1}(t_0)$ and $\varphi^{-1}(t_1)$. Let us choose g in $C(T)$ such that $0\leqslant g\leqslant 1$ and $g(t) = j$ for t in some neighbourhood, say U_j, of t_j $(j = 0, 1)$. Let $V_j = \varphi^{-1}(U_j)$. Let $|\mu|$ be the unique non-negative element in $M(S)$ such that $|\mu| = h\mu$ for some unimodular Borel function h on S (for any Borel function k, $k\mu$ denotes the usual product of function and measure; as a functional on $C(S)$, $k\mu(f) = \int_S k(s)f(s)\mu(ds)$ for f in $C(S)$). Clearly $|\mu|(V_j) > 0$, because $V_j\cap\varphi^{-1}(t_j)\neq\emptyset$ and $|\mu|$ is concentrated on the same set as μ. Let $\mu_0 = (1_S-\varphi^\circ g)\mu$ and $\mu_1 = \varphi^\circ g\mu$. Then $\mu = \mu_0+\mu_1$; $\|\mu_j\|\geqslant|\mu|(V_j) > 0$; $\mu_j\in B$ $(j = 0, 1)$. Therefore μ does not belong to B^e. Hence for each μ in B^e there is a unique t_μ in T such that μ is concentrated on $\varphi^{-1}(t_\mu)$.

Now, for $\varepsilon > 0$ let us choose μ_ε in B^e such that

(9.9.3) $$\mu_\varepsilon(f) > \sup_{\mu\in B^e}|\mu(f)|-\varepsilon.$$

Let $t_\varepsilon = t_{\mu_\varepsilon}$ and let z_ε and r_ε denote the centre and the radius respectively of the smallest circle in the complex plane circumscribed on the set $f\big(\varphi^{-1}(t_\varepsilon)\big)$. It is easily seen that for every non-empty set in the plane there exists the unique circle with that property. Since $\mu_\varepsilon(1_S) = 0$ and $\|\mu_\varepsilon\| = 1$ (because $\mu_\varepsilon\in B^e$) and since μ_ε is concentrated on $\varphi^{-1}(t_\varepsilon)$, we have

(9.9.4) $$|\mu_\varepsilon(f)| = |\mu_\varepsilon(f-z_\varepsilon\cdot 1_S)| = \Big|\int_{\varphi^{-1}(t_\varepsilon)}\big(f(s)-z_\varepsilon\big)\mu_\varepsilon(ds)\Big| \leqslant \|\mu_\varepsilon\|\sup_{s\in\varphi^{-1}(t_\varepsilon)}|f(s)-z_\varepsilon| = r_\varepsilon.$$

Thus, letting ε tend to zero, and combining (9.9.2) with (9.9.3) and (9.9.4) we get

$$\varrho\big(f, \varphi^\circ[C(T)]\big) \leqslant \sup_{t\in T} r(t). \tag{9.9.5}$$

Now, let us fix t in T and let $g \in C(T)$. Then the definition of $r(t)$ implies that there is s in $\varphi^{-1}(t)$ (depending on g and t) such that

$$\|f-\varphi^\circ g\| \geqslant |f(s)-(\varphi^\circ g)(s)| = |f(s)-g(t)| \geqslant r(t).$$

Hence

$$\varrho\big(f, \varphi^\circ[C(T)]\big) = \inf_{g\in C(T)} \|f-\varphi^\circ g\| \geqslant \sup_{t\in T} r(t). \tag{9.9.6}$$

Clearly (9.9.5) and (9.9.6) imply (9.9.1). That completes the proof.

9.10. COROLLARY. *If $\varphi\colon S \to T$ is an epimorphism and $o(\varphi) \leqq 2$, then for every f in $C(S)$*

$$\varrho\big(f, \varphi^\circ[C(T)]\big) = \sup \frac{1}{2}|f(s')-f(s'')|, \tag{9.10.1}$$

where the supremum is taken over all pairs (s', s'') in $S\times S$ such that $\varphi(s') = \varphi(s'')$.

9.11. LEMMA. *Let $\varphi\colon S \to T$ be of Cantor type and let the set F_φ be infinite. Then the quotient space $C(S)/\varphi^\circ[C(T)]$ is isometric to the space c_0. The isometry $w\colon \big(C(S)/\varphi^\circ[C(T)]\big) \to c_0$ is defined by*

$$w([f]) = \left(\frac{f(s_{2k-1})-f(s_{2k})}{2}\right)_{k=1,2,\ldots} \quad \textit{for} \quad [f]\in C(S)/\varphi^\circ[C(T)] \tag{9.11.1}$$

where $[f]$ denotes the coset class of $f\in C(S)$, and the sequence (s_k) consists of all elements of F_φ ordered in such a way that $\varphi(s_{2k-1}) = \varphi(s_{2k})$ for $k = 1, 2, \ldots$

Proof. It follows from (9.7.3) that the set F_φ (defined in the statement of Proposition 9.8) is at most countable. Since F_φ is infinite, we can order all elements of F_φ in an infinite sequence $(s_k)_{k=1,2,\ldots}$. Moreover, by (9.7.2), it is possible to order it in such a way that $\varphi(s_{2k-1}) = \varphi(s_{2k})$ for $k = 1, 2, \ldots$ Let us observe that for every f in $C(S)$ the right side of (9.11.1) depends only on the coset class $[f]$ of f. Indeed, if we choose f_1 and f_2 in $C(S)$ such that $[f_1] = [f_2]$, then $f_1-f_2 = \varphi^\circ g$ for some $g\in C(T)$. Hence

$$(f_1-f_2)(s_{2k-1}) = (f_1-f_2)(s_{2k}) = g(t_k),$$
$$\text{where} \quad t_k = \varphi(s_{2k-1}) = \varphi(s_{2k}) \ (k = 1, 2, \ldots).$$

Therefore

$$f_1(s_{2k-1})-f_1(s_{2k}) = f_2(s_{2k-1})-f_2(s_{2k}) \quad (k = 1, 2, \ldots).$$

Next we will check that $w([f])$ belongs to c_0 for every $f \in C(S)$. Indeed, since S is compact, every f in $C(S)$ is uniformly continuous. Therefore for a given $\eta > 0$ there is $\delta = \delta(\eta, f) > 0$ such that if $d(s', s'') < \delta$, then $|f(s') - f(s'')| < \eta$ for any s' and s'' in S. Since by (9.7.3) the set $F_\varphi(\delta)$ is finite, there is an index N such that if $n > N$, then $s_n \notin F_\varphi(\delta)$. Therefore, if $k > N$, then $d(s_{2k-1}, s_{2k}) < \delta$, and consequently $|f(s_{2k-1}) - f(s_{2k})| < \eta$. Hence $\lim_k \frac{f(s_{2k-1}) - f(s_{2k})}{2} = 0$, equivalently $w([f]) \in c_0$.

Next we will establish that w is a linear isometry. By the definition of the norm in the quotient space, we have

$$\|[f]\| = \inf_{g \in C(T)} \|f - \varphi^\circ g\| = \varrho\big(f, \varphi^\circ[C(T)]\big).$$

Thus, by Corollary 9.10,

$$\|[f]\| = \sup_{1 \leqslant k < +\infty} \frac{1}{2} |f(s_{2k-1}) - f(s_{2k})| = \|w([f])\|.$$

Hence w is an isometry. The linearity of w is obvious.

Finally we will show that the range of w is the whole space c_0. For each pair (k, n) of positive integers we define $f_k^{(n)}$ in $C(S)$ such that

$$\|f_k^{(n)}\| = f^{(n)}(s_{2k-1}) = -f^{n}(s_{2k}) = 1,$$

$$f_k^{(n)}(s_m) = 0 \quad \text{for} \quad m \leqslant n \text{ and } m \neq 2k-1,\ m \neq 2k.$$

Clearly

$$\|[f_k^{(n)}]\| = \|w([f_k^{(n)}])\| = 1,$$

and

$$\lim_n \frac{1}{2} [f_k^{(n)}(s_{2m-1}) - f_k^{(n)}(s_{2m})] = \begin{cases} 0 & \text{for} \quad m \neq k, \\ 1 & \text{for} \quad m = k. \end{cases}$$

Thus, by the characterization of weak convergence in c_0 (cf. Dunford-Schwartz [1], p. 339), the sequence $\big(w([f_k^{(n)}])\big)_{n=1}^\infty$ weakly converges in c_0 to the k-th unit vector, say e_k $(k = 1, 2, \ldots)$. Since w is a linear isometry, the range of w is a closed convex subset of c_0. Consequently it is weakly closed. Therefore e_k belongs to the range of w $(k = 1, 2, \ldots)$. Thus the range of w contains the smallest linear subspace spanned by all unit vectors, i.e. the whole space c_0. That completes the proof.

Proof of Proposition 9.8. By Corollary 2.3 it is enough to show that if P is a projection from $C(S)$ onto $\varphi^\circ[C(T)]$, then $\|P\| \geqslant m$.

We observe first that if P is a projection from a Banach space X onto its subspace Y and if v denotes the restriction to $\ker P$ of the quotient map $x \to [x]$ from X onto X/Y, then v is a linear homeomorphism of $\ker P$ onto X/Y such that

$$(9.8.1) \qquad \|x\| \geqslant \|vx\| \geqslant (1+\|P\|)^{-1}\|x\| \quad \text{for} \quad x \in \ker P.$$

The left-hand side inequality is clear, because $\|x\| \geqslant \|[x]\| = \|vx\|$ for all x in $\ker P$. Now, for $\varepsilon > 0$ and for $x \in \ker P$ let $x' \in [x]$ be choosen in such a way that $\|x'\| < \|[x]\| + \varepsilon = \|vx\| + \varepsilon$. Clearly $x' - x \in Y$ and $Px = 0$. Hence $x = x' - Px'$. Therefore $\|x\| \leqslant \|x'\| + \|P\|\,\|x'\|$. Thus $(1+\|P\|)^{-1}\|x\| \leqslant \|x'\| < \|vx\| + \varepsilon$. Since this is true for all $\varepsilon > 0$, we get the right-hand side inequality of (9.8.1).

We shall use the previous observation in the case where $X = C(S)$ and $Y = \varphi^\circ[C(T)]$. In this case, by Lemma 9.11, the quotient space X/Y is isometric to c_0. Let e_n denote the n-th unit vector c_0. Let us set

$$f_n = v^{-1}w^{-1}e_n \quad (n = 1, 2, \ldots),$$

where w is defined by (9.11.1). Then, by the definition of v, $f_n \in \ker P$. Furthermore, by (9.11.1),

$$(9.8.2) \qquad f_n(s_{2n-1}) - f_n(s_{2n}) = 2 \quad (n = 1, 2, \ldots),$$

where the sequence $(s_n)_{n=1,2,\ldots}$ is defined in the statement of Lemma 9.11.

Let us set

$$q(s', s'', f) = \begin{cases} s' & \text{for} \quad |f(s')| \geqslant |f(s'')| \\ s'' & \text{for} \quad |f(s'')| > |f(s')| \end{cases} \quad \big((s', s'') \in S \times S;\ f \in C(S)\big),$$

$$K(s, \delta) = \{s' \in S : d(s, s') < \delta\} \quad \text{for} \quad \delta > 0 \text{ and for } s \in S,$$

where $d(\cdot, \cdot)$ denotes the metric of S.

Let $\varepsilon > 0$. Let us choose an index n_1 in such a way that $s_{2n_1} \in F_\varphi^{[m]}$ (the existence of n_1 follows from the assumption that $F_\varphi^{[m]}$ is non-empty). Let $q_1 = q(s_{2n_1-1}, s_{2n_1}, f_{n_1})$. Pick $\delta_1 > 0$ so that if $d(s', s'') < \delta_1$, then $|f_{n_1}(s') - f_{n_1}(s'')| < \varepsilon/(m+1)$. Next define n_2 so that

$$s_{2n_2} \in F_\varphi^{[m-1]} \cap K(q_1, 2^{-1}\delta_1) \quad \text{and} \quad d(s_{2n_2-1}, s_{2n_2}) < 2^{-1}\delta_1.$$

(Such n_2 exists, because $q_1 \in F_\varphi^{[m]}$ and therefore it is the limit of a sequence of different points belonging to $F_\varphi^{[m-1]}$. Thus in $K(q_1, 2^{-1}\delta_1)$ there are infinitely many points belonging to $F_\varphi^{[m-1]}$. But only finite number of them belong to $F_\varphi(2^{-1}\delta_1)$ (by (9.7.3)). Next we pick a positive $\delta_2 < \delta_1$ so that if $d(s', s'') < \delta_2$, then $|f_{n_2}(s') - f_{n_2}(s'')| < \varepsilon/(m+1)$ for $(s', s'') \in S \times S$. We put $q_2 = q(s_{2n_2-1}, s_{2n_2}, f_{n_2})$. Repeating this procedure we define inductively a finite sequences of indices $n_1, n_2, \ldots, n_{m+1}$ and of positive numbers $\delta_1 > \delta_2 > \ldots > \delta_m$ such that

(9.8.3) $$s_{2n_\nu} \in F_\varphi^{[m-\nu+1]},$$

(9.8.4) if $d(s', s'') < \delta_\nu$, then $|f_{n_\nu}(s') - f_{n_\nu}(s'')| < \dfrac{\varepsilon}{m+1}$

for (s', s'') in $S \times S$ $(\nu = 1, 2, \ldots, m)$,

(9.8.5) $$d(s_{2n_\nu - 1}, s_{2n_\nu}) < 2^{-1}\delta_{\nu-1} \quad (\nu = 2, 3, \ldots, m+1),$$

(9.8.6) $$s_{2n_\nu} \in \bigcap_{\mu=1}^{\nu-1} K(q_\mu, 2^{-1}\delta_\mu) \quad \text{for} \quad \nu = 2, 3, \ldots, m+1,$$

where $q_\nu = q(s_{2n_\nu - 1}, s_{2n_\nu}, f_\nu)$ for $\nu = 1, 2, \ldots, m+1$.

It follows from (9.8.5) and (9.8.6) that $q_{m+1} \in K(q_\nu, 2^{-1}\delta_\nu)$ for $\nu = 1, 2, \ldots, m$. Hence by (9.8.4)

$$|f_{n_\nu}(q_{m+1}) - f_{n_\nu}(q_\nu)| < \frac{\varepsilon}{m+1} \quad \text{for} \quad \nu = 1, 2, \ldots, m+1.$$

Thus

$$\sum_{\nu=1}^{m+1} |f_{n_\nu}(q_{m+1})| \geqslant \sum_{\nu=1}^{m+1} |f_{n_\nu}(q_\nu)| - \sum_{\nu=1}^{m+1} |f_{n_\nu}(q_\nu) - f_{n_\nu}(q_{m+1})| \geqslant \sum_{\nu=1}^{m+1} |f_{n_\nu}(q_\nu)| - \varepsilon.$$

On the other hand, combining (9.8.2) with the definition of the points q we infer that $|f_{n_\nu}(q_\nu)| = \max\big(|f_{n_\nu}(s_{2n_\nu - 1})|, |f_n(s_{2n_\nu})|\big) \geqslant 1$ $(\nu = 1, 2, \ldots, m+1)$. Hence

$$\sum_{\nu=1}^{m+1} |f_{n_\nu}(q_{m+1})| \geqslant m+1-\varepsilon.$$

Now we pick complex numbers a_ν such that $|a_\nu| = 1$ and $a_\nu f_{n_\nu}(q_{m+1}) = |f_{n_\nu}(q_{m+1})|$ $(\nu = 1, 2, \ldots, m+1)$. Then

(9.8.7) $$\Big\|\sum_{\nu=1}^{m+1} a_\nu f_{n_\nu}\Big\| \geqslant \Big|\sum_{\nu=1}^{m+1} a_\nu f_{n_\nu}(q_{m+1})\Big| = \sum_{\nu=1}^{m+1} |f_{n_\nu}(q_{m+1})| \geqslant m+1-\varepsilon.$$

Let $x = \sum_{\nu=1}^{m+1} a_\nu f_{n_\nu}$. Since w is an isometry, the identity $vf_n = w^{-1}e_n$ $(n = 1, 2, \ldots)$ together with the well-known propertes of the unit vectors in c_0 imply

$$1 = \max_\nu |a_\nu| = \Big\|\sum_{\nu=1}^{m+1} a_\nu e_\nu\Big\| = \Big\|w^{-1}\Big(\sum_{\nu=1}^{m+1} a_\nu e_\nu\Big)\Big\| = \|vx\|.$$

On the other hand, combining (9.8.1) with (9.8.7) we get

$$1 = \|vx\| \geqslant (1+\|P\|)^{-1}\|x\| = \frac{m+1-\varepsilon}{1+\|P\|}.$$

Let ε tend to zero, and we get $1+\|P\| \geqslant m+1$. Hence $\|P\| \geqslant m$. That completes the proof.

9.12. COROLLARY. *There is no linear averaging operator for the Cantor map* $h: \mathscr{C} \to I$ *where* $h = \sum_{i=1}^{\infty} 2^{-1}\xi_i$ *for* $\xi = (\xi_i) \in \mathscr{C}$.

Proof. Clearly h is of Cantor type. The set F_h consists of all $\xi = (\xi_i) \in \mathscr{C}$ such that either almost all $\xi_i = 0$, or almost all $\xi_i = 1$, and $0 \neq \xi \neq 1$. (If we represent $\mathscr{C}$ as the subset of I consisting of all triodic fractions with figures equal either 0 or 2, then F_h consists of all end-points of all open intervals being components of $I \setminus \mathscr{C}$.) Therefore $F_h^{[m]} = F_h$ $(m = 1, 2, \ldots)$. Thus, according to Proposition 9.8, there is no linear averaging operator for h.

9.13. THEOREM (Amir [1], [2]). *If S is an infinite compact metric space, then the following conditions are equivalent*:

(9.13.1) *Some derived set of S (cf. Kelley [1], p. 42) of finite order is empty*;

(9.13.2) *$C(S)$ is linearly homeomorphic to the space c of all convergent sequences*;

(9.13.3) *$C(S)$ is a $\mathfrak{P}'_\lambda$ space for some $\lambda \geqslant 1$.*

Proof. We recall that if ϑ is an ordinal, then $[\vartheta]$ denotes the space of all ordinals $\leqslant \vartheta$ endowed with the usual order topology (cf. Kelley [2], pp. 57, 266-271).

(9.13.1) $\Rightarrow$ (9.13.2). Let k be the first ordinal with the property that $S^{(k)} = \emptyset$. Since S is infinite and satisfies (9.13.1), $1 < k < \omega$. Therefore $\chi(S) = k^\omega = 2^\omega = \chi([\omega])$. (For the definition of $\chi(S)$ see 8.6). Thus, by Corollary 8.7, $C(S)$ is linearly homeomorphic to $C([\omega])$. Clearly $C([\omega])$ can be identified with the space c.

(9.13.2) $\Rightarrow$ (9.13.3). By a theorem of Sobczyk [1] (see also Pełczyński [3], p. 217, Dean [2]) the space c_0 of all scalar-valued sequences convergent to zero is a $\mathfrak{P}'_2$ space. Since c is linearly homeomorphic to c_0 (cf. Banach [1], pp. 182-184), (9.13.2) implies that $C(S)$ is linearly homeomorphic to c_0. To complete the proof of the implication it is enough to use the following general observation (cf. Pełczyński [3], Proposition 1).

(*) *If X is a $\mathfrak{P}'_\lambda$ space for some $\lambda > 1$ and if X_1 is linearly homeomorphic to X, then X_1 is a $\mathfrak{P}'_\mu$ space for some $\mu \geqslant 1$.*

Proof. We have to show that X_1 regarded as a subspace of an arbi trary separable Banach space, say Y_1, admits a projection $\pi_1: Y_1 \to X_1$ (onto) with $\|\pi_1\| \leqslant \mu$ where μ depends only on X_1. Let $\|\cdot\|_1$ denote the original norm in Y_1. By the assumption there is a linear homeomorphism, say v, from X onto X_1 (which may be regarded also as a linear homeomorphic embedding of X into Y_1). Hence there are positive constants $a = \|v^{-1}\|^{-1}$ and $b = \|v\|$ such that $a\|x\| \leqslant \|vx\|_1 \leqslant b\|x\|$ for $x \in X$. We introduce in Y_1 the new norm $\|\cdot\|$ being the Minkowski functional of the convex body W, where

$$W = \{z \in Y_1 : z = tvx + (1-t)y_1, \text{ where } 0 \leqslant t \leqslant 1;\ x \in X,\ \|x\| \leqslant 1;\ y_1 \in Y_1;\ \|y_1\| \leqslant a\},$$

i.e. we put

$$\|y\| = \inf\left\{k > 0 : \frac{y}{k} \in W\right\} \quad \text{for} \quad y \in Y_1.$$

Let Y denote the space Y_1 under the new norm $\|\cdot\|$. Then Y is linearly homeomorphic to Y_1. Precisely, one can easily check that the identity map $i: Y \to Y_1$ with $iy = y$ for $y \in Y$ is a linear homeomorphism satisfying the inequalities

$$a\|y\| \leqslant \|iy\|_1 \leqslant b\|y\| \quad \text{for} \quad y \in Y.$$

Furthermore $i^{-1}v$ is an isometric embedding of X into Y. Since X is a $\mathfrak{P}'_\lambda$ space, there is a projection $\pi: Y \to X$ (onto) with $\|\pi\| \leqslant \lambda$. One can easily check that $\pi_1 = i\pi i^{-1}$ is the desired projection from Y_1 onto X_1 with $\|\pi_1\| \leqslant \lambda b a^{-1} = \lambda\|v\|\,\|v^{-1}\|$.

non (9.13.1) $\Rightarrow$ non (9.13.3). This implication is an immediate consequence of the following three lemmas

9.14. LEMMA. *If X is a $\mathfrak{P}'_\lambda$ space and if Z is a complemented subspace of X, then Z is a $\mathfrak{P}'_\lambda$ space.*

9.15. LEMMA. *If S is a compact metric space which does not satisfy* (9.13.1), *i.e. if all derived sets $S^{(k)}$ are non-empty for $k < \omega$, then $C(S)$ contains a complemented subspace isometric to $C([\omega^\omega])$.*

9.16. LEMMA. *$C([\omega^\omega])$ is not a $\mathfrak{P}'_\lambda$ space for any $\lambda \geqslant 1$.*

Proof of Lemma 9.14. Let Z_1 be a complementary subspace to Z in X, and let p and p_1 be projections from X onto Z and onto Z_1 respectively such that $\ker p = Z_1$ and $\ker p_1 = Z$. Let u be an isometric embedding of Z into a separable Banach space Y. Then v, with $vx = (upx, p_1x)$ for $x \in X$ is a linear homeomorphism from X into $Y \times Z_1$. Since X is a $\mathfrak{P}'_\lambda$ space, (*) implies (cf. the proof of the previous implication) that vX is a $\mathfrak{P}'_\mu$ space for $\mu = \|v\|\,\|v^{-1}\|\lambda$. Hence there is a projection π from $Y \times Z_1$ onto vX with $\|\pi\| \leqslant \mu$. Now one can easily verify that $q\pi j$ is the desired projection from Y onto uZ, where j is the natural isometric embedding of Y into $Y \times Z_1$ (i.e. $jy = (y, 0)$ for $y \in Y$) and q is the natural projection from $Y \times Z_1$ onto Y (i.e. $q[(y, z_1)] = y$ for $(y, z_1) \in Y \times Z_1$). That completes the proof.

Proof of Lemma 9.15. We observe first that it is enough to show that S contains a subset, say S_0, homeomorphic to $[\omega^\omega]$. Indeed, that implies that there is a regular extension operator from $C(S_0)$ into $C(S)$

(because $[\omega^\omega]$ being metrizable is, by Theorem 6.6, a Dugundji space). Thus, by Corollary 2.4, $C([\omega^\omega])$ is isometric to a complemented subspace of $C(S)$.

Let us consider two cases. 1° S is uncountable. Then, since S is metric, it contains a homeomorphic copy of the Cantor discontinuum (cf. Hausdorff [1], p. 136-138) and therefore a homeomorphic copy of every zero-dimensional compact metric space, in particular a copy of $[\omega^\omega]$. 2° S is countable. Then, by a result of Mazurkiewicz and Sierpiński (cf. Kuratowski [2], p. 58), S is homeomorphic to the space $[\omega^\vartheta\cdot n]$, where ϑ is the last countable ordinal number such that the derived set $S^{(\vartheta)}$ is non-empty and n is the number of elements in $S^{(\vartheta)}$ (clearly $S^{(\vartheta)}$ is finite). By the assumption of the Lemma, $\vartheta \geqslant \omega$. Therefore $\omega^\vartheta\cdot n \geqslant \omega^\vartheta \geqslant \omega^\omega$. Hence the space S homeomorphic to $[\omega^\vartheta\cdot n]$ contains a component homeomorphic to $[\omega^\omega]$. That completes the proof.

Proof of Lemma 9.16. First we shall define inductively maps $\varphi_n: [\omega^n\cdot 2]\to[\omega^n]$ of Cantor type such that

$$F^{[n]}_{\varphi_n} = \{\omega^n\}\cup\{\omega^n\cdot 2\}, \tag{9.16.1}$$

$$\varphi_m(x) = \varphi_n(x) \quad \text{for} \quad x\leqslant \omega^m\cdot 2 \ (1\leqslant m<n;\ n = 1,2,\ldots). \tag{9.16.2}$$

Let us put

$$\begin{aligned}\varphi_1(\omega) &= \varphi_1(\omega\cdot 2) = \omega,\\ \varphi_1(2k-1) &= \varphi_1(2k) = 2k-1,\\ \varphi_1(\omega+2k-1) &= \varphi_1(\omega+2k) = 2k \qquad (k = 1,2,\ldots).\end{aligned}$$

Clearly the map $\varphi_1: [\omega\cdot 2]\to[\omega]$ is of Cantor type and $F^{[1]}_{\varphi_1} = \{\omega\}\cup\{\omega\cdot 2\}$.

Let us suppose that for $1\leqslant m<n$ the maps of Cantor type satisfying (9.16.1) and (9.16.2) have been already defined. Let us set

$$\begin{aligned}\varphi_n(\omega^n\cdot 2) &= \varphi_n(\omega^n) = \omega^n,\\ \varphi_n\big(\omega^{n-1}(2k-2)+x\big) &= \varphi_n\big(\omega^{n-1}(2k-1)+x\big)\\ &= \omega^{n-1}(2k-2)+\varphi_{n-1}(x),\\ \varphi_n\big(\omega^n+\omega^{n-1}(2k-2)+x\big) &= \varphi_n\big(\omega^n+\omega^{n-1}(2k-1)+x\big)\\ &= \omega^{n-1}(2k-1)+\varphi_{n-1}(x)\\ &\qquad \text{for } 0<x\leqslant\omega^{n-1}\ (k = 1,2,\ldots).\end{aligned} \tag{9.16.3}$$

If follows from inductive hypothesis and (9.16.3) that

$$F^{[n-1]}_{\varphi_n}\supset\bigcup_{k=1}^{\infty}\{\omega^{n-1}k\}\cup\bigcup_{k=1}^{\infty}\{\omega^n+\omega^{n-1}k\}.$$

Therefore, by definition,

$$F^{[n]}_{\varphi_n}\supset\{\omega^n\}\cup\{\omega^n\cdot 2\}.$$

Observe that if $\varphi: S \to T$ is any map of Cantor type, then $F_\varphi^{[n]} \subset S^{(n)}$, where $S^{(n)}$ denotes the n-th derived set of S. Since the n-th derived set of the space $[\omega^n \cdot 2]$ is exactly the two-point set $\{\omega^n\} \cup \{\omega^n \cdot 2\}$, we have the inclusion $F_{\varphi_n}^{[n]} \subset \{\omega^n\} \cup \{\omega^n \cdot 2\}$. Therefore $F_{\varphi_n}^{[n]} = \{\omega^n\} \cup \{\omega^n \cdot 2\}$. Clearly $\varphi_n(x) = \varphi_{n-1}(x)$ for $0 < x \leqslant \omega^{n-1} \cdot 2$. Furthermore if $o(\varphi_{n-1}) = 2$, then $o(\varphi_n) = 2$. Finally, it is easy to verify that if φ_{n-1} satisfies (9.7.3), then φ_n does. Hence φ_n is of Cantor type. That completes the induction.

Now, let us define the epimorphism $\varphi_\omega : [\omega^\omega] \to [\omega^\omega]$ by

$$\varphi_\omega(\omega^\omega) = \omega^\omega,$$
$$\varphi_\omega(x) = \varphi_n(x) \quad \text{for} \quad 0 < x \leqslant \omega^n \cdot 2 \ (n = 1, 2, \ldots).$$

Since all φ_n are of Cantor type, (9.16.2) implies that φ_ω is also of Cantor type. By (9.16.1), $F_{\varphi_\omega}^{[n]}$ is not empty for $n = 1, 2, \ldots$ Thus, by Proposition 9.8, the map $\varphi_\omega : [\omega_\omega] \to [\omega^\omega]$ has no linear averaging operators. Equivalently (by Corollary 2.3) there is no projection from $C([\omega^\omega])$ onto its subspace $\varphi_\omega^0[C([\omega^\omega])]$ isometric to $C([\omega^\omega])$. Therefore $C([\omega^\omega])$ is not a $\mathfrak{P}'_\lambda$ space for any $\lambda \geqslant 1$. That completes the proof.

NOTES AND REMARKS

Ad § 2. The existence of an extension for an arbitrary real valued continuous function defined on a closed subset of a metric space M was proved by Tietze [1] (cf. also Hausdorff [2]), and generalized by Urysohn [1] to the case of normal topological spaces. The case where M is an n-dimensional Euclidean space was treated earlier by several authors; L. E. J. Brouwer [1], § 4, [2], H. Bohr (see Carathéodory [1], § 541-542), Carathéodory [2], §§ 541-542, De la Vallé Poussin [1] p. 319, and Lebesgue [1], p. 379.

The extension problem for continuous function is closely related to several topological theories. After Borsuk's papers [2], [3], it became clear that many extension problems for continuous functions can be restated in terms of the theory of retracts. We refer the reader to the books of Borsuk [4] and Hu [2] and to references in this books for further information. Borsuk's homotopy extension theorem and other connections between extension theory and homotopy theory can be found in Hu [1] and references given there. For the relation to obstruction theory we refer the reader to the pioneer paper of Eilenberg [1] and to the book of Steenrod [1]. In E. Michael [4], [5], [6], [7] the connection between extension theorems and selection theorems is discussed. Roughly speaking, every selection theorem contains as a special case some extension theorem. Several topological notions previously intrinsically defined are either equivalent to some extension or selection properties, e.g. the characterization of dimension (cf. Hurewicz and Wallman [1], p. 83 in the metric case and Nagata [1] in the more general case), collectionwise normality (cf. Dowker [1], Bing [1]). For other similar results see Arens [1], Hanner [1], [2], [3], Michael [1], [5].

Linear extension operators are usually called operators of simultaneous extension. The first result in the theory is due to Borsuk [1]. For further results see Kakutani [1], Dugundji [1], who proved the existence of regular linear operator of extension for arbitrary closed subset of metric space, Arens [1], Michael [1], Pełczyński [1], [2], Borges [1], Michael and Pełczyński [1], and Semadeni's expository papers [1], [2]. Clearly linear extension operators, as well as linear exaves, can be defined for arbitrary topological spaces and for special classes of functions on these spaces (cf. Pełczyński [1]).

If $\varphi : S \to T$ is a homeomorphic embedding and S, T are non-metrizable compact spaces, then in general there is no linear extension operators for φ (Arens [1], Day [2], Semadeni [1], Corson and Lindenstrauss [1], cf. also Corollary 8.14 in the present paper. The related examples are also given by Michael [1], Klee [1]).

Recently Corson and Lindenstrauss [1] constructed for $k = 1, 2, \ldots$ a homeomorphic embedding $\varphi_k : S_k \to T_k$ (S_k, T_k — compact spaces) such that φ_k has linear extension operators, but the norm of every such operator is $\geqslant 2k-1$ (cf. also Example 6 in this Notes and Remarks).

Extensions and linear extensions of smooth functions to smooth functions. The basic results on extension of differentiable functions are due to Whitney [1] (cf. also Hestenes [1], Calderón [1]). The formulas of Whitney (cf. Whitney [1], [2]) define a linear extension in the case of finitely differentiable functions defined on an arbitrary closed subset of the Euclidean space E^N but not in the case of infinitely differentiable functions.

Operators of linear extension for infinitely many time differentiable functions have been constructed recently by several authors under various assumptions on the sets of arguments (cf. Mitjagin [1], Seeley [1], Aronszajn [1], Adams, Aronszajn and Smith [1], Ogrodzka [1]. Ogrodzka's method is essentially based upon some previously unpublished result of Ryll-Nardzewski). Some counterexamples are given in Mitjagin [1] and Adams, Aronszajn and Smith [1]. Ogrodzka [1] treated the case of linear extension operators in vector bundless. It seems that the general theory of linear exaves can be extended to this case.

Extensions of continuous functions defined on a closed subset of a (compact) topological space T to the function belonging to a given closed linear subspace of $C(T)$ are treated by Bishop [1], Glicksberg [2], Pełczyński [1], [2], Gamelin [1], Michael and Pełczyński [1]. These results generalized previous results concerning disc algebra due to Rudin [2] and Carleson [1].

An interesting result on extending a metric is due to Hausdorff [3]; see also Arens [1], Bing [2], Kuratowski [3].

Several authors have studied the problem of extending a function with given modulus of continuity (especially a Lipschitz function), which is initially defined on a subset of a given metric space, to a function with the same modulus of continuity defined on the whole space (cf. McShane

[1], Kirszbraun [1], Banach [2], Mickle [1], Valentine [1], [2], [3], Cipszer and Geher [1], Aronszajn and Panitchpakdi [1]). For more detailed information we refer to the expository paper by Danzer, Grünbaum and Klee [1] and Isbell [1].

There is rather unsatisfactory information as to whether there exist linear extension operators taking uniformly continuous functions into uniformly continuous functions or Lipschitzian functions into Lipschitzian functions. The next Proposition and Corollaries show that for arbitrary metric spaces such operators need not exist. This is related to the recent results of Lindenstrauss [2] concerning non-linear projections (cf. Corollary D in this Notes and Remarks).

PROPOSITION A. *Let X be a Banach space and let Y be a closed linear subspace of X. Let $C_u(X)$ and $C_u(Y)$ denote spaces of all uniformly continuous real-valued functions on X and Y respectively. Let $v: C_u(Y) \to C_u(X)$ be a linear extension operator which is continuous provided $C_u(Y)$ and $C_u(X)$ both carry the same one of the following three topologies: topology of simple convergence, topology of compact convergence, topology of uniform convergence* (cf. Bourbaki [2], pp. 4-5). *Then there is a linear operator $w: C_u(Y) \to X^*$ such that the restriction of w to Y^* is a linear homeomorphic embedding of the Banach space Y^* into the Banach space X^* such that $(\omega y^*)(y) = y^*y$ $(y^* \in Y^*,\ y \in Y)$.*

Proof. Let $\int\limits_A \ldots da$ denote an invariant mean on the space $B(A)$ of all bounded real-valued functions on an Abelian group A. (cf. Hewitt and Ross [1], pp. 230-245). Let us put

$$(wf)(z) = \int\limits_X \Big(\int\limits_Y [(vf)(x+y+z)-(vf)(x+y)]dy\Big)dx, \qquad f \in C_u(Y),\ z \in X.$$

It can be easily verify that the operator w has the desired properties.

COROLLARY B. *If a Banach space X together with its closed linear subspace Y satisfy the assumption of Proposition* A, *then the pair* (X, Y) *satisfies each of the following equivalent conditions*

(i) *there is a bounded linear operator $w: Y^* \to X^*$ such that $(wy^*)(y) = y^*y$ for $y^* \in Y^*$ and for $y \in Y$,*

(ii) *there is a bounded linear projection π from X^* onto the annihilator* $Y^{\perp} = \{x^* \in X^* : x^*(y) = 0$ for $y \in Y\}$,

(iii) *there is a bounded linear operator $u: X \to Y^{**}$ such that $uy = y$ for $y \in Y$* (3).

(3) Here and in the next Corollaries we identify a Banach space E with its cannonical image in E^{**}.

Proof. In view of Proposition A it as enough to show that for every pair consisting of a Banach space X and its subspace Y the conditions (i), (ii) and (iii) are equivalent each to other.

(i) $\Rightarrow$ (ii). Let $r: X^* \to Y^*$ be the restriction operator, i.e. $(ry^*)(y) = y^*y$ for every y in Y and for every y^* in Y^*. We put $\pi x^* = x^* - wrx^*$ for $x^* \in X^*$.

(ii) $\Rightarrow$ (i). Let $a: Y^* \to X^*$ be an operator (not necessarily linear and continuous) such that y^* is an extension of y^* and $\|ay^*\| = \|y^*\|$ for $y^* \in Y^*$. We put $wy^* = ay^* - \pi ay^*$ for $y^* \in Y^*$. Clearly w is a bounded operator from Y^* into X^*. Let y_1^* and y_2^* be arbitrary linear functionals in Y^*. Then $a(y_1^* + y_2^*) - ay_1^* - ay_2^* \in Y^\perp$ (because ay^* is an extension of y^*). Therefore

$$\pi[a(y_1^* + y_2^*) - ay_1^* - ay_2^*] = a(y_1^* + y_2^*) - ay_1^* - ay_2^*.$$

On the other hand, by linearity of π,

$$\pi[a(y_1^* + y_2^*) - ay_1^* - ay_2^*] = \pi a(y_1^* + y_2^*) - \pi ay_1^* - \pi ay_2^*.$$

Combining those formulas with the definition of w we get

$$w(y_1^* + y_2^*) = wy_1^* + wy_2^*.$$

That proves the linearity of w. Clearly wy^* is an extension of y^*, because $\pi ay^* \in Y^\perp$ for every $y^* \in Y^*$. That completes the proof of the implication.

(i) $\Rightarrow$ (iii). Define u as the restriction to X of the adjoint operator $w^*: X^{**} \to Y^{**}$.

(iii) $\Rightarrow$ (i). Define w as the restriction to Y^* of the adjoint operator $u^*: Y^{***} \to X^*$.

COROLLARY C. *If the pair (X, Y) satisfies the assumption of Proposition* A *and if there exists a bounded linear projection p form Y^{**} onto Y, then there exists a bounded linear projection q from X onto Y.*

Proof. Put $q = pu$, where $u: X \to Y^{**}$ is defined in (iii).

COROLLARY D (cf. Lindenstrauss [2]). *If there exists a uniform retraction from a Banach space X onto its closed linear subspace Y, then the pair (X, Y) satisfies the equivalent conditions* (i), (ii), *and* (iii). *In particular if there exists a bounded linear projection from Y^{**} onto Y, then there exists a bounded linear projection from X onto Y.*

Proof. Observe that if φ is a uniform retraction from X onto Y, then $\varphi^\circ: C_u(Y) \to C_u(X)$ is a linear extension operator satisfying the assumption of Proposition A. Then apply Corollaries B and C.

Remark. Let Y be an infinite dimensional reflexive subspace of the space $X = C(S)$, S–arbitrary compact space. Clearly the identity operator $Y \to Y$ can be regarded as a projection from $Y^{**} = Y$ onto Y. On the other hand there is no bounded linear projection from $X = C(S)$ onto Y

(cf. Grothendieck [2], Pełczyński [3]). Therefore, by Corollary C, there is no linear extension operator from $C_u(Y)$ into $C_u(X)$ which is continuous provided both spaces $C_u(Y)$ and $C_u(X)$ carry one of the topologies described in Proposition A.

Proposition 2.2 is well known (cf. Borsuk [1], Bessaga and Pełczyński [1], Pełczyński [1], [3], Dean [1], Semadeni [1], [2]).

For a result similar to Proposition 2.6 cf. Dean [1] and Amir [3].

Problem 1. May one replace in (2.8.1) and (2.8.3) the terms "Banach space" by "normed linear space" or, more generally, by "locally convex linear space"? In the last case by $C(S, E)$ we mean the space of all continuous mapping from a compact space S into a locally convex linear space E with the topology of uniform convergence induced by a base of all neighbourhoods of 0 of the form

$$\mathscr{V}_V = \big(f \in C(S, E) : f(s) \in V\big)$$

where V is a neighbourhood of zero in E.

Proposition 2.9 has been observed by E. Michael (oral communication) and is published here with his permission.

Regular averaging operators were introduced by Birkhoff [1] and have been investigated by Kelley [1], Davis [1], Wright [1], Lloyd [1], [2], Moy [1], G. C. Rota [1], Brainerd [1], [2], Michael [2], [3], Corson and Lindenstrauss [2]. For further information we refer the reader to the expository paper by Birkhoff [2]. Michael, Corson and Lindenstrauss used different terminology but considered exactly the same kind of operators (cf. Notes and Remarks to § 3). Averaging operators are a "limit case" of Reynolds operators which arose from the operator used by the British engineer, Osborne Reynolds [1], in his classic calculation of the average stresses due to turbulent momentum convection. The mathematical investigation of Reynolds operators was initiated by Kampé de Fériet [1], [2]. For further information we refer the reader to the paper of G. C. Rota [2] and the references in this paper.

Actually Birkhoff defined an averaging operator in $C(S)$ as a regular linear operator $A : C(S) \to C(S)$ such that

$$\text{(B)} \qquad A(fAg) = Af \cdot Ag \quad \text{for} \quad f, g \in C(S).$$

It can easily be shown that A is a projection whose range is a self-adjoint subalgebra of $C(S)$. Hence there is a compact space T and an epimorphism $\varphi : S \to T$ such that $\varphi^\circ[C(T)] = A[C(S)]$. Thus the projection A

with $\|A\| = 1$ uniquely determines the regular averaging operator $u : C(S) \to C(T)$ for φ such that $A = \varphi^\circ \circ u$. Conversely if $u : C(S) \to C(T)$ is a regular averaging operator for an epimorphism $\varphi : S \to T$, then applying Corollary 2.3 and Proposition 4.1 one can easily show that $A = \varphi^\circ u$: $C(S) \to C(S)$ is a regular operator satisfying (B).

EXAMPLE 1. Non-normal exave. Let $S = T = \{1\} \cup \{2\}$ be the two-point discrete space and let $\varphi : S \to T$ be defined by $\varphi(1) = \varphi(2) = 1$. Let us set $u(x(1), x(2)) = \big((x(1)+x(2))/2, x(1)\big)$ for $x = (x(1), x(2)) \in C(\{1\} \cup \{2\})$. Then u is a non-normal regular exave for φ.

Let $\varphi : S \to T$ be a map and let $M : C(S) \to C(T)$ be a continuous operator (not necessary linear) such that $M(fg) = M(f)\cdot M(g)$ for f, g in $C(S)$ and $\varphi^\circ M \varphi^\circ = \varphi^\circ$. Then M will be called a *multiplicative exave* for φ.

It seems to be interesting to investigate what maps have multiplicative exaves.

Problem 2. Does there exists a multiplicative extension operator from $C(\partial B_n)$ into $C(B_n)$, where B_n is the unit ball in n-dimensional Euclidean space and ∂B_n is the boundary of B_n — the $(n-1)$-dimensional sphere?

We are able to show that there is no multiplicative extension operator from $C(\partial B_2)$ into $C(B_2)$.

Ad § 3. Proposition 3.3 is due to H. Yoshizawa [1]. Propositions 3.3 and 3.4 can easily be extended to the case where S and T are arbitrary completely regular spaces.

Localization principle. As far as we know the idea of localization was first used by Lichtenstein [1] and was applied by several authors in the case of smooth extensions (cf. McShane [2], Hestenes [1], Adams-Aronszajn and Smith [1], Ogrodzka [1]).

Ad § 4. Let u be a regular averaging operator for an epimorphism $\varphi : S \to T$. Then, Proposition 4.1 implies that u satisfies the inequality (cf. Michael [2])

$$(+) \qquad \inf_{s \in \varphi^{-1}(t)} f(s) \leqslant uf(t) \leqslant \sup_{s \in \varphi^{-1}(t)} f(s), \qquad \text{for} \quad f \text{ in } C_R(S) \text{ and } t \in T.$$

Conversely if $u_R : C_R(S) \to C_R(T)$ is a linear operator satisfying (+), then u_R is a regular averaging operator from $C_R(S)$ onto $C_R(T)$ and therefore, by Proposition 2.9, it can be extended to a regular averaging operator from $C(S)$ onto $C(T)$. Therefore a result of Michael [2], Theorem 1.1, may be restated as follows:

If S and T are compact metric spaces and if φ is an open map from S onto T, then φ admits a regular averaging operator.

Corson and Lindenstrauss [2], pp. 501-504, showed that the above result may fail for non-metrizable compact spaces. However they proved that for some special non-metrizable compact spaces the assertion of Michael's theorem still holds.

The map $\Psi: \mathscr{C} \times \mathscr{C} \to I$ constructed in Milutin Lemma 5.5 is not open, because every open map from zero-dimensional space has zero-dimensional image. On the other hand, by Lemma 5.5, Ψ has a regular averaging operator.

Problem 3. Let S and T be compact metric spaces. Give a necessary and sufficient condition (in topological terms) in order that an epimorphism $\varphi: S \to T$ admit a regular averaging operator.

EXAMPLE 2. Let $S = I = [0, 1]$ and let T be the unit circle on the complex plane. Let $\varphi(s) = e^{2\pi i s}$ for $s \in I$. Then there is no regular averaging operator for φ. Precisely, if u is a linear averaging operator for φ, then $\|u\| \geqslant 2$.

Proof. According to Corollary 2.3 it is enough to show that if p is a projection from $C(I)$ onto its subspace $E = \varphi^{\circ}[C(T)]$, then $\|p\| \geqslant 2$. Since E is a maximal hyperplane of $C(I)$, p is of the from, $p(f) = f - \mu(f)g$ for f in $C(I)$, where μ is a linear functional on $C(I)$ with $\ker\mu = E$ and $g \in C(I)$ is picked such that $\mu(g) = 1$. Thus, replacing (if necessary) μ by $a\mu$ and g by $a^{-1}g$ where a is a suitable complex number, we may assume that $\mu = \frac{1}{2}(\delta_1 - \delta_0)$, where δ_i denotes the unit point mass at the point i $(i = 0, 1)$. Hence $p(f) = f - \frac{1}{2}[f(1) - f(0)]g$ for $f \in C(I)$, where $\frac{1}{2}\big(g(1) - g(0)\big) = \frac{1}{2}(\delta_1 - \delta_0)(g) = 1$. Now for a positive $\varepsilon < 1$ we pick a point s_0 in the open interval $(0, 1)$ so that $g(s_0) > \|g\| - \varepsilon$. Clearly there exists f_ε in $C(I)$ such that $\|f_\varepsilon\| = \|g\|$; $f_\varepsilon(0) = g(0)$; $f_\varepsilon(1) = g(1)$; $f_\varepsilon(s_0) = -g(s_0)$. Then $\|p(f_\varepsilon)\| \geqslant |p(f_\varepsilon)(s_0)| = 2|g(s_0)| > 2\|g\| - 2\varepsilon$. Thus $\|p\| \geqslant 2(\|g\| - \varepsilon)/\|g\|$, because $\|f_\varepsilon\| = \|g\|$. Let ε tend to zero, and we have $\|p\| \geqslant 2$. That completes the proof.

Observe that for $[p_0(f)](s) = f(s) - \frac{1}{2}\big(f(1) - f(0)\big)(2s - 1)$ for $s \in I$ and $f \in C(I)$ we have $\|p_0\| = 2$. Therefore φ has a linear averaging operator of norm 2.

Problem 4. Let a map $\varphi: S \to T$ has a linear exave of norm < 2. Does there exist a regular exave for φ?

Proposition 4.2 is due to Arens [1] (see also Corson and Lindenstrauss [1]).

EXAMPLE 3. Let $\Psi: \mathscr{C} \times \mathscr{C} \to I$ be the map defined by (5.5.2) in the proof of Milutin Lemma 5.5. Let $\varphi: \mathscr{C} \to \mathscr{C} \times \mathscr{C}$ be the homeomorphic embedding defined by $\varphi\xi = (\xi, 0)$ for $\xi \in \mathscr{C}$. Then it is easily seen that $\Psi\varphi = h: \mathscr{C} \to I$ is the Cantor map. Therefore, by Corollary 9.12, there is no linear averaging operator for h. On the other hand φ, being a coretraction has a regular extension operator, and, by Milutin Lemma 5.5, Ψ has a regular averaging operator. This example shows that

1° In Proposition 4.3 the assumption $\varphi_1 Q = \varphi^{-1} T_0$ cannot be dropped.

2° Proposition 4.4 fails if we change the order in which we compose a linear exave with linear extension operator (resp. with linear averaging operator).

Example 4. Let $Q = \{c, d\}$, $S = \{I, II, III, IV\}$, $T = \{a, b\}$ be discrete four and two point spaces. Let $\varphi_1(c) = I$, $\varphi_1(d) = II$; $\varphi(I) = \varphi(II) = a$, $\varphi(III) = \varphi(IV) = b$. Then φ_1 is a homeomorphic embedding of Q into S, and φ is an epimorphism from S onto T. Clearly $\varphi_1(Q) = \varphi^{-1}(\{a\})$. Let us set

$$u(f) = \big(f(c), f(d), f(c), 2f(d)\big) \quad \text{for} \quad f = \big(f(c), f(d)\big) \in C(Q),$$

$$v(g) = \left(\frac{g(I)+g(II)}{2} + g(III) - g(IV),\ \frac{g(III)+g(IV)}{2}\right)$$

$$\text{for} \quad g = \big(g(I), g(II), g(III), g(IV)\big) \in C(S).$$

It is easily to verify that u is a linear extension operator for φ_1 and v is a linear averaging operator for φ. However, uv is not a linear exave for $\varphi\varphi_1$, because

$$(\varphi\varphi_1)^\circ uv(\varphi\varphi_1)^\circ(1_T) = (2, 2) \neq (\varphi\varphi_1)^\circ 1_T = 1_Q = (1, 1).$$

This example shows that in Proposition 4.3 the assumption of regularity of linear exaves u and v is essential.

Ad § 5. Dyadic spaces have been introduced by P. S. Alexandroff [1] and have been investigated by Šanin [1], Esenin-Volpin [1], Marczewski (Szpilrajn) [1], Ivanovskiĭ [1], Kuzminov [1], Engelking and Pełczyński [1], Efimov and Engelking [1], Engelking [1], Efimov [1], [2], [3], Mardešič and Papič [1], Alexandroff and Ponomarev [1] and Ponomarev [1].

Lemma 5.5 was proved by Milutin in his thesis in 1952 but was published only recently (cf. Milutin [1], [2]).

The first example of a dyadic space which does not have [B.S.P.] is due to Engelking [1]. The example given below is due to H. H. Corson and is published here with his permission.

Example 5. Let $T_{\mathfrak{m}}$ be the space obtained from $D^{\mathfrak{m}}$ by identification of two different points of $D^{\mathfrak{m}}$, say η and ζ. Let $\mathfrak{h} = \iota/\{\eta\} \cup \{\zeta\}$ be the natural epimorphism from $D^{\mathfrak{m}}$ onto $T_{\mathfrak{m}}$. Then, according to the remark after Proposition 2.6, for every $\varepsilon > 0$ there exists a linear averaging operator for $\mathfrak{h}$ of norm $\leqslant 2+\varepsilon$. Hence $T_{\mathfrak{m}}$ is an almost Milutin space. However, if $\mathfrak{m} > \aleph_0$, then $T_{\mathfrak{m}}$ does not have [B.S.P.] and therefore, by Corollary 5.11, $T_{\mathfrak{m}}$ is not a Milutin space.

Proof. For $\xi \in D^{\mathfrak{m}}$ let ξ_0 denote that coordinate which is different for η and ζ. Without loss of generality we may assume that $\eta_0 = 0$ and $\zeta_0 = 1$. Let us put for $j = 0, 1$,

$$A_j = \{\xi \in D^{\mathfrak{m}} : \xi_0 = j\},$$
$$U_j = \mathfrak{h}A_j \setminus \{p\}$$

where $p = \mathfrak{h}\eta = \mathfrak{h}\zeta$.

Clearly $U_1 \cap U_2 = \emptyset$. Since A_j are closed and since $U_1 = T_{\mathfrak{m}} \setminus \mathfrak{h}A_0$ and $U_0 = T_{\mathfrak{m}} \setminus \mathfrak{h}A_1$, U_j are open. The pair (U_0, U_1) cannot be separated by open F_σ-sets. This is an immediate consequence of the following facts

1° if $\mathfrak{m} \geqslant \aleph_0$, then A_j is homeomorphic to $D^{\mathfrak{m}}$ $(j = 0, 1)$,

2° the restriction of $\mathfrak{h}$ to A_j is a homeomorphism $(j = 0, 1)$,

3° if $\mathfrak{m} > \aleph_0$ and ξ is an arbitrary point of $D^{\mathfrak{m}}$, then $D^{\mathfrak{m}} \setminus \{\xi\}$ is not an F_σ.

The proof of 1° and 2° is trivial. For 3° see e.g. Corson [1].

The proof of Proposition 5.10 in the form presented in this paper has been obtained by the author jointly with H. H. Corson.

Problem 5. Let φ be an epimorphism from $D^{\mathfrak{m}}$ onto $T_{\mathfrak{m}}$ and let $\mathfrak{m} > \aleph_0$. Is it true that $\|u\| > 2$ for every linear averaging operator for φ? (cf. Problem 4).

Problem 6. Give an example of a dyadic space which is not an almost Milutin space? We conjecture that the product $\mathop{\boldsymbol{P}}\limits_{2 \leqslant N < +\infty} T_{\mathfrak{m}}^{(N)}$ for $\aleph > \mathfrak{m}_0$ possesses the properties in question. For the definition of $T_{\mathfrak{m}}^{(N)}$ see Notes and Remarks to § 6.

Problem 7. Construct for every $a > 1$ an almost Milutin space S_a with the property that every linear averaging operator for an arbitrary epimorphism $\varphi : D^{\mathfrak{m}} \to S_a$ is of norm $> a$.

Problem 8. Give a topological characterization of Milutin spaces.

Ad § 6. The fact that every compact metric space is a Dugundji space has been observed by Arens (cf. Arens [1], Theorem 5.2).

Example 6. The space $T_{\mathfrak{m}}$ defined in Example 5 is for $\mathfrak{m} > \aleph_0$ an almost Dugundji space, but it is not a Dugundji space. Precisely

(a) if $\psi : T_{\mathfrak{m}} \to I^{\mathfrak{m}}$ is a homeomorphic embedding, then every linear extension operator for ψ is of norm $\geqslant 3$,

(b) if $\varphi : T_{\mathfrak{m}} \to T$ is an arbitrary homeomorphic embedding of $T_{\mathfrak{m}}$ into a compact space T, then there exists a linear extension operator for φ of norm $\leqslant 3$.

Proof. (a) follows from the Remark to Proposition 5.10 and the fact that $I^{\mathfrak{m}}$ has [B.S.P.] but $T_{\mathfrak{m}}$ does not have [B.S.P.] (cf. Example 5).

According to the Remark to Proposition 6.2 to prove (b) it is enough to show that every homeomorphic embedding $\psi : T_{\mathfrak{m}} \to I^{\mathfrak{m}}$ admits a linear extension operator of norm $\leqslant 3$.

Let the epimorphism $\mathfrak{h}: D^{\mathfrak{m}} \to T_{\mathfrak{m}}$ and the sets A_j $(j = 0, 1)$ have the same meaning as in Example 5 and let $p = \mathfrak{h}A_0 \cap \mathfrak{h}A_1$ be the unit point of $T_{\mathfrak{m}}$ such that $\mathfrak{h}^{-1}(p)$ is a two-point set. Let ψ_j be the restriction of ψ to $\mathfrak{h}A_j$ $(j = 0, 1)$. First we show that there is a regular extension operator for ψ_j, say u_j, such that

$$(u_j f)(t) = f(p) \quad \text{for } f \in C(\mathfrak{h}A_j) \text{ and for } t \in \mathfrak{h}A_{j*},$$

where $j^* = (j+1) \bmod 2$ $(j = 0, 1)$.

Let $\iota_j = \iota/\mathfrak{h}A_j$ denote the natural epimorphism from $I^{\mathfrak{m}}$ onto the space $I^{\mathfrak{m}}/\mathfrak{h}A_j$ obtained by glueing together of all points of $\mathfrak{h}A_j$. It is easily seen that $\varphi_j = \iota_{j*} \circ \psi_j$ is a homeomorphic embedding of $\mathfrak{h}A_j$ into $I^{\mathfrak{m}}/\mathfrak{h}A_{j*}$. Since $\mathfrak{h}A_j$ is homeomorphic to $D^{\mathfrak{m}}$ and since $D^{\mathfrak{m}}$ is a Dugundji space, the map φ_j has a regular extension operator, say u_j'. Let $u_j = (\iota_{j*})^{\circ} u_j'$. It can easily be verified that $u_j : C(\mathfrak{h}A_j) \to C(I^{\mathfrak{m}})$ has the desired property.

Finally we put

$$uf = u_0 f_0 + u_1 f_1 - u_1[(\psi^{\circ} u_0 f_0)_1] \quad \text{for} \quad f \in C(T_{\mathfrak{m}})$$

where for $g \in C(T_{\mathfrak{m}})$ by g_j we denote the restriction of g to $\mathfrak{h}A_j$ $(j = 0, 1)$.

One can easily check then that u is a linear extension operator for ψ with $\|u\| \leqslant \|u_0\| + \|u_1\| + \|u_1\| \, \|\psi^{\circ}\| \, \|u_0\| = 3$. That completes the proof.

Example 6 admits the following generalization obtained jointly by H. H. Corson and the author.

Let N be an integer $\geqslant 2$. Denote by $T_{\mathfrak{m}}^{(N)}$ the space obtained from $D^{\mathfrak{m}}$ by identification of the points of some N-point subset of $D^{\mathfrak{m}}$. Clearly $T_{\mathfrak{m}} = T_{\mathfrak{m}}^{(2)}$.

If $\mathfrak{m} > \aleph_0$, then

(a$^{(N)}$) if $\psi : T_{\mathfrak{m}}^{(N)} \to I^{\mathfrak{m}}$ is a homeomorphic embedding, then every linear extension operator for ψ is of norm $\geqslant 2N-1$;

(b$^{(N)}$) if $\varphi : T_{\mathfrak{m}}^{(N)} \to T$ is an arbitrary homeomorphic embedding of $T_{\mathfrak{m}}^{(N)}$ into a compact space T, then there exists a linear extension operator for φ of norm $2N-1$.

Problem 9. Let S be an almost Dugundji space of topological weight $\mathfrak{m}$. Is it true that there exists an odd integer $\eta(S)$ such that

1° for every compact space T and for every homeomorphic embedding $\varphi: S \to T$ there is a linear extension operator of norm $\leqslant \eta(S)$,

2° for every homeomorphic embedding $\psi: S \to I^{\mathfrak{m}}$ every linear extension for ψ is of norm $\geqslant \eta(S)$?

Problem 10. Show that if $\mathfrak{m} > \aleph_0$, then every averaging operator for arbitrary epimorphism from $D^{\mathfrak{m}}$ onto $T_{\mathfrak{m}}^{(N)}$ is of norm $\geqslant N+1$.

Problem 11 (cf. Corson and Lindenstrauss [1]). Let $\varphi: S \to T$ be a homeomorphic embedding (S, T compact spaces). Let $\eta(\varphi)$ denote the g.l.b. of the set of norms of all linear extension operators for φ. Is $\eta(\varphi)$ an odd integer?

Problem 12. Let $\varphi: S \to T$ be an epimorphism. Let $\zeta(\varphi)$ denote the g.l.b. of the set of all linear averaging operators for φ. Is $\zeta(\varphi)$ an integer?

Problem 13. Give a topological characterization of Dugundji spaces.

Problem 14. Is every Dugundji space (every almost Dugundji space) dyadic?

All known examples of compact non-dyadic spaces are not Dugundji spaces (cf. Proposition 8.11 and Corollary 8.14). On the other hand, if $\mathfrak{m} > \aleph_0$, then the product $\boldsymbol{P}_{2 \leqslant N < +\infty} T_{\mathfrak{m}}^{(N)}$ is a dyadic space, but it is not an almost Dugundji space (because of ($\mathrm{a}^{(\mathrm{N})}$)).

Problem 15. Does the class of all Milutin spaces (almost Milutin spaces) coincide with the class of all Dugundji spaces (almost Dugundji spaces)?

Problem 16. Let S be a compact zero-dimensional space. Are the following conditions equivalent?

(i) S is a retract of $D^{\mathfrak{m}}$,

(ii) S is a dyadic space and S has [B.S.P.],

(iii) S is a Milutin space,

(iv) S is a Dugundji space.

Condition (i) implies each of the remaining conditions. The implication (i) $\Rightarrow$ (ii) was first proved directly in Engelking [1]. The implication (iii) $\Rightarrow$ (ii) follows from Corollary 5.11.

The next problem is related to Problem 16.

Problem 17. Let a homeomorphic embedding $\varphi: S \to T$ of a compact space S into a zero-dimensional compact space T admit a regular extension operator. Is φ a coretraction, i.e. is φS a retract of T?

The affirmative answer to Problem 17 implies (by Proposition 3.3) that the existence of a regular extension operator for a homeomorphic embedding of zero-dimensional compact spaces is equivalent to the existence of linear multiplicative extension operator. This is somewhat related to the works of Phelps [1] and Bonsall, Lindenstrauss and Phelps [1].

Problem 18. Let S be a Dugundji space (an almost Dugundji space). Is it true that for each s in S there is a closed neighbourhood F_s which is a Dugundji space (an almost Dugundji space)?

Ad § 7. Proposition 7.2 is closely related to the result of Rudin [1] on the existence of the invariant projection (see also Glicksberg [1] and Rosenthal [1]). It is also related to the results of A. Ionescu Tulcea [1] concerning liftings commuting with a group of maps (see also C. Ionescu Tulcea [1]).

Problem 19. Does the conclusion of Proposition 7.2 remain true under the assumption that G is an arbitrary locally compact group?

Probably using the notion of invariant mean as did de Leeuw (presented in Glicksberg [1]), one can generalize Proposition 7.2 to the case where G is an arbitrary locally compact abelian group.

Theorem 7.5 can also be proved by using the Pontryagin representation of a compact topological group as an inverse limit of a Lie series (cf. Pontryagin [1], p. 327, Ivanovskiĭ [1]; see also the proof of Proposition 8.10).

Perhaps the same idea may be useful in solving the following problems.

Problem 20. Is every compact coset-space of a locally compact topological group a Milutin space (an almost Milutin space)?

Problem 21. Is every compact topological group, or more generally, every compact coset-space of a locally compact topological group, a Dugundji space (an almost Dugundji space)?

The results of this paragraph can be partially generalized to the case of principal fibre spaces using the existence (in certain special situations) of local cross sections (see e.g. Serre [1], Borel [1], Mostert [1], Sklyarenko [1], Michael [4]).

Ad § 8. The proof of Proposition 8.3 uses a special case of the abstract decomposition scheme (described in Bessaga [1], p. 283). The decomposition method was discovered in Borsuk [1] and has been developed for various purposes in Pełczyński [3], [4], Bessaga and Pełczyński [2], [3] and Kadec and Levin [1].

Theorem 8.5 is due to Milutin [1]. (cf. also Milutin [2]). This settled a question of Banach (cf. Banach [1, p. 185]). Linear topological classification of spaces of continuos functionus on countable compact spaces is given in Bessaga and Pełczyński [1].

Problem 22. Let S be a compact space of weight $\mathfrak{n}$. Assume

(*) S cannot be represented as a countable union of its closed subsets of topological weights less than the topological weight of S.

Also let S have one of the following properties:

(i) S is a Dugundji space (an almost Dugundji space).

(ii) S is a dyadic space, less generally S is a Milutin space.

(iii) S is an absolute retract.

Does S contain a closed subset homeomorphic to $D^{\mathfrak{n}}$?

Is $C(S)$ linearly homeomorphic to $C(D^{\mathfrak{n}})$?

Clearly if the topological weight of S, say $\mathfrak{n}$, cannot be represented as a countable sum of cardinals less than $\mathfrak{n}$, then (*) is satisfied automatically. The next example shows that assumption (*) is, in general, essential.

Example 7. Let $(\mathfrak{n}_\nu)_{\nu=0}^{+\infty}$ be an increasing sequence of cardinals. Let $\mathfrak{n} = \sup_{0\leqslant\nu<+\infty} \mathfrak{n}_\nu$. Let S be a compact space such that $S = \bigcup_{\nu=0}^{\infty} S_\nu$, S_ν are closed in S and $\mathfrak{n}_\nu$ is the topological weight of S_ν $(\nu = 0, 1, \ldots)$. Then

(o) no closed subset of S is homeomorphic to $D^{\mathfrak{n}}$,

(oo) $C(S)$ is not linearly homeomorphic to $C(D^{\mathfrak{n}})$

Proof. We shall show that there is no linear homeomorphism from $C(D^{\mathfrak{n}})$ into $C(S)$. This fact clearly implies (oo), and it also implies (o) because, if $D^{\mathfrak{n}}$ were homeomorphic to a subset of S, then $C(S)$ would be a factor of $C(S)$ (cf. Proposition 8.4).

Let $u: C(D^{\mathfrak{n}}) \to C(S)$ be an arbitrary linear map. Let E be the closed linear subspace of $C(D^{\mathfrak{n}})$ spanned by the family of functions $(f_a)_{a\in A}$, where $f_a(\xi) = 2\xi_a - 1$ for $\xi = (\xi_b)_{b\in A}$ and A is a set of indices of cardinality $\mathfrak{n}$. We shall show that even the restriction of u to E is not a linear homeomorphism. First we observe that E is linearly homeomorphic to the space $l_1(A)$ of all scalar valued functions $t = (t_a)_{a\in A}$ such that $\|t\| = \sum_{a\in A} |t_a| < +\infty$. This follows from the inequality

$$\sum_{i=1}^{m} |t_{a_i}| \geqslant \Big\| \sum_{i=1}^{m} t_{a_i} f_{a_i} \Big\| \geqslant \frac{\sqrt{2}}{2} \sum_{i=1}^{m} |t_{a_i}| \quad \text{for } t = (t_a) \in l_1(A) \text{ and}$$
$$\text{for every finite subset } \{a_1, a_2, \ldots, a_m\} \subset A \ (m = 1, 2, \ldots).$$

Let $R_n : C(S) \to C\big(\bigcup_{\nu=0}^{n} S_\nu\big)$ be the operation of restriction of functions to $\bigcup_{\nu=0}^{n} S_\nu$ and let $u_n = R_n u$ $(n = 0, 1, \ldots)$. Since the topological weight of $\bigcup_{\nu=0}^{n} S_\nu$ is $\sum_{\nu=0}^{n} \mathfrak{n}_\nu$ which is less than $\mathfrak{n}$, the density character of $C(\bigcup_{\nu=0}^{n} S_\nu)$

is less than the density character of $l_1(A)$ which is equal to $\mathfrak{n}$. Thus no u_n is a linear homeomorphism. Hence there are g_n in E such that

$$\|g_n\| = 1; \quad \sup\{s \in \bigcup_{\nu=0}^{n} S_\nu | |(ug_n)(s)|\} = \|u_n g_n\| < \frac{1}{n+1} \quad (n = 0, 1, \ldots).$$

Therefore the sequence (ug_n) weakly converges to 0 in $C(S)$ (because $\|ug_n\| \leqslant \|u\|$ $(n = 0, 1, \ldots)$ and $\lim\limits_n (ug_n)(s) = 0$ for $s \in S = \bigcup_{\nu=0}^{\infty} S_\nu$). Since $\|g_n\| = 1$ $(n = 0, 1, \ldots)$, the sequence (g_n) does not converge weakly to 0 in E. (We use the fact in $l_1(A)$, and therefore in E, weak and norm convergences of sequences are equivalent (cf. Day [1], p. 33). That shows that the restriction of u to E is not a linear homeomorphism.

Let us specify Example 7 as follows. Let S be the one-point compactification of the discrete sum $\bigcup_{\nu=1}^{\infty} D^{\mathfrak{n}_\nu}$. Then $S = \bigcup_{\nu=0}^{\infty} S_\nu$, where S_0 is the one-point set consisting of "the point at infinity" and $S_\nu = D^{\mathfrak{n}_\nu}$ for $\nu = 1, 2, \ldots$ Clearly S satisfies the assumption of Example 7 while being a Milutin space and a Dugundji space.

Problem 22 is a particular case of the following:

Problem 23. Let S be a compact space satisfying $(*)$ and such that $C(S)|C(D^{\mathfrak{n}})$. Is $C(S)$ linearly homeomorphic to $C(D^{\mathfrak{n}})$ ($\mathfrak{n}$ is the topological weight of S)?

Problem 24. Does there exist a compact space S such that $C(S)|C(D^{\mathfrak{n}})$, in particular $C(S)$ is linearly homeomorphic to $C(D^{\mathfrak{n}})$, but S is not dyadic?

The next problem seems to be a proper generalization of the results of Bessaga and Pełczyński [1].

Problem 25. Give a complete linear topological classification of $C(S)$ spaces such that $S = \bigcup_{\nu=1}^{\infty} S_\nu$, where S_ν is closed in S and homeomorphic to $D^{\mathfrak{n}_\nu}$ $(\nu = 1, 2, \ldots)$, S — compact.

Problem 26. Generalize Theorem 8.9 and Proposition 8.10 to the case where S is a compact coset space of a locally compact topological group.

Some results of Hulanicki [2], [3] and Jones [1] (see also Hewitt and Ross [1], p. 79) on cardinality and metrizability of compact topological groups can easily be deduced from our Proposition 8.10.

Problem 27. Give an example of a compact non-dyadic space S such that $C(S)$ possesses the properties (a) and (b) stated before Proposition 8.11.

Following Banach [1], p. 242, we assign to each pair (X, Y) of normed linear spaces a real number $a(X, Y)$ with $1 \leqslant a(X, Y) < +\infty$ as follows

$a(X, Y) = +\infty$ whenever X is not linearly homeomorphic to Y,

$a(X, Y) = \inf\|u\|\,\|u^{-1}\|$ if X is linearly homeomorphic to Y; the infimum is taken over all linear homeomorphisms u from X onto Y.

Let S_1, S_2 be compact spaces. We put

$$a(S_1, S_2) = a\big(C(S_1), C(S_2)\big).$$

Problem 28. Is it true that if $a(S_1, S_2) < +\infty$, then $a(S_1, S_2)$ is an integer?

Problem 29. Compute the following numbers

1° $a(I, \mathscr{C})$,

2° $\beta(\mathscr{C}) = \sup\{a(S, \mathscr{C}) \mid S$ compact metric uncountable$\}$,

3° $\beta^* = \sup\{a(S_1, S_2) \mid S_1, S_2$ compact metric uncountable$\}$.

Let us note that recently Amir [4] proved that if $a(S_1, S_2) < 2$, then S_1 and S_2 are homeomorphic. On the other hand it follows from the analysis of the proof of Theorem 8.5 that $\beta(\mathscr{C}) \leqslant 12$. Hence $2 \leqslant \beta^* \leqslant 144$.

Ad § 9. Proposition 9.9 was conjectured by Z. Semadeni. The proof given in the present paper is based upon de Branges [1] proof of the Stone-Weierstrass approximation theorem (cf. also Glicksberg [2]). The real analogue of Proposition 9.9 has been considered by several authors. Clearly for $f \in C_R(S)$ formula (9.9.1) gives

$$(*) \qquad \varrho\big(f, \varphi^\circ[C_R(T)]\big) = \sup_{t \in T} \sup_{\varphi(s_1)=\varphi(s_2)=t} 2^{-1}|f(s_1) - f(s_2)|.$$

The formula (∗) was announced without proof in Pełczyński [5]. An elegant proof is due to S. Mazur (cf. Semadeni [2], p. 20). The same problem has been considered independently by Kripke and Holmes [1].

Recently Olech [1] gave an alternative proof of Proposition 9.9 in the complex case as well as in the case of some vector valued functions. He also showed that for each $f \in C(S)$ there is a g in $C(T)$ such that $\varrho\big(f, \varphi^\circ[C(T)]\big) = \|f - \varphi^\circ g\|$. The same result for the real case is due to Mazur.

Corollary 9.12 has been proved by M. I. Kadec in an unpublished part of his thesis (about 1949) and independently in a recent paper by Foiaş and Singer [1] who restated it as follows:

There is no projection of $D(I)$ onto its subspace $C(I)$, where $D(I)$ is the space of all bounded functions on I continuous at each non-dyadic point, continuous on the right and on the left at each point and having for every $\varepsilon > 0$ only a finite number of "jumps" greater than ε. This is related to Example 2 in Corson [2], p. 13.

Another example of an epimorphism $\varphi: S \to T$, S and T compact metric spaces, which has no linear averaging operators is given in Arens [2], Theorem 3.5.

Problem 30. Let S be an infinite compact metric space. Are the conditions (9.13.1)-(9.13.3) equivalent to the following conditions?

(9.13.4). There is an epimorphism $\varphi: \mathscr{C} \to T$ which does not have linear averaging operators.

(9.13.5) There is a compact space T and an epimorphism $\varphi: S \to T$ which does not have linear averaging operators.

The construction described in Lemma 9.16 together with Proposition 9.8 show that $C([\omega^n])$ is not a $\mathfrak{P}'_\lambda$ space for $\lambda < n$. One can show that $C([\omega^n])$ is a $\mathfrak{P}'_{n+2}$ space. Modyfying the proof of Proposition 9.8 and the construction of Lemma 9.16, one could probably show that $n+2$ is the exact constant, i.e. that $C([\omega^n])$ is not a $\mathfrak{P}'_\lambda$ space for $\lambda < n+2$ (for $n = 1$, cf. McWilliams [1]).

APPENDIX: CATEGORY-THEORETICAL APPROACH

The basic notions of the theory of categories can be found in the monographs by Brinkmann and Puppe [1] and Mitchell [1].

Let $\mathscr{T}$ and $\mathscr{L}$ be two categories. Objects of $\mathscr{T}$ (denoted by $S, T, Q, \ldots$) will be called "spaces" and morphisms of $\mathscr{T}$ (denoted by $\varphi, \psi, \ldots$) will be called "maps". Morphisms of $\mathscr{L}$ (denoted by $u, v, w, \ldots$) will be called "operators".

Let F be a fixed contravariant functor from $\mathscr{T}$ into $\mathscr{L}$.

DEFINITION 1. An operator $u: F(S) \to F(T)$ is called an *F-exave* for a map $\varphi: S \to T$ if

$$F(\varphi)uF(\varphi) = F(\varphi). \tag{1}$$

Let us observe that if $F(\varphi)$ is an epimorphism([4]), then (1) is equivalent to the condition

$$F(\varphi)u = \mathrm{id}_{F(S)}; \tag{2}$$

f $F(\varphi)$ is a monomorphism, then (1) is equivalent to the condition

$$uF(\varphi) = \mathrm{id}_{F(T)}. \tag{3}$$

DEFINITION 2. An operator u satisfying (2) is called an *F-extension operator* for φ. An operator u satisfying (3) is called an *F-averaging operator* for φ.

In the most important examples the contravariant functor F has some additional properties.

DEFINITION 3. A contravariant functor F from $\mathscr{T}$ into $\mathscr{L}$ is said to be of *Banach-Stone type* if the following conditions are satisfied:

([4]) Let us recall that a morphism α in a category $\mathscr{A}$ is said to be an epimorphism (resp. a monomorphism) if for arbitrary morphisms β and γ of $\mathscr{A}$ the condition $\beta\alpha = \gamma\alpha$ implies $\beta = \gamma$ ($\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$).

If A is an object of $\mathscr{A}$, then id_A denotes the identity morphism of A.

	$\mathcal{T}$			$\mathcal{L}$		
	Symbol	object	morphisms	symbol	objects	morphisms
1	Comp.	compact Hausdorff spaces	continuous transformations	a) **C**(Comp.) b) $\mathbf{C}_R$(Comp) c) $\mathbf{C}_+$(Comp)	spaces of all continuous a) complex valued b) real valued c) non-negative functions on compact Hausdorff spaces with the topology of uniform convergence	a) and b) bounded linear operators c) continuous affine operators
2	,,	— ,, —	— ,, —	a) reg. **C**(Comp) b) reg. $\mathbf{C}_R$(Comp) c) reg. $\mathbf{C}_+$(Comp)	the same as in 1 a), b), c)	a) and b) regular operators c) regular affine operators
3	,,	— ,, —	— ,, —	a) mult. ad. **C**(Comp) b) mult. ad. $\mathbf{C}_R$(Comp) c) mult. ad. $\mathbf{C}_+$(Comp)	the same as in 1 a), b), c)	a) and b) linear multiplicative operators c) affine multiplicative operators
4	,,	— ,, —	— ,, —	a) mult. **C**(Comp) b) mult. $\mathbf{C}_R$(Comp) c) mult. $\mathbf{C}_+$(Comp)	the same as in 1 a), b), c)	continuous multiplicative operators
5	,,	— ,, —	— ,, —	$\mathbf{C}_G$(Comp)	topological groups of all continuous maps from a compact Hausdorff space into a given topological group G with pointwise multiplication as group operation and with the topology of uniform convergence.	continuous group homomorphisms
6	Tot. disc. Comp.	totally disconnected compact Hausdorff spaces	continuous transformations	α) $\mathbf{C}_{\text{int}}$(t-d. Comp.) β) mult. ad. $\mathbf{C}_{\text{int}}$(t-d. Comp.) γ) mult. ad. $\mathbf{C}_{\text{int}}$(t-d. Comp.)	topological groups of all continuous integer valued functions on totally disconnected compact Hausdorff spaces with the topology of uniform convergence	continuous operators, α) additive β) multiplicative γ) multiplicative-additive

7	**Unif.**	uniform spaces	uniformey continuous transformations	$j - \mathbf{C}_i^k$(Unif.) where $i = a, b, c$, $j = \alpha, \beta, \gamma$, $k = A, B, C$ (27 different categories)	Spaces of all uniformly continuous functions on uniform spaces a) complex valued b) real valued c) non-negative with one of the following topologies (cf. Bourbaki [2], pp. 4–5) A) topology of simple convergence B) topology of uniform convergence on compact sets C) topology of uniform convergence	continuous operators α) linear or affine in case c) β) multiplicative γ) multiplicative-additive or affine-multiplicative in case c)
8	k-**Ma-nif.**	k-differentiable locally compact manifolds $k = 1, 2, \ldots, +\infty$	k-differentiable transformations $k = 1, 2, \ldots, +\infty$	a, α) $\mathbf{C}^k$(Manif.) a, β) mult. $\mathbf{C}^k$(Manif.) a, γ) mult. ad. $\mathbf{C}_k$(Manif.) b, α) $\mathbf{C}_R$(Manif.) b, β) mult. $\mathbf{C}_R$(Manif.) b, γ) mult. ad. $\mathbf{C}_R$(Manif.)	Spaces of all k-time differentiable functions a) complex valued b) real valued with the topology of almost uniform convergence of functions and their k-first derivatives	continuous operators α) linear β) multiplicative γ) linear multiplicative
9	**Bool**	Boolean algebras	Boolean homomorphisms	a, α) **Meas** a, β) **Metric Meas** b, α) **R-Meas** b, β) **Metric R-Meas**	Banach spaces of all a) complex valued b) real valued measures with bounded variation with the norm $\|\mu\|$ = total variation of μ	α) bounded linear operators β) bounded linear operators of norm ≤ 1

(1) There is a faithful([5]) functor σ from $\mathscr{T}$ into the category **Ens** of sets as objects and with arbitrary mappings as morphisms.

(2) There is a set M depending only on $\mathscr{L}$ such that for every object T of $\mathscr{T}$ there is the unique object of $\mathscr{L}$, denoted by $F(T)$ and being a set of functions from $\sigma(T)$ into M. Conversely if X is an object of $\mathscr{L}$, then $X = F(T)$ for some object T of $\mathscr{T}$.

(3) For all objects $F(S)$ and $F(T)$ the set $[F(T), F(S)]_{\mathscr{L}}$ of all morphisms of $\mathscr{L}$ from $F(T)$ to $F(S)$ consists of some transformations from $F(T)$ into $F(S)$. If $\varphi: S \to T$ is a morphism of $\mathscr{T}$, then $\varphi^{\circ} \in [F(T), F(S)]_{\mathscr{L}}$, where φ° denotes the induced operator defined by $\varphi^{\circ} g = g \circ \sigma(\varphi)$ for $g: \sigma(T) \to M$.

(4) F assigns to each object S of $\mathscr{T}$ the object $F(S)$ of $\mathscr{L}$ and to each map $\varphi: S \to T$ which is a morphism of $\mathscr{T}$ the induced operator φ^{0} (i.e. $F(\varphi) = \varphi^{\circ}$).

We list in the table on pp. 76-77 various examples of functors of Banach-Stone type. Clearly in each case it is enough to describe categories $\mathscr{T}$ and $\mathscr{L}$, because the functor of Banach-Stone type from $\mathscr{T}$ into $\mathscr{L}$, if it exists, is uniquely determined by the given categories $\mathscr{T}$ and $\mathscr{L}$, and by the functor σ which in all these examples assigns to the topological space T the set of its elements.

Let us complete the list of examples by the following functor which is not exactly of Banach-Stone type.

Let $\mathscr{T}$ = **Euclid. Vect. Bund.** be the category whose objects are vector bundles with compact bases, with fibres being finite dimensional Euclidean spaces and with $Q = \lim\operatorname{ind} Q_n$ — the infinite orthogonal group as the transformations group. Morphisms of **Euclid. Vect. Bund.** are continuous transformations from one bundle to another which map bases into bases, fibres into fibres and which acts on fibres as partial isometries (composition of unitary transformation with orthogonal projection). For a given bundle B (being an object of **Euclid. Vect. Bund.**), let $\mathrm{Sec}(B)$ denote the space of all global sections of B with natural (i.e. pointwise) operation of addition and multiplication by scalars and with the topology of uniform convergence. Under an admissible norm $\mathrm{Sec}(B)$ may be regarded as a Banach space. Let $\mathscr{L}$ = **Sec (Euclid. Vect. Bund.)** be the category whose objects are $\mathrm{Sec}(B)$ for B being objects of **Euclid. Vect. Bund.**, and whose morphisms are bounded linear operations. The functor F assigns to each bundle B the space $\mathrm{Sec}(B)$ and to each morphism $\varphi: B_1 \to B_2$ the induced operator $\varphi^{\circ}: \mathrm{Sec}(B_2) \to \mathrm{Sec}(B_1)$ defined by $\varphi^{\circ}\pi = \pi \circ \varphi$ for $\pi \in \mathrm{Sec}(B_2)$.

([5]) A functor K from a category $\mathscr{A}$ into a category $\mathscr{B}$ is said to be *faithful* (cf. Mitchell [1], p. 51) if for every objects A and B in $\mathscr{A}$ the function $[A, B]_{\mathscr{A}} \to [K(A), K(B)]_{\mathscr{B}}$, induced by K, is univalent. (By $[A, B]_{\mathscr{A}}$ we denote the set of all morphisms in a category $\mathscr{A}$ from A to B.)

One can easily define the analogous functor from the category of k-times differentiable bundles into the category of their k-times differentiable global sections.

For a given functor F from $\mathscr{T}$ into $\mathscr{L}$ we can develop the theory of F-exaves along the same line as it is done in the present paper for the Banach-Stone functors from the category **Comp** into the categories **C**(**Comp**), **reg C**(**Comp**) and **mult. ad. C**(**Comp**). The analogies are most significant in the case where $\mathscr{T}$ is a subcategory(6) of the category of topological spaces and $\mathscr{L}$ is a subcategory of the category of linear topological spaces.

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(6) A category $\mathscr{B}$ is said to be a subcategory of a category $\mathscr{A}$ if every object of $\mathscr{B}$ is an object of $\mathscr{A}$ and every morphism of $\mathscr{B}$ is a morphism of $\mathscr{A}$ and the rules of composition of morphisms in $\mathscr{A}$ and in $\mathscr{B}$ are the same.

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