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On two-point boundary value problems for systems of ordinary non-linear, first-order differential equations

by A. LASOTA (Kraków)

Abstract. By using the Brouwer open mapping theorem, an existence theorem for the two-point boundary value problem is proved. The theorem is stated in the form of non-linear Fredholm alternative: the uniqueness of solutions implies the existence.

1. It is well known that for any system of ordinary linear differential equations the uniqueness of solutions of a linear boundary value problem implies the existence of solutions. It has also been discovered recently that the same is true for some boundary value problems related to non-linear differential equations of the second [4] and the third order [2] (see also [1]). The purpose of this paper is to establish this kind of interdependence for a non-linear system

(1) $$x' = f(t, x), \quad a \leqslant t \leqslant b,$$

and the boundary value conditions

(2) $$Ax(a)+Bx(b) = r,$$

where r, x and f are d-dimensional vectors and A, B are $d\times d$-matrices. We shall assume that $f(t, x)$ is defined in the strip

$$D = [a, b]\times R^d$$

and satisfies the following condition:

(C) For every point $(t_0, r_0)\in D$ there exists exactly one solution of equation (1), defined on $[a, b]$ and such that $x(t_0) = r$.

In the sequel $|\cdot|$ denotes the Euclidean norm in the space R^d. The norm in the space $R^{d\times d}$ of $d\times d$-matrices is defined by the formula:

$$\|A\| = \sup\{|Ax| : |x| = 1\}.$$

THEOREM 1. *Let a triple $(f, \mathscr{A}, \mathscr{B})$ be given, where $f\colon D\to R^d$ is a continuous function satisfying condition* (C) *and where $\mathscr{A}$, $\mathscr{B}$ are sets of $d\times d$-matrices. Let at least one of the sets $\mathscr{A}$, $\mathscr{B}$ be open.*

Assume that for each $A \in \mathscr{A}$, $B \in \mathscr{B}$ and $r \in R^d$ there exists at most one solution of the boundary value problem (1)–(2). *Then for each $A \in \mathscr{A}$, $B \in \mathscr{B}$ and $r \in R^d$ there exists exactly one solution of* (1)–(2).

Proof. Suppose that $\mathscr{A}$ is open. When $\mathscr{B}$ is open, the proof is similar. Fix $A \in \mathscr{A}$, $B \in \mathscr{B}$ and consider the mapping $u\colon R^d \to R^d$ given by the formula

$$u(q) = Ay(a, q) + By(b, q), \tag{3}$$

where $y(t, q)$ is the solution of equation (1) satisfying the initial condition $y(a, q) = q$. By assumption (C) such a solution exists and $u(q)$ is a continuous function of q. We claim that $u(R^d) = R^d$. Suppose that is not so and choose a point $p \notin u(R^d)$. Let us write

$$\bar{t} = \inf\{0 \leqslant t \leqslant 1\colon (1-t)u(0) + tp \notin u(R^d)\},$$

$$\bar{p} = (1-\bar{t})u(0) + \bar{t}p.$$

Since by the Brouwer open mapping theorem the set $u(R^d)$ is open, we have $\bar{t} > 0$ and $\bar{p} \notin u(R^d)$. Now choose a sequence $t_n \to \bar{t}$ $(0 < t_n < \bar{t})$ and write

$$p_n = (1-t_n)u(0) + t_n p.$$

From the definition of $\bar{t}$ it follows that $p_n \in u(R^d)$ and we may define the sequence $q_n = u^{-1}(p_n)$. We shall consider two cases:

1° The sequence q_n is convergent to a point $\bar{q} \in R^d$.

2° The sequence q_n is not convergent.

In the first case we have by the continuity of u

$$u(\bar{q}) = \lim_{n\to\infty} u(q_n) = \lim_{n\to\infty} p_n = \bar{p}.$$

Thus $\bar{p} = u(q) \in u(R^d)$, which is impossible. In the second case there exists a positive number ε and two increasing sequences of integers h_n and k_n such that

$$|q_{h_n} - q_{k_n}| \geqslant \varepsilon.$$

On the other hand, we have

$$|u(q_{h_n}) - u(q_{k_n})| = |p_{h_n} - p_{k_n}| \to 0.$$

Denote by δ the radius of a neighbourhood of the matrix A which is contained in $\mathscr{A}$ and choose an integer m such that

$$\frac{|p_{h_m} - p_{k_m}|}{|q_{h_m} - q_{k_m}|} < \delta.$$

It is easy to construct a matrix $\tilde{A}$ (not necessarily unique) which satisfies the following conditions:

$$\tilde{A}(q_{h_m} - q_{k_m}) = p_{h_m} - p_{k_m}, \quad \|\tilde{A}\| < \delta. \tag{4}$$

From the definition of p_n and q_n it follows that

$$A(q_{h_m}-q_{k_m})+B\big(y(b,q_{h_m})-y(b,q_{k_m})\big) = p_{h_m}-p_{k_m}. \tag{5}$$

Now write

$$r_m = (A-\tilde{A})y(a,q_{h_m})+By(b,q_{h_m}).$$

From (4), (5) and the identity $y(a,q) = q$ we obtain

$$r_m = (A-\tilde{A})y(a,q_{k_m})+By(b,q_{k_m}).$$

This means that $y(t,q_{k_m})$ and $y(t,q_{h_m})$ are two different solution of equation (1) satisfying the same boundary value condition of the form

$$(A-\tilde{A})x(a)+Bx(b) = r_m.$$

Since $A-\tilde{A}\in\mathscr{A}$ and $B\in\mathscr{B}$, this contradicts the uniqueness. Thus the second case is also impossible. We have proved our claim that $u(R^d) = R^d$. Therefore for each $r\in R^d$ there exists a solution q of the equation $u(q) = r$. The corresponding function $y(t,q)$ is the desired solution of the problem (1)–(2). The proof is complete.

2. It is easy to prove that the solution x of the boundary value problem (1)–(2) in Theorem 1 depends continuously upon A, B and r. We may state even a more general result. Denote by $\mathscr{F}$ the space of all continuous functions $f\colon D\to R^d$ with the topology of the uniform convergence on compact sets and denote by C^1 the space of all continuously differentiable functions $x\colon [a,b]\to R^d$ with the usual norm

$$\|x\|_{C^1} = \sup\{|x(t)|\colon a\leqslant t\leqslant b\}+\sup\{|x'(t)|\colon a\leqslant t\leqslant b\}.$$

We have the following theorem:

THEOREM 2. *Let a triple $\mathscr{F}_0$, $\mathscr{A}$, $\mathscr{B}$) be given, where $\mathscr{F}_0$ is a subset of $\mathscr{F}$ and where $\mathscr{A}$, $\mathscr{B}$ are sets of $d\times d$ matrices. Assume that*

(i) *for each $f\in\mathscr{F}_0$ condition* (C) *is satisfied,*

(ii) *at least one of the sets $\mathscr{A}$, $\mathscr{B}$ is open.*

Assume, moreover, that for each $f\in\mathscr{F}_0$, $A\in\mathscr{A}$, $B\in\mathscr{B}$ and $r\in R^d$ there exists at most one solution of (1)–(2). *Then for each $f\in\mathscr{F}_0$, $A\in\mathscr{A}$, $B\in\mathscr{B}$ and $r\in R^d$ there exists exactly one solution $x(t;f,A,B,r)$ of* (1)–(2) *and the mapping*

$$\mathscr{F}_0\times\mathscr{A}\times\mathscr{B}\times R^d\ni(f,A,B,r)\to x(\cdot;f,A,B,r)\in C^1 \tag{6}$$

is continuous.

Proof. The existence of the solution $x(\cdot;f,A,B,r)$ follows from Theorem 1. In order to prove the continuity of the mapping (6), note that by the classical continuous dependence theorem the function $u = u(q;f,A,B)$ defined by formula (3) is continuous. Since for each fixed triple $(f, A, B) \in \mathscr{F}_0$ the mapping $q \to u(q, f, A, B)$ is a homeomorfism, it is easy to verify that the solution $q = q(f, A, B, r)$ of the equation $u = u(q; f, A, B) = r$ depends continuously upon (f, A, B, r). Thus the corresponding solution

$$x(\cdot; f, A, B, r) = y\big(\cdot; q(f, A, B, r)\big)$$

of (1)–(2) is a continuous function of (f, A, B, r).

3. Now we are going to apply Theorem 1 in two special cases. The first application is stimulated by an old paper of Krasnosielski–Pierov [3]; the second one is related to an example given in [5].

For $u, v \in R^d$ we denote by $\langle u, v\rangle$ the scalar product of u and v. For any matrix $A \in R^d$ we write

$$\|A\|_0 = \|A\| = \sup\{|Ax| : |x| = 1\},$$

$$\|A\|_1 = \inf\{|Ax| : |x| = 1\},$$

$$\|A\|_2 = \sup\{\langle Ax, x\rangle : |Ax| = 1\} \qquad (\text{if } \|A\|_0 > 0).$$

THEOREM 3. *Let f: $D \to R^d$ be a continuous function satisfying condition* (C) *and such that*

$$\langle f(t, x) - f(t, y), x - y\rangle \geqslant 0 \quad \textit{for } t \in [a, b]; x, y \in R^d.$$

If $\|A\|_0 < \|B\|_1$, then for each $r \in R^d$ the problem (1)–(2) *has exactly one solution.*

Proof. Fix B and write

$$\mathscr{A} = \{A \in R^{d\times d}: \|A\|_0 < \|B\|_1\}, \qquad \mathscr{B} = \{B\}.$$

The set $\mathscr{A}$ is evidently open. Thus it remains to prove that for each $A \in \mathscr{A}$ and $r \in R^d$ the problem (1)–(2) admits at most one solution. Suppose, this is not so and let x and y be two different solutions of (1)–(2). From (1) it follows that

$$\frac{d}{dt}|x-y|^2 = 2\langle x' - y', x - y\rangle = 2\langle f(t, x) - f(t, y), x - y\rangle \geqslant 0$$

and consequently

$$|x(a) - y(a)| \leqslant |x(b) - y(b)|. \tag{7}$$

On the other hand, from (2) it follows that

$$A\big(x(a) - y(a)\big) = B\big(y(b) - x(b)\big)$$

and consequently

$$\|A\|_0 |x(a) - y(a)| \geqslant \|B\|_1 |x(b) - y(b)|.$$

From this and (7) we obtain

$$\|A\|_0 |x(a)-y(a)| \geqslant \|B\|_1 |x(a)-y(a)|.$$

Since $x \neq y$ and f satisfies (C), we have $x(a) \neq x(b)$. Thus $\|A\|_0 \geqslant \|B\|_1$, which contradicts the assumption.

THEOREM 4. *Assume that f: $D \to R^d$ is continuous and satisfies the Lipschitz condition*

$$|f(t,x)-f(t,y)| \leqslant \varphi(t)|x-y| \quad \textit{for } t \in [a,b];\ x, y \in R^d. \tag{8}$$

If

$$\int_a^b \varphi(t)\,dt < \arccos \|-B^{-1}A\|_2, \tag{9}$$

then for each $r \in R^d$ the boundary value problem (1)–(2) *admits exactly one solution.*

Proof. Fix B and write

$$\mathscr{A} = \{A\colon (9) \text{ holds}\}, \quad \mathscr{B} = \{B\}.$$

The set $\mathscr{A}$ is open. Moreover, the continuity of f and the Lipschitz condition (8) imply (C). Thus in order to finish the proof it is sufficient to verify the uniqueness of (1)–(2) with $A \in \mathscr{A}$. Denote by x and y two different solutions of (1)–(2) and write $z = x-y$. From (1), (8) and (9) it follows that

$$L \stackrel{\text{def}}{=} \int_a^b \left|\left(\frac{z}{|z|}\right)'\right| dt \leqslant \int_a^b \frac{|z'|}{|z|}\,dt \leqslant \int_a^b \varphi(t)\,dt < \arccos\|-B^{-1}A\|_2.$$

The integral L denotes the length of the arc

$$\left\{\frac{z(t)}{|z(t)|} : a \leqslant t \leqslant b\right\}$$

joining the vectors $z(a)/|z(a)|$ and $z(b)/|z(b)|$ on the unit sphere. The length of the shortest arc which joins the same vectors is equal to

$$\arccos \frac{\langle z(a), z(b)\rangle}{|z(a)|\,|z(b)|}.$$

Thus we have

$$\arccos \frac{\langle z(a), z(b)\rangle}{|z(a)|\,|z(b)|} < \arccos\|-B^{-1}A\|_2$$

or

$$\frac{\langle z(a), z(b)\rangle}{|z(a)|\,|z(b)|} > \|-B^{-1}A\|_2. \tag{10}$$

On the other hand, from (2) it follows that $Az(a) = -Bz(b)$ and consequently

$$\frac{\langle -B^{-1}Az(a), z(a)\rangle}{|-B^{-1}Az(a)|\,|z(a)|} = \frac{\langle z(a), z(b)\rangle}{|z(a)|\,|z(b)|}.$$

Now by the definition of $\|\cdot\|_2$ we obtain

$$\frac{\langle z(a), z(b)\rangle}{|z(a)|\,|z(b)|} \leqslant \|-B^{-1}A\|_2,$$

which contradicts (10).

Added in proof: It was recently shown [6] that Theorem 1 and 2 may be stated for the system (1) and the boundary value condition $Lx = r$, where L is a linear operator from the space of continuously differentiable functions defined on $[a, b]$ into R^d.

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