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Cosine families of unbounded normal operators

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Abstract. The cosine families of unbounded operators in a Hilbert space are defined and then the families consisting of normal operators are investigated. Under certain growth conditions the existence of a dense set of analytic vectors for the second generator is proved. Then that is used to obtain a spectral representation of the given family as the hyperbolic cosine functions of the essentially normal second generator.

1. Introduction. The bounded cosine families of operators in Banach spaces are now well known and have been studied in various papers, starting from the monograph by Sova [5]. In [6], Sova investigated normal cosine families of bounded operators in the Hilbert spaces. It turned out that the necessary and sufficient condition for regular cosine family to consist of normal operators is to have a normal second generator. Now the operators of a cosine family always commute and when they are bounded it follows that they are of the form $\mathscr{C}(t) = \int_{\boldsymbol{R}} \operatorname{coh}(ts)\, E(ds)$ with a certain spectral measure E (see Kurepa [2]).

The cosine families of unbounded operators were studied from the point of view of differential equations in the paper by Grabmüller [1] and the spectral representation of a cosine family of stronly commuting normal operators, under additional assumptions, has been found by Maltese in [3]. The aim of the present paper is to study the cosine families of unbounded normal operators *via* their analytic properties, that is, using the theorems that give sufficient conditions for formally normal operators to be essentially normal, whenever they have a dense set of analytic vectors (see [7]).

2. Unbounded cosine family and its second generator. Suppose that we have a family $\mathscr{C} = \{\mathscr{C}(t)\colon t \in \boldsymbol{R}\}$ of linear operators in a Hilbert space H. Define the set $D(\mathscr{C})$ as follows:

$$D(\mathscr{C}) = \left\{ x \in \bigcap_{p,q \in \boldsymbol{R}} D[\mathscr{C}(p)\mathscr{C}(q)]\colon \begin{array}{l} \text{(i) } 2\mathscr{C}(t)\mathscr{C}(s)x = \mathscr{C}(t+s)x + \mathscr{C}(t-s)x \\ \quad \text{for any } t, s \in \boldsymbol{R}, \\ \text{(ii) } \boldsymbol{R} \ni t \to \mathscr{C}(t)x \text{ is continuous.} \end{array} \right.$$

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Condition (i) is the d'Alembert equation for the cosine families and condition (ii) is a regularity requirement. We consider here cosine families defined on the whole real axis, for they are even functions, $\mathscr{C}(t) = \mathscr{C}(-t)$, and this can be used to extend them from the positive reals onto all reals.

Now the family $\mathscr{C}$ we will call a *cosine family of unbounded operators* iff the set $D(\mathscr{C})$ is a dense subspace of the given Hilbert space H and $\mathscr{C}(0) \subset I$. The set $D(\mathscr{C})$ we will call the *domain of the cosine family* $\mathscr{C}$.

Let us remark here that the d'Alembert equation assures that the domain of the cosine family is invariant for any operator from this family.

Later on we will need two other sets

$$D_1(\mathscr{C}) = \{x \in D(\mathscr{C}) \colon \exists_{\omega,M>0} \forall_{t\in \mathbf{R}} \|\mathscr{C}(t)x\| \leqslant M\exp(\omega|t|)\},$$

$$D_2(\mathscr{C}) = \{x \in D(\mathscr{C}) \colon \exists_{\omega,M>0} \forall_{t\in \mathbf{R}} \|\mathscr{C}(t)x\| \leqslant M\exp(\omega t^2)\}.$$

Some fundamental properties of the cosine families follow immediately from the definition. For example, putting $t = 0$ in the d'Alembert equation, we obtain ($\mathscr{D} = D(\mathscr{C})$)

$$\forall_{t\in \mathbf{R}} \forall_{x\in\mathscr{D}} \mathscr{C}(t)x = \mathscr{C}(-t)x$$

and from this follows

$$\forall_{t,s\in \mathbf{R}} \forall_{x\in\mathscr{D}} \mathscr{C}(t)\mathscr{C}(s)x = \mathscr{C}(s)\mathscr{C}(t)x.$$

The ability to use the integral calculus is essential in the studies of cosine families. Hence, one needs to put on the operators $\mathscr{C}(t)$ certain conditions concerning their closability or continuity. Four our purposes, essential is only the behaviour of $\mathscr{C}(t)$ on the cosine domain $\mathscr{D}$. Therefore, we will put our condition in the terms of a topology on $\mathscr{D}$. The *cosine topology* or *$\mathscr{C}$-topology* on $\mathscr{D}$ induced by the family $\mathscr{C}$ we define as the topology given by the family of increasing norms n_K,

$$n_K(x) := \sup_{|t|\leqslant K} \|\mathscr{C}(t)x\|, \qquad K \geqslant 0,\ x \in \mathscr{D}.$$

As $n_0(x) = \|x\|$, it follows that the cosine topology is stronger than the Hilbert space topology. The other immediate conclusion is that any $\mathscr{C}(s)$, $s \in \mathbf{R}$, is continuous in the $\mathscr{C}$-topology. Indeed, there is $n_K(\mathscr{C}(s)x) \leqslant n_{K+|s|}(x)$ as $\mathscr{C}(t)\mathscr{C}(s)x = \frac{1}{2}\mathscr{C}(t+s)x + \frac{1}{2}\mathscr{C}(t-s)x$ for any $x \in \mathscr{D}$.

The cosine family $\mathscr{C}$ we will call a *closed cosine family* iff the (metrizable) $\mathscr{C}$-topology is a Fréchet topology, that is, if $\mathscr{D}$ is under it a complete space.

PROPOSITION 1. *Any cosine family consisting of closed operators is a closed cosine family.*

Proof. We want to show that the cosine domain $\mathscr{D}$ is $\mathscr{C}$-complete. Let then $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathscr{D}$, that is, for any $K \geqslant 0$, $\varepsilon > 0$, there is a number N such that if $n, m \geqslant N$, then $n_K(x_n - x_m) < \varepsilon$.

As $\|\mathscr{C}(t)x\| \leqslant n_{|t|}(x)$ for any $t \in \boldsymbol{R}$, $x \in \mathscr{D}$, so for any $t \in \boldsymbol{R}$ $\{\mathscr{C}(t)x_n\}$ is a Cauchy sequence in the Hilbert space norm, and hence it has a limit, say y_t. But $x_n = \mathscr{C}(0)x_n$ tend to y_0 and as the operators $\mathscr{C}(t)$, $t \in \boldsymbol{R}$, are closed, this implies $y_0 \in D(\mathscr{C}(t))$ and $\mathscr{C}(t)y_0 = y_t$ for any $t \in \boldsymbol{R}$.

We will now show that $y_0 \in \mathscr{D}$ and $x_n \to y_0$ in $\mathscr{C}$-topology.

Fix any $s \in \boldsymbol{R}$. We have

$$\mathscr{C}(t)\mathscr{C}(s)x_n = \tfrac{1}{2}[\mathscr{C}(t+s)x_n + \mathscr{C}(t-s)x_n] \to \tfrac{1}{2}y_{t+s} + \tfrac{1}{2}y_{t-s},$$

and

$$\mathscr{C}(s)x_n \to y_s = \mathscr{C}(s)y_0.$$

Once again it follows that

$$\mathscr{C}(s)y_0 \in D(\mathscr{C}(t)) \text{ and } \mathscr{C}(t)\mathscr{C}(s)y_0 = \tfrac{1}{2}\mathscr{C}(t+s)y_0 + \tfrac{1}{2}\mathscr{C}(t-s)y_0.$$

Of course, $\mathscr{C}(0)y_0 = y_0$.

Take now the limit with $m \to \infty$ in the Cauchy condition. We get

$$\mathop{\forall}_{K \geqslant 0}\ \mathop{\forall}_{\varepsilon > 0}\ \mathop{\exists}_{N}\ \mathop{\forall}_{n \geqslant N}\ \sup_{|t| \leqslant K} \|\mathscr{C}(t)y_0 - \mathscr{C}(t)x_n\| \leqslant \varepsilon.$$

This implies that on any compact interval the function $\boldsymbol{R} \ni t \to \mathscr{C}(t)y_0$ is continuous as a uniform limit of the continuous functions $\boldsymbol{R} \ni t \to \mathscr{C}(t)x_n$. The second thing we obtain from the above estimation is that $x_n \to y_0$ in the $\mathscr{C}$-topology. Our proposition is proved.

With any cosine family there is connected its *second generator*. In our case, we define it as the operator $\mathscr{C}^{:}$ given by

$$D(\mathscr{C}^{:}) = \{x \in \mathscr{D}\colon \text{there exists } \mathscr{C}\text{-}\lim_{h \to 0} 2/h^2[\mathscr{C}(h)x - x]\}$$

and

$$\mathscr{C}^{:}x = \mathscr{C}\text{-}\lim_{h \to 0} 2/h^2[\mathscr{C}(h)x - x].$$

As the $\mathscr{C}$-topology is a complete topology on $\mathscr{D}$ we have by definition $\mathscr{C}^{:}x \in \mathscr{D}$.

3. Elementary properties of the second generator. From now on we assume, if not stated otherwise, that the given cosine family $\mathscr{C}$ is closed. This section we begin with a few simple remarks.

Let start with noticing that the function $t \to \mathscr{C}(t)x$ is for any $x \in \mathscr{D}$ continuous in the $\mathscr{C}$-topology. Indeed, $\|\mathscr{C}(s)(\mathscr{C}(t)x - \mathscr{C}(t_0)x)\|$ is not greater than

$$\tfrac{1}{2}(\|\mathscr{C}(s+t)x - \mathscr{C}(s+t_0)x\| + \|\mathscr{C}(s-t)x - \mathscr{C}(s-t_0)x\|).$$

But $t \to \mathscr{C}(t)x$ as continuous is uniformly continuous on compact intervals. Hence for any $\varepsilon > 0$ there will be $\sup_{|s| \leqslant K} \|\mathscr{C}(s)(\mathscr{C}(t)x - \mathscr{C}(t_0)x)\| \leqslant \varepsilon$ for sufficiently small $|t - t_0|$, as $s+t$, $s+t_0$, $s-t$, $s-t_0 \in [K-t_0, K+t_0]$. That proves the $\mathscr{C}$-continuity of our function.

Suppose now that we have a real continuous function φ defined on an interval J, finite or not. We have just shown that the function $J \ni s \to \varphi(s)\mathscr{C}(s)x$ is $\mathscr{C}$-continuous for any $x \in \mathscr{S}$. Define now the operator $M(\varphi)$ as one that has the domain equal to

$$\{x \in \mathscr{S}: \underset{K \geqslant 0}{\forall} \int_J |\varphi(s)| \cdot n_{K+|s|}(x)\, ds < \infty\}$$

and the values equal to $M(\varphi)x = \int_J \varphi(s)\mathscr{C}(s)x\, ds$.

Knowing that $n_K(\mathscr{C}(s)x) \leqslant n_{K+|s|}(x)$ we see that the integral defining $M(\varphi)x$ exists as the Riemann integral in the $\mathscr{C}$-topology, and therefore its value belongs to $\mathscr{S}$. The reason is that for any $\mathscr{C}$-continuous function $X: J \to \mathscr{S}$ such that $\int_J n_K(X(t))\,dt < \infty$, $K \geqslant 0$, there exists the Riemann type integral $\int_J X(t)\,dt$.

Proposition 2. *For any $x \in D(M(\varphi))$ and $t \in \mathbf{R}$ there is $\mathscr{C}(t)x \in D(M(\varphi))$ and $M(\varphi)\mathscr{C}(t)x = \mathscr{C}(t)M(\varphi)x$. If the interval J is compact, $J = [-a, a]$ for certain $a > 0$, then $D(M(\varphi)) = \mathscr{S}$ and $M(\varphi)$ is a $\mathscr{C}$-continuous operator. For any fixed $t \in \mathbf{R}$ and $x \in D(M(\varphi))$ there exists the integral $\int_J \varphi(s)\mathscr{C}(s+t)x\,ds$ and is equal to $\int_{J-t} \varphi(s-t)\mathscr{C}(s)x\,ds$.*

Proof. There holds $n_{K+|s|}(\mathscr{C}(t)x) \leqslant n_{K+|t|+|s|}(x)$ and therefore if $x \in D(M(\varphi))$ then $\mathscr{C}(t)x \in D(M(\varphi))$. But $\mathscr{C}(t)$ is $\mathscr{C}$-continuous and $M(\varphi)x$ is a $\mathscr{C}$-limit of partial sums, and $\mathscr{C}(t)$ commutes with any $\mathscr{C}(s)$ that appears in that partial sum. This shows that $M(\varphi)\mathscr{C}(t)x = \mathscr{C}(t)M(\varphi)x$.

If the interval J is equal to $[-a, a]$, then $n_{K+|s|}(x) \leqslant n_{K+a}(x)$ for $s \in J$ and hence one obtains the estimation $n_K(M(\varphi)x) \leqslant 2a \sup_{s \in J}|\varphi(s)| \times \times n_{K+a}(x)$. Thus $M(\varphi)$ is $\mathscr{C}$-continuous.

To complete the proof consider $n_K(\mathscr{C}(s+t)x)$. This is estimated by $n_{K+|t|+|s|}(x)$ which shows the existence of the integral $\int_J \varphi(s)\mathscr{C}(s+t)x\,ds$ in the $\mathscr{C}$-topology and one only needs linearly change the variable.

Turn now our attention to the second generator. Let us start with a simple remark on its commutativity with $\mathscr{C}(t)$, $t \in \mathbf{R}$. $\mathscr{C}(t)$ and $\mathscr{C}(h)$ commute on $\mathscr{S}$ for any $t, h \in \mathbf{R}$. Therefore if only $x \in D(\mathscr{C}^{:})$ then

$$n_K[2h^{-2}(\mathscr{C}(h)-I)\mathscr{C}(t)x - \mathscr{C}(t)\mathscr{C}^{:}x] \leqslant n_{K+|t|}[2h^{-2}(\mathscr{C}(h)-I)x - \mathscr{C}^{:}x]$$

which proves that for any $t \in \mathbf{R}$ $\mathscr{C}(t)D(\mathscr{C}^{:}) \subset D(\mathscr{C}^{:})$ and $\mathscr{C}^{:}\mathscr{C}(t)x = \mathscr{C}(t)\mathscr{C}^{:}x$.

But we have a more interesting relation between $\mathscr{C}(t)$ and $\mathscr{C}^{:}$ that is formulated in the following proposition.

Proposition 3. *Let $x \in D(\mathscr{C}^{:})$, where $\mathscr{C}^{:}$ is the second generator of a closed* *cosine family* $\mathscr{C}$. *Then for any* $t \in \boldsymbol{R}$ *we have* $\int\limits_0^t (t-s)\mathscr{C}(s)x\,ds \in D(\mathscr{C}^{\cdot})$ *and*

$$\mathscr{C}^{\cdot}\int\limits_0^t (t-s)\mathscr{C}(s)x\,ds = \int\limits_0^t (t-s)\mathscr{C}(s)\mathscr{C}^{\cdot}x\,ds = \mathscr{C}(t)x - x.$$

Proof. Fix $K \geqslant 0$ and any $u \in \boldsymbol{R}$, $|u| \leqslant K$. We want to estimate the norm

$$\Big\|\mathscr{C}(u)\big[2h^{-2}(\mathscr{C}(h)-I)\int\limits_0^t (t-s)\mathscr{C}(s)x\,ds - \mathscr{C}(t)x + x\big]\Big\|.$$

Denote by y the vector $\mathscr{C}(u)x$. From Proposition 2 it follows that

$$\mathscr{C}(u)\int\limits_0^t (t-s)\mathscr{C}(s)x\,ds = \int\limits_0^t (t-s)\mathscr{C}(s)\mathscr{C}(u)x\,ds.$$

Using this and the d'Alembert equation we can estimate our norm by

$$\Big\|h^{-2}\int\limits_0^t (t-s)\big(\mathscr{C}(s+h)+\mathscr{C}(s-h)-2\mathscr{C}(s)\big)y\,ds - \mathscr{C}(t)y + y\Big\|.$$

Linearly changing variables and reducing similar integrals one can obtain, after some tedious calculations, that the above norm is equal to the norm of the following expression

$$-2h^{-1}\int\limits_0^h \big(\mathscr{C}(s)y - y\big)ds + 2h^{-2}\int\limits_0^h s\big(\mathscr{C}(s)y-y\big)ds + h^{-1}\int\limits_{t-h}^{t+h}\big(\mathscr{C}(s)y - \mathscr{C}(t)y\big)ds$$

$$+h^{-2}\int\limits_t^{t+h}(t-s)\big(\mathscr{C}(s)y-\mathscr{C}(t)y\big)ds - h^{-2}\int\limits_{t-h}^{t}(t-s)\big(\mathscr{C}(s)y-\mathscr{C}(t)y\big)ds.$$

The norm of the above depends only on $\sup\limits_{|s|\leqslant|h|}\|\mathscr{C}(s)y-y\|$ and $\sup\limits_{|s-t|\leqslant|h|}\|\mathscr{C}(s)y - \mathscr{C}(t)y\|$, where $y = \mathscr{C}(u)x$, $|u| \leqslant K$.

This leads to the estimation

$$n_K\big[2h^{-2}(\mathscr{C}(h)-I)\int\limits_0^t (t-s)\mathscr{C}(s)x\,ds - \mathscr{C}(t)x + x\big]$$

$$\leqslant 4\sup_{|s|\leqslant|h|} n_K\big(\mathscr{C}(s)x - \mathscr{C}(0)x\big) + 3\sup_{|s-t|\leqslant|h|} n_K\big(\mathscr{C}(s)x - \mathscr{C}(t)x\big).$$

The uniform $\mathscr{C}$-continuity of the function $t \to \mathscr{C}(t)x$ on compact intervals allows us to conclude that

$$\mathscr{C}^{\cdot}\int\limits_0^t (t-s)\mathscr{C}(s)x\,ds = \mathscr{C}(t)x - x.$$

The second equality in the conclusion we obtain by taking the limit under the integral, and that is justified by the estimation, for $0 \leqslant s \leqslant t$,

$$n_K\left[(t-s)\mathscr{C}(s)\left(2h^{-2}(\mathscr{C}(h)x-x)\right)\right] \leqslant |t|\, n_{K+|t|}\left[2h^{-2}(\mathscr{C}(h)x-x)\right].$$

The representation of $\mathscr{C}(t)x$ that we have just shown turns out to be very useful. It is the main tool in the proofs of the next two remarks.

Remark 4. $\mathscr{C}^{\cdot}$ is a $\mathscr{C}$-closed operator.

Proof. Let $x_n \in D(\mathscr{C}^{\cdot})$, $\mathscr{C}\text{-}\lim x_n = x \in \mathscr{D}$ and $\mathscr{C}\text{-}\lim \mathscr{C}^{\cdot} x_n = y \in \mathscr{D}$. We want to show that $x \in D(\mathscr{C}^{\cdot})$ and $\mathscr{C}^{\cdot}x = y$. Fix now $\varepsilon > 0$ and $K \geqslant 0$. We look for $h_0 > 0$ such that $|h| \leqslant h_0$ implies $n_K\left[2h^{-2}(\mathscr{C}(h)x-x)-y\right] \leqslant \varepsilon$. The representation from Proposition 3 gives

$$\begin{aligned} n_K\left[2h^{-2}(\mathscr{C}(h)-I)(x_n-x_m)\right] &= n_K\left[2h^{-2}\int_0^h (h-s)\mathscr{C}(s)\mathscr{C}^{\cdot}(x_n-x_m)\,ds\right] \\ &\leqslant 2h^{-2}\int_0^h (h-s)\, n_K\left[\mathscr{C}(s)\mathscr{C}^{\cdot}(x_n-x_m)\right] ds \\ &\leqslant 2h^{-2}\cdot h\cdot h\cdot n_{K+|h|}(\mathscr{C}^{\cdot}x_n-\mathscr{C}^{\cdot}x_m) \leqslant 2n_{K+1}(\mathscr{C}^{\cdot}x_n-\mathscr{C}^{\cdot}x_m) \end{aligned}$$

if we take $|h| \leqslant 1$. Passing to the limit with m we obtain

$$n_K\left[2h^{-2}(\mathscr{C}(h)-I)(x_n-x)\right] \leqslant n_{K+1}(\mathscr{C}^{\cdot}x_n-y).$$

Let now N be such that $n_{K+1}(\mathscr{C}^{\cdot}x_N-y) \leqslant \varepsilon/6$ and $n_K(\mathscr{C}^{\cdot}x_N-y) \leqslant \varepsilon/3$. For this N let h_0 be such that for any h, $|h| \leqslant h_0$, holds $n_K\left[2h^{-2}(\mathscr{C}(h)-I)x_N - \mathscr{C}^{\cdot}x_N\right] \leqslant \varepsilon/3$. Then $n_K\left[2h^{-2}(\mathscr{C}(h)x-x)-y\right]$ will be estimated by

$$n_K\left[2h^{-2}(\mathscr{C}(h)-I)(x-x_N)\right]+n_K\left[2h^{-2}(\mathscr{C}(h)x_N-x_N)-\mathscr{C}^{\cdot}x_N\right]+n_K(\mathscr{C}^{\cdot}x_N-y)$$

and will be less then ε.

This shows that $y = \mathscr{C}\text{-}\lim_{h\to 0} 2h^{-2}(\mathscr{C}(h)x-x)$. Hence $x \in D(\mathscr{C}^{\cdot})$ and $\mathscr{C}^{\cdot}x = y$, what was to be proved.

Remark 5. Let $x \in D(\mathscr{C}^{\cdot})$ and $f_x(t) = \mathscr{C}(t)x$, $t \in \boldsymbol{R}$. Then the function f_x is twice continuously $\mathscr{C}$-differentiable and $f_x'(t) = \int_0^t \mathscr{C}(s)\mathscr{C}^{\cdot}x\,ds$, $f_x''(t) = \mathscr{C}(t)\mathscr{C}^{\cdot}x = \mathscr{C}^{\cdot}\mathscr{C}(t)x$. Hence $f_x'(0) = 0$ and $f_x''(0) = \mathscr{C}^{\cdot}x$.

Proof. Proposition 3 allows us to write

$$f_x(t) = x + \int_0^t (t-s)\mathscr{C}(s)\mathscr{C}^{\cdot}x\,ds = x + t\int_0^t \mathscr{C}(s)\mathscr{C}^{\cdot}x\,ds - \int_0^t s\,\mathscr{C}(s)\mathscr{C}^{\cdot}x\,ds$$

and from this immediately follows our remark.

The above remark connects our and the standard definitions of the second generator. We see that it corresponds to the second derivative. More clearly this is to be seen in the next proposition. The assumptions may seem too strong, but that will be enough for our purposes.

PROPOSITION 6. *Let $\varphi\colon \boldsymbol{R}\to\boldsymbol{R}$ be a three times continuously differentiable real function and let $x\in\mathcal{D}$ be such that for any $K\geqslant 0$, $i=0, 1, 2, 3$ and sufficiently small h there is*

$$\int_{-\infty}^{\infty}\sup_{|u-s|\leqslant h}|\varphi^{(i)}(u)|\,n_{K+|s|}(x)\,ds<\infty .$$

Then $M(\varphi)x\in D(\mathscr{C}^{\cdot})$ and $\mathscr{C}^{\cdot}M(\varphi)x = M(\varphi'')x$.

Proof. One can use the d'Alembert equation and a linear change of variables in order to obtain

$$n_K\big[2h^{-2}(\mathscr{C}(h)-I)M(\varphi)x-M(\varphi'')x\big]$$
$$= n_K\Big[\int_{-\infty}^{\infty}\big(h^{-2}(\varphi(s+h)+\varphi(s-h)-2\varphi(s))-\varphi''(s)\big)\mathscr{C}(s)x\,ds\Big].$$

Expanding φ into the Taylor series up to the second degree terms we see that the above is estimated by

$$\tfrac{1}{6}h\cdot 2\int_{-\infty}^{\infty}\sup_{|u-s|\leqslant h}|\varphi'''(u)|\,n_{K+|s|}(x)\,ds .$$

It is enough now to pass to the limit with $h\to 0$.

4. Normal cosine families of unbounded operators. In this section we will consider a specific situation when a given cosine family consists of normal or formally normal operators. We already know that the operators of a cosine family commute pointwise on the cosine domain. Pointwise commuting on an invariant domain is not sufficient for unbounded normal operators to strongly commute, that is to have commuting spectral measures. Analytic properties, for example the existence of a dense set of analytic or quasianalytic vectors can solve the problem. Hence the exponential growth domains $D_1(\mathscr{C})$ and $D_2(\mathscr{C})$ play an important role in what follows.

Let begin with the concept of the formally normal operator. Having a Hilbert space H and in it a linear domain $\mathcal{D}$, dense in H, we consider a set $\mathscr{L}^{\#}(\mathcal{D})$ of operators from $\mathcal{D}$ into $\mathcal{D}$ which we define in the following way. An operator A belongs to $\mathscr{L}^{\#}(\mathcal{D})$ iff for any $y\in\mathcal{D}$ there exists $z\in\mathcal{D}$ such that for any $x\in\mathcal{D}$ there holds $\langle Ax, y\rangle=\langle x, z\rangle$, and then we write z as $A^{\#}y$. It is easily seen that $A^{\#}=A^{*}|\mathcal{D}$, the restriction of the Hilbert space adjoint to the subspace $\mathcal{D}$. The set $\mathscr{L}^{\#}(\mathcal{D})$ consists of all operators for which the subspace $\mathcal{D}$ is invariant and in the same time is invariant for their adjoints.

An operator $A \in \mathscr{L}^{\#}(\mathscr{D})$ we will call formally normal on $\mathscr{D}$ iff for any $x \in \mathscr{D}$ there is $\|Ax\| = \|A^{\#}x\|$, and this is equivalent to $AA^{\#}x = A^{\#}Ax$ for any $x \in \mathscr{D}$. Of course if an operator T in H is normal, $\mathscr{D} \subset D(T) = D(T^*)$, $T\mathscr{D} \subset \mathscr{D}$ and $T^*\mathscr{D} \subset \mathscr{D}$, then the operator $A = T|\mathscr{D}$ is formally normal on $\mathscr{D}$. The inverse needs not to be true, a formally normal operator can have no normal extension. However, under specific conditions it always has and we will use this later.

Suppose now that we have a cosine family $\tilde{\mathscr{C}} = \{\tilde{\mathscr{C}}(t): t \in \boldsymbol{R}\}$ such that each $\tilde{\mathscr{C}}(t)$ is normal and $\tilde{\mathscr{C}}(t)^*\mathscr{D} \subset \mathscr{D}$, where $\mathscr{D} = \mathscr{D}(\tilde{\mathscr{C}})$ is the domain of the cosine family $\tilde{\mathscr{C}}$. The situation would be the same if we have a closed cosine family $\mathscr{C} = \{\mathscr{C}(t): t \in \boldsymbol{R}\}$ of formally normal operators from $\mathscr{L}^{\#}(\mathscr{D})$. We simply put $\mathscr{C}(t) = \tilde{\mathscr{C}}(t)|\mathscr{D}$. From our point of view such a proceder is justified, as of any concern is only what is happening on the common cosine domain of the given family. Hence our next theorem is formulated in the more general terms of formally normal, rather than normal operators. It solves a natural question arising about the relations of the original family and the one defined as $\{\mathscr{C}(t)^{\#}: t \in \boldsymbol{R}\}$.

THEOREM 7. *Suppose that $\mathscr{C} = \{\mathscr{C}(t): t \in \boldsymbol{R}\}$ is a closed cosine family with the domain $\mathscr{D} = D(\mathscr{C})$ and that each $\mathscr{C}(t) \in \mathscr{L}^{\#}(\mathscr{D})$ and is formally normal on $\mathscr{D}$. Then the family defined as $\mathscr{C}_{\#} = \{\mathscr{C}(t)^{\#}: t \in \boldsymbol{R}\}$ is a closed cosine family with the same domain $\mathscr{D}$ and the $\mathscr{C}$-topology is identical with the $\mathscr{C}_{\#}$-topology on $\mathscr{D}$.*

Furthermore if $\mathscr{C}^{\cdot\cdot}$ and $\mathscr{C}^{\cdot\cdot}_{\#}$ are the corresponding second generators, then $D(\mathscr{C}^{\cdot\cdot}) = D(\mathscr{C}^{\cdot\cdot}_{\#})$ and for any $x, y \in D(\mathscr{C}^{\cdot\cdot})$ one has $\langle \mathscr{C}^{\cdot\cdot}x, y\rangle = \langle x, \mathscr{C}^{\cdot\cdot}_{\#}y\rangle$ and $\|\mathscr{C}^{\cdot\cdot}x\| = \|\mathscr{C}^{\cdot\cdot}_{\#}x\|$. The operators $\mathscr{C}(t)$, $t \in \boldsymbol{R}$, doubly commute on $\mathscr{D}$, that is for any $s, t \in \boldsymbol{R}$ and $x \in \mathscr{D}$ there is

$$\mathscr{C}(t)\mathscr{C}(s)x = \mathscr{C}(s)\mathscr{C}(t)x \quad \textit{and} \quad \mathscr{C}(t)^{\#}\mathscr{C}(s)x = \mathscr{C}(s)\mathscr{C}(t)^{\#}x.$$

Proof. The operators $\mathscr{C}(t)$ and $\mathscr{C}(s)^{\#}$ act on $\mathscr{D}$ and therefore it is sufficient to take the $\#$-adjoint of the d'Alembert equation to obtain

$$2\mathscr{C}(t)^{\#}\mathscr{C}(s)^{\#} = \mathscr{C}(t+s)^{\#} + \mathscr{C}(t-s)^{\#} \quad \text{on } \mathscr{D}.$$

$\mathscr{D}$ is obviously equal to $\bigcap_{t,s,R} D(\mathscr{C}_{\#}(t)\mathscr{C}_{\#}(s))$, being invariant for any $\mathscr{C}(t)^{\#}$. To prove that $\mathscr{D} = D(\mathscr{C}_{\#})$ we only need the continuity condition (ii) from the definition of the cosine domain.

The formal normality of $\mathscr{C}(t)$ gives $\|\mathscr{C}(t)x\| = \|\mathscr{C}_{\#}(t)x\|$ for any $t \in \boldsymbol{R}$ and $x \in \boldsymbol{R}$ and $x \in \mathscr{D}$. This shows that the $\mathscr{C}_{\#}$-topology is identical with the $\mathscr{C}$-topology.

Returning to the continuity condition. Whenever $t \to t_0$ we have

$$\|\mathscr{C}_{\#}(t)x\| = \|\mathscr{C}(t)x\| \to \|\mathscr{C}(t_0)x\| = \|\mathscr{C}_{\#}(t_0)x\|$$

and for any $y \in \mathscr{D}$

$$\langle \mathscr{C}_{\#}(t)x, y\rangle = \langle \mathscr{C}(t)^{\#}x, y\rangle = \langle x, \mathscr{C}(t)y\rangle \to \langle x, \mathscr{C}(t_0)y\rangle = \langle \mathscr{C}_{\#}(t_0)x, y\rangle.$$

Thus when $t \to t_0$ $\mathscr{C}_{\#}(t)x$ are norm bounded vectors tending weakly on $\mathscr{D}$ to $\mathscr{C}_{\#}(t_0)x$, and with norms tending to $\|\mathscr{C}_{\#}(t_0)x\|$. Therefore $\mathscr{C}_{\#}(t)x \to \mathscr{C}_{\#}(t_0)x$ as $t \to t_0$. The continuity condition is now shown and that proves that $\mathscr{D} = D(\mathscr{C}_{\#})$.

To prove the assertions on the second generators we would like to have an equality $\mathscr{C}_{\#}(t)\mathscr{C}(s)x = \mathscr{C}(s)\mathscr{C}_{\#}(t)x$, $s, t \in \boldsymbol{R}$, as then $n_K^{\#}[2h^{-2}(\mathscr{C}_{\#}(h)-I)x-2h_1^{-2}(\mathscr{C}_{\#}(h_1)-I)x]$ will be equal to $n_K[2h^{-2}(\mathscr{C}(h)-I)x-2h_1^{-2}\times\times(\mathscr{C}(h_1)-I)x]$ and therefore the one will tend to zero at the same time as the other. This will imply the equality of the domains of the second generators. The other two properties are then obtained by taking appriopriate point limits and using the equality $\langle\mathscr{C}(t)x, y\rangle = \langle x, \mathscr{C}(t)^{\#} y\rangle$.

We will now prove the needed double commutativity.

We know that the operators $\mathscr{C}(t)$ and $\mathscr{C}(s)$ commute on $\mathscr{D}$ for any $t, s \in \boldsymbol{R}$, and that any $\mathscr{C}(t)$, $t \in \boldsymbol{R}$, being formally normal commutes with its $\#$-adjoint, $\mathscr{C}(t)\mathscr{C}(t)^{\#}x = \mathscr{C}(t)^{\#}\mathscr{C}(t)x$ for any $x \in \mathscr{D}$.

For any $u \in \boldsymbol{R}, u \neq 0$, consider the set Z_u of all $t \in \boldsymbol{R}$ such that $\mathscr{C}(u)\mathscr{C}(t)^{\#}x = \mathscr{C}(t)^{\#}\mathscr{C}(u)x$ for any $x \in \mathscr{D}$. It is not empty as it always contains 0 and u. The d'Alembert equation implies that $t+s$ is in Z_u, whenever t, s and $t-s$ are. On the other hand $2\mathscr{C}(u/2)^2 = \mathscr{C}(u)+I$ and therefore if t is in $Z_{u/2}$ then it is in Z_u. What we have said implies that any number of the form $k2^{-n}u$, where $k \in \boldsymbol{Z}$, $n \in \boldsymbol{N}$, is the set Z_u. Such numbers form a dense subset of $\boldsymbol{R}$. Take then for an arbitrary $s \in \boldsymbol{R}$ a sequence $s_n \to s$, $s_n \in Z_u$. For any $x \in D$ there is $\mathscr{C}_{\#}(s)\mathscr{C}(u)x = \lim \mathscr{C}_{\#}(s_n)\mathscr{C}(u)x$. But $s_n \in Z_u$ and hence $\mathscr{C}_{\#}(s)\mathscr{C}(u)x = \lim \mathscr{C}(u)\mathscr{C}_{\#}(s_n)x$. At the same time

$$\|\mathscr{C}(u)\mathscr{C}_{\#}(s_n)x\| = \|\mathscr{C}_{\#}(u)\mathscr{C}_{\#}(s_n)x\| = \|\mathscr{C}_{\#}(s_n)\mathscr{C}_{\#}(u)x\|$$

tends to

$$\|\mathscr{C}_{\#}(s)\mathscr{C}_{\#}(u)x\| = \|\mathscr{C}_{\#}(u)\mathscr{C}_{\#}(s)x\| = \|\mathscr{C}(u)\mathscr{C}_{\#}(s)x\|.$$

For any $y \in \mathscr{D}$

$$\langle \mathscr{C}(u)\mathscr{C}_{\#}(s_n)x, y\rangle = \langle \mathscr{C}(s_n)^{\#}x, \mathscr{C}(u)^{\#}y\rangle$$

and tends to

$$\langle \mathscr{C}(s)^{\#}x, \mathscr{C}(u)^{\#}y\rangle = \langle \mathscr{C}(u)\mathscr{C}(s)^{\#}x, y\rangle.$$

Once again these two imply that $\mathscr{C}(u)\mathscr{C}_{\#}(s_n)x \to \mathscr{C}(u)\mathscr{C}_{\#}(s)x$ in the norm topology of H, and therefore $\mathscr{C}_{\#}(s)\mathscr{C}(u)x = \mathscr{C}(u)\mathscr{C}_{\#}(s)x$. The theorem is now proved.

In the previous section we considered, for a real continuous function φ, the integral operator $M(\varphi)$. One can in a parallel way define $M_{*}(\varphi)$ by $M_{*}(\varphi)x = \int \varphi(s)\mathscr{C}_{*}(s)x\,ds$. We see that the both integral operators are defined on the same subset of $\mathscr{D}$ and that for any x, y from this subset there is

$$\langle M(\varphi)x, y\rangle = \langle x, M_{*}(\varphi)y\rangle.$$

We could specified here the properties and mutual relations of these two integral operators. But we will need them only for a special kind of functions, namely for the functions $g_r(s) = (\pi r)^{-1/2}\exp(-s^2/r)$, $r > 0$, and all their derivates. We will concentrate our attention on thetset $D_2(\mathscr{C})$, where the rate of growth of $\|\mathscr{C}(t)x\|$ is estimated by $M\exp(\omega t^2)$, with certain $\omega, M > 0$. We start with showing that this set is invariant under $M(g_r)$. In order to do that fix x, ω, M as above. Let g be a function of the form $g(s) = w(s)\exp(-s^2/r)$, where $r^{-1} > 3\omega$ and w is a polynomial. Then $M(g)x \in D_2(\mathscr{C})$.

Indeed,

$$\begin{aligned}\int |g(s)|\, n_{K+|s|}(x)\,ds &\leqslant \int |w(s)|\exp(-s^2/r)\,M\exp\big(\omega(K+|s|^2\big)\,ds\\ &= M\int |w(s)|\exp\big[(\omega - r^{-1})s^2 + 2K\omega|s| + \omega K^2\big]\,ds\end{aligned}$$

and as $\omega - r^{-1} < 0$ the last integral is finite for any $K \geqslant 0$. It follows that x is in the domain of $M(g)$.

There is $\mathscr{C}(t)M(g)x = M(g)\mathscr{C}(t)x$ and this leads to the estimations

$$\begin{aligned}\|\mathscr{C}(t)M(g)x\| = \|M(g)\mathscr{C}(t)x\| &\leqslant \int |g(s)|\cdot\|\mathscr{C}(s)\mathscr{C}(t)x\|\,ds\\ &\leqslant \int |w(s)|\exp(-s^2/r)\cdot 2^{-1}\big(\|\mathscr{C}(t+s)x\| + \|\mathscr{C}(t-s)x\|\big)\,ds\\ &\leqslant 2^{-1}M\int |w(s)|\exp(-s^2/r)\big[\exp\big(\omega(t+s)^2\big) + \exp\big(\omega(t-s)^2\big)\big]\,ds\\ &\leqslant M\exp(\omega t^2)\int |w(s)|\exp\big((\omega - r^{-1})s^2\big)\exp(2\omega|t|\cdot|s|)\,ds.\end{aligned}$$

By the assumption $r^{-1} - \omega > 2\omega$ so the above is estimated by

$$M\exp(\omega t^2)\int |w(s)|\exp(-\omega s^2)\exp(-\omega s^2 + 2\omega|t|\cdot|s|)\,ds.$$

The function $\exp(-\omega s^2 + 2\omega|t|\cdot|s|)$ has the maximum equal to $\exp(\omega t^2)$ and the integral $\int |w(s)|\exp(-\omega s^2)\,ds$ is finite. Therefore there exist $m > 0$ such that

$$\|\mathscr{C}(t)M(g)x\| \leqslant m\exp(2\omega t^2), \text{ hence } M(g)x \in D_2(\mathscr{C}).$$

The same is true if we take $M_{*}(g)$ instead of $M(g)$. In that way we come to the conclusion that for any $r, q < (6\omega)^{-1}$ and any $i, j \in N$ we have $x \in D\big(M_{*}(g_q^{(i)})M(g_r^{(j)})\big)$ if only $\|\mathscr{C}(t)x\| \leqslant M\exp(\omega t^2)$.

To simplify notation we will consider g_r and g_q instead of $g_r^{(j)}$ and $g_q^{(i)}$. Let $y = M(g_r)x$.

Then $y \in D(\mathscr{C}^{\cdot})$ and $\mathscr{C}^{\cdot} y = M(g_r'')x$. Now $y \in D(M_{\#}(g_q))$ and $M_{\#}(g_q)y$ is a $\mathscr{C}$-limit of the partial sums of the form

$$y_n = \sum_k (s_{k+1}^{(n)} - s_k^{(n)}) g_q(s_k^{(n)}) \mathscr{C}(s_k^{(n)})^{\#} y.$$

On the other hand $\mathscr{C}(s)^{\#}$ is $\mathscr{C}$-continuous and hence $\mathscr{C}(s)^{\#} y \in D(\mathscr{C}^{\cdot})$ and $\mathscr{C}^{\cdot}\mathscr{C}(s)^{\#} y = \mathscr{C}(s)^{\#}\mathscr{C}^{\cdot} y$ for any $s \in \mathbf{R}$. Applying it to y_n we obtain

$$\mathscr{C}^{\cdot} y_n = \sum_k (s_{k+1}^{(n)} - s_k^{(n)}) g_q(s_k^{(n)}) \mathscr{C}(s_k^{(n)})^{\#} \mathscr{C}^{\cdot} y.$$

But $\mathscr{C}^{\cdot} y = M(g_r'')x$ and belongs to the domain of $M_{\#}(g_q)$, and therefore the last sums tend in the $\mathscr{C}$-topology to $M_{\#}(g_q)\mathscr{C}^{\cdot} y$. The second generator is closed in the $\mathscr{C}$-topology and this observation implies that $M_{\#}(g_q) y \in D(\mathscr{C}^{\cdot})$ and $\mathscr{C}^{\cdot} M_{\#}(g_q) y = M_{\#}(g_q)\mathscr{C}^{\cdot} y$.

Put now together all that we have said.

PROPOSITION 8. *Let $\mathscr{C}$ be a cosine function as in Theorem 7. Let, for $x \in D_2(\mathscr{C})$, $M, \omega > 0$ be such that $\|\mathscr{C}(t)x\| \leqslant M\exp(\omega t^2)$. Define g_r by $g_r(s) = (\pi r)^{-1/2}\exp(-s^2/r)$. Suppose that $q, r < (6\omega)^{-1}$, and i, j are any natural numbers. Then there is*

(i) $\quad x \in D(M_{\#}(g_q^{(i)}) M(g_r^{(j)}))$ *and* $M_{\#}(g_q^{(i)}) M(g_r^{(j)}) x \in D_2(\mathscr{C})$;

$$M_{\#}(g_q^{(i)}) M(g_r^{(j)}) x \in D(\mathscr{C}^{\cdot}) \cap D(\mathscr{C}^{\cdot}_{\#}) \quad \textit{and}$$

(ii) $$\mathscr{C}^{\cdot} M_{\#}(g_q^{(i)}) M(g_r^{(j)}) x = M_{\#}(g_q^{(i)}) M(g_r^{(j+2)}) x,$$

$$\mathscr{C}^{\cdot}_{\#} M_{\#}(g_q^{(i)}) M(g_r^{(j)}) x = M_{\#}(g_q^{(i+2)}) M(g_r^{(j)}) x;$$

(iii) $$\mathscr{C}\text{-}\lim_{r\to 0}(\mathscr{C}\text{-}\lim_{q\to 0} M_{\#}(g_q) M(g_r) x) = x.$$

Proof. Only property (iii) is as yet not proved. As all we prove about $\mathscr{C}$ transfers itself immediately onto $\mathscr{C}_{\#}$ it will be sufficient to show that for any $y \in D_2(\mathscr{C})$ there is $\mathscr{C}\text{-}\lim_{r\to 0} M(g_r)y = y$. The function g_r is normalized, $\int g_r(s)\,ds = 1$ and therefore $y = \int g_r(s) y\,ds$. This gives

$$n_K(M(g_r)y - y) = n_K\left(\int g_r(s)(\mathscr{C}(s)y - y)\,ds\right) \leqslant \int g_r(s)\, n_K(\mathscr{C}(s)y - y)\,ds$$

for any fixed $K \geqslant 0$. Take an arbitrary $\varepsilon > 0$. The $\mathscr{C}$-continuity of the function $s \to \mathscr{C}(s)y$ assures the existence of $h > 0$ such that $n_K(\mathscr{C}(s)y - y) \leqslant \varepsilon$ if $|s| \leqslant h$. We will split the integral into two parts, one over the set on which the above estimation holds and the other over the rest of reals. Remembering that

$$n_K(\mathscr{C}(s)y - y) \leqslant 2n_{K+|s|}(y) \leqslant 2M\exp(\omega(K+|s|)^2)$$

we can write

$$n_K(M(g_r)y-y) \leqslant \int_{|s|\leqslant h} g_r(s)\, n_K(\mathscr{C}(s)y-y)\,ds + \int_{|s|>h} g_r(s)\, n_K(\mathscr{C}(s)y-y)\,ds$$

$$\leqslant \varepsilon \int_{|s|\leqslant h} g_r(s)\,ds + 2M\exp(\omega K^2) \int_{|s|\geqslant h} g_r(s)\exp(\omega s^2+2\omega K|s|)\,ds.$$

Put $s = r^{1/2}t$ and $M_1 = 2M\pi^{-1/2}\exp(\omega K^2)$. Then we get

$$n_K(M(g_r)y-y) \leqslant \varepsilon\cdot 1 + M_1 \int_{|t|\geqslant hr^{-1/2}} \exp(-t^2-\omega rt^2+2K\omega r^{1/2}|t|)\,dt.$$

The function under the integral is, for $r<1$, bounded by the summable function $\exp(-t^2+2K\omega|t|)$.

As the sets over which we take the integral tend to the empty set when $r\to 0$, there exists $r_0>0$ such that for $r<r_0$ the second component of our estimation is no greater then ε. Hence $n_K(M(g_r)y-y)\leqslant 2\varepsilon$. It follows that indeed $\mathscr{C}\text{-}\lim_{r\to 0} M(g_r)y = y$.

5. Analytic vectors of the second generator. The standard definition says that a vector $x\in H$ is called an *analytic vector for an operator A in a Hilbert space H* if $x\in\bigcap_{n\in N} D(A^n)$ and there exists $s>0$ such that the series $\sum s^n/n!\cdot\|A^n x\|$ is convergent. By the analogy we will say that $x\in\mathscr{D}$ is a *$\mathscr{C}$-analytic vector for the second generator $\mathscr{C}^{:}$* if $x\in\bigcap_{n\in N} D(\mathscr{C}^{:n})$ and there exists $s>0$ such that for any $K\geqslant 0$ the series $\sum s^n/n!\cdot n_K(\mathscr{C}^{:n}x)$ is convergent.

If $x\in\mathscr{S}$ is a $\mathscr{C}$-analytic vector of the second generator, then the cosine function $\mathscr{C}(t)x$ expands to a power series.

Proposition 9. *If $\mathscr{C}=\{\mathscr{C}(t)\colon t\in\boldsymbol{R}\}$ is a closed cosine family and x is a $\mathscr{C}$-analytic vector for the second generator $\mathscr{C}^{:}$, then $\mathscr{C}(t)x = \sum_{n=0}^{\infty} t^{2n}/(2n)!\cdot\mathscr{C}^{:n}x$ and the series is absolutely convergent in the $\mathscr{C}$-topology.*

Proof. We know that x is in the domain of any power of $\mathscr{C}^{:}$ hence applying several times Proposition 3 we get

$$\mathscr{C}(t)x = x+\int_0^t (t-s)\big[\mathscr{C}^{:}x+\int_0^s (s-u)\,\mathscr{C}(u)\,\mathscr{C}^{:2}x\,du\big]\,ds = \ldots$$

$$= \sum_{n=0}^{N-1} t^{2n}/(2n)!\cdot\mathscr{C}^{:n}x + R_N.$$

Where the remainder R_N is given for any $N\in\boldsymbol{N}$ by

$$R_N = \int_0^t\int_0^{s_1}\ldots\int_0^{s_{N-1}} (t-s_1)(s_1-s_2)\ldots(s_{N-1}-u)\,\mathscr{C}(u)\,\mathscr{C}^{:N}x\,du\,ds_{N-1}\ldots ds_1$$

and therefore

$$n_K(R_N) \leqslant \sup_{0\leqslant u\leqslant t} n_K\big(\mathscr{C}(u)\mathscr{C}^{:N}x\big)\cdot \int_0^t \ldots \int_0^{s_{N-1}} (t-s_1)\ldots(s_{N-1}-u)\,du\ldots ds_1$$

hence

$$n_K(R_N) \leqslant \sup_{0\leqslant u\leqslant t} n_{K+|u|}(\mathscr{C}^{:N}x)\,t^{2N}/(2N)! \leqslant n_{K+|t|}(\mathscr{C}^{:N}x)\,t^{2N}/(2N)!.$$

But the series $\sum s^N/N!\cdot n_{K+|t|}(\mathscr{C}^{:N}x)$ is convergent and for sufficiently big N there is $t^{2N}/(2N)! \leqslant s^N/N!$. Hence follows the convergence of the series $\sum t^{2N}/(2N)!\cdot n_{K+|t|}(\mathscr{C}^{:N}x)$, where $t\in\boldsymbol{R}$ is arbitrary and fixed. It now follows that the remainder R_N tends to zero as $N\to\infty$ and that shows that the required representation of $\mathscr{C}(t)x$ as a power series holds.

From what has been said we do not know yet whether the second generator has any $\mathscr{C}$-analytic vectors, even whether it is densely defined. However, we already know that it is defined on vectors of the form $M(g_r)x$, where $x\in D_2(\mathscr{C})$. Such vectors turn out to be as well analytic for the second generator. But for our purposes we will need slightly more.

PROPOSITION 10. *Let $x\in D_2(\mathscr{C})$ and let g_r and g_q be as in Proposition 8. Then $y = M_{\#}(g_q^{(2i)})M(g_r^{(2j)})x$, $i, j\in\boldsymbol{N}$, is a $\mathscr{C}$-analytic vector at the same time for $\mathscr{C}^{:}$ and for $\mathscr{C}^{:}_{\#}$.*

Proof. We will prove that y is $\mathscr{C}$-analytic for $\mathscr{C}^{:}_{\#}$, as for any $s\in\boldsymbol{R}$ $\mathscr{C}^{:}_{\#}\mathscr{C}^{:}\mathscr{C}(s)y = \mathscr{C}^{:}\mathscr{C}^{:}_{\#}\mathscr{C}(s)y = M_{\#}(g_q^{(2i+2)})M(g_r^{(2j+2)})C(s)x$ and $\|\mathscr{C}^{:}_{\#}z\| = \|\mathscr{C}^{:}z\|$, implying $n_K(\mathscr{C}^{:n}_{\#}y) = n_K(\mathscr{C}^{:n}y)$. Hence the $\mathscr{C}$-analytic vectors for $\mathscr{C}^{:}_{\#}$ and $\mathscr{C}^{:}$ coincide. The set of such vectors being invariant under $\mathscr{C}^{:}_{\#}$ and $\mathscr{C}^{:}$ allows us to prove the $\mathscr{C}$-analyticity for $y = M_{\#}(g_q)M(g_r)x$ only. Let $z = M(g_r)x$. We know that $\mathscr{C}^{:n}_{\#}y = M_{\#}(g_q^{(2n)})z$ and there exist $a, \omega' > 0$ such that $\|\mathscr{C}(t)z\| \leqslant a\exp(\omega' t^2)$ and $q^{-1} > 2\omega'$, $a, \omega' > 0$. Therefore we can estimate

$$\begin{aligned} n_K(\mathscr{C}^{:n}_{\#}y) &= n_K\big(\textstyle\int g_q^{(2n)}(s)\,\mathscr{C}(s)^{*}z\,ds\big) \leqslant \int |g_q^{(2n)}(s)|\,n_{K+|s|}(z)\,ds \\ &\leqslant \textstyle\int |g_q^{(2n)}(s)|\cdot a\exp\big(\omega(K+|s|)^2\big)\,ds. \end{aligned}$$

Now change the variable $s = t(q/2)^{1/2}$ and denote by $W_n(t)$ the Hermite polynomials $(d^n/dt^n)\exp(-t^2/2) = W_n(t)\exp(-t^2/2)$. Then the following holds

$$n_K(\mathscr{C}^{:n}_{\#}y) \leqslant a\exp(\omega' K^2)\,\pi^{-1/2}\int |W_{2n}(t)|\exp(-t^2/2)(2/q)^n\exp(\omega' q t^2/2 + b|t|)\,dt,$$

where $b = (2q)^{1/2}\omega' K$. The next step is to estimate this by

$$a\exp(\omega' K^2)\,\pi^{-1/2}(2/q)^n\big[\textstyle\int |W_{2n}(t)|^2\exp(-t^2/2)\,dt\big]^{1/2}\times$$
$$\times\big[\textstyle\int \exp(-t^2/2 + \omega' q t^2 + 2b|t|)\,dt\big]^{1/2}.$$

The second integral is finite for we assumed that $q^{-1} > 2\omega'$, and does not depend on n. The first integral is equal to $(2n)!(2\pi)^{1/2}$. Thus there exists a constant $c > 0$ such that $n_K(\mathscr{C}^{\cdot n}_{\#} y) \leqslant c(2/q)^n[(2n)!]^{1/2}$. It immediately follows that for $|s| < q/4$ the series $\sum s^n/n! \cdot n_K(\mathscr{C}^{\cdot n}_{\#} y)$ is convergent, that is that y is a $\mathscr{C}$-analytic vector for $\mathscr{C}^{\cdot}_{\#}$.

For the bounded cosine families the analytic vectors of the given above form have been found by Nelson and Triggani in paper [4].

6. Cosine families of normal operators. In the bounded case the norm of the operators of the cosine family is bounded by an exponential function, $\|\mathscr{C}(t)\| \leqslant M\exp(\omega|t|)$, and therefore the set $D_1(\mathscr{C})$ is equal to the whole space. On the other hand it is not difficult to check, in the unbounded case, that if x is a $\mathscr{C}$-analytic vector for $\mathscr{C}^{\cdot}$, then x belongs to $D_2(\mathscr{C})$. Therefore if we want a cosine family to have any analytic properties the set $D_2(\mathscr{C})$ should be big enough. Hence the assumptions in our main theorems.

THEOREM 11. *Let $\mathscr{C} = \{\mathscr{C}(t)\colon t \in \boldsymbol{R}\}$ be a closed cosine family defined on $\mathscr{D} = D(\mathscr{C})$. Let $\mathscr{C} \subset \mathscr{L}^{\#}(\mathscr{D})$ and let each $\mathscr{C}(t)$, $t \in \boldsymbol{R}$, be formally normal. Suppose that the set $D_2(\mathscr{C})$ is dense. Then the second generator $\mathscr{C}^{\cdot}$ is essentially normal.*

Denote by E the spectral measure of $\overline{\mathscr{C}^{\cdot}}$. Then for any $x \in D_2(\mathscr{C})$ there is $\mathscr{C}(t)x = \int_{\boldsymbol{C}} \operatorname{coh}(t\lambda^{1/2})E(d\lambda)x$, where $\operatorname{coh}(t\lambda^{1/2}) = \sum_{n=0}^{\infty} t^{2n}/(2n)!\,\lambda^n$ (does not actually depend on square root of λ).

Proof. Let consider the set

$$\mathscr{E} = \operatorname{lin\,span}\left\{M_{\#}(g_q^{(2i)})\,M(g_r^{(2j)})\,x\colon \quad \begin{array}{l} x \in D_2(\mathscr{C}),\ \|\mathscr{C}(t)x\| \leqslant M\exp(\omega t^2), \\ q,\, r < (6\omega)^{-1},\ i, j \in \boldsymbol{N}. \end{array}\right.$$

We have already shown that $\mathscr{E}$ is an invariant space for $\mathscr{C}^{\cdot}$ and for $\mathscr{C}^{\cdot}_{\#}$ (Proposition 8), that the vectors from $\mathscr{E}$ are $\mathscr{C}$-analytic for $\mathscr{C}^{\cdot}$ and for $\mathscr{C}^{\cdot}_{\#}$ (Proposition 10), and that for any $x, y \in \mathscr{E}$ there is $\langle \mathscr{C}^{\cdot}x, y\rangle = \langle x, \mathscr{C}^{\cdot}_{\#}y\rangle$ and $\|\mathscr{C}^{\cdot}x\| = \|\mathscr{C}^{\cdot}_{\#}x\|$ (Theorem 7). We also know that $\lim_{r\to 0}\lim_{q\to 0} M_{\#}(g_q)M(g_r)x = x$ for any $x \in D_2(\mathscr{C})$ and that $D_2(\mathscr{C})$ is dense in our Hilbert space H. It follows that $\mathscr{E}$ is an invariant, analytic and dense domain for the operators $\mathscr{C}^{\cdot}|\mathscr{E}$ and $\mathscr{C}^{\cdot}_{\#}|\mathscr{E}$, that are formally normal and $\#$-adjoint on $\mathscr{E}$. Theorem 1, p. 35, from the paper of Stochel and Szafraniec [7] states that in such a case the operators $\mathscr{C}^{\cdot}|\mathscr{E}$ and $\mathscr{C}^{\cdot}_{\#}|\mathscr{E}$ are essentially normal. But there is $\overline{\mathscr{C}^{\cdot}|\mathscr{E}} \subset \overline{\mathscr{C}^{\cdot}} \subset (\mathscr{C}^{\cdot}_{\#})^* \subset (\mathscr{C}^{\cdot}_{\#}|\mathscr{E})^*$ and the first and the last terms are normal. Thus the inclusions must in fact be equalities which implies $\overline{\mathscr{C}^{\cdot}} = \overline{\mathscr{C}^{\cdot}|\mathscr{E}}$ and is normal.

Let E be a spectral measure such that $\overline{\mathscr{C}^{\cdot}} = \int_{\boldsymbol{C}} \lambda E(d\lambda)$. Any $x \in \mathscr{E}$ is a $\mathscr{C}$-analytic vector for $\mathscr{C}^{\cdot}$ so we can use Proposition 9 in order to obtain $\mathscr{C}(t)x = \sum t^{2n}/(2n)! \cdot \mathscr{C}^{\cdot n} x$. On the other hand there is

$$|\sum_{k=N}^{\infty} \lambda^k t^{2k}/(2k)!|^2 \leqslant \sum_{n=N}^{\infty} |\lambda|^n t^{2n}/(2n)! \sum_{k+l=n} (2n)!/[(2k)!(2l)!]$$
$$\leqslant \sum_{n=N}^{\infty} |\lambda|^n (2t)^{2n}/(2n)!$$

and hence

$$\int_{C} |\sum_{k=N}^{\infty} \lambda^k t^{2k}/(2k)!|^2 \|E(d\lambda)x\|^2 \leqslant \int_{C} \sum_{n=N}^{\infty} (2t)^{2n}/(2n)! |\lambda|^n \|E(d\lambda)x\|^2$$
$$\leqslant \sum_{n=N}^{\infty} (2t)^{2n}/(2n)! [\int_{C} |\lambda|^{2n} \|E(d\lambda)x\|^2]^{1/2} \cdot [\int_{C} 1 \cdot \|E(d\lambda)x\|^2]^{1/2}$$
$$= \sum_{n=N}^{\infty} (2t)^{2n}/(2n)! \cdot \|\mathscr{C}^{\cdot n} x\| \cdot \|x\| \to 0 \quad \text{as } N \to \infty$$

and is finite. It follows that $x \in D(\int_{C} \mathrm{coh}(t\lambda^{1/2}) E(d\lambda))$ and that

$$\|\int_{C} \mathrm{coh}(t\lambda^{1/2}) E(d\lambda) x - \sum_{k=0}^{N-1} t^{2k}/(2k)! \cdot \mathscr{C}^{\cdot k} x\|^2$$
$$= \|\int_{C} \sum_{k=N}^{\infty} t^{2k}/(2k)! \lambda^k E(d\lambda) x\|^2 = \int_{C} |\sum_{k=N}^{\infty} t^{2k}/(2k)! \lambda^k|^2 \|E(d\lambda) x\|^2,$$

hence tends to zero as $N \to \infty$. This implies that

$$\int_{C} \mathrm{coh}(t\lambda^{1/2}) E(d\lambda) x = \sum_{k=0}^{\infty} t^{2k}/(2k)! \cdot \mathscr{C}^{\cdot k} x = \mathscr{C}(t) x$$

for any x that is a $\mathscr{C}$-analytic vector for $\mathscr{C}^{\cdot}$, in particular for any $x \in \mathscr{E}$.

We also know that any $y \in D_2(\mathscr{C})$ is a limit of vectors from $\mathscr{E}$, $y = \lim_{r \to 0} \lim_{q \to 0} M_{\#}(g_q) M(g_r) y$ and therefore

$$\mathscr{C}(t) y = \lim_{r \to 0} \lim_{q \to 0} (\int_{C} \mathrm{coh}(t\lambda^{1/2}) E(d\lambda)) M_{\#}(g_q) M(g_r) y$$

(we applied the $\mathscr{C}$-continuity of $\mathscr{C}(t)$). It is enough now to use the observation that the spectral integral is a closed operator to conclude that $y \in D(\int_{C} \mathrm{coh}(t\lambda^{1/2}) E(d\lambda))$ and that $\mathscr{C}(t) y = \int_{C} \mathrm{coh}(t\lambda^{1/2}) E(d\lambda) y$.

THEOREM 12. *Suppose that $\mathscr{C} = \{\mathscr{C}(t)\colon t \in \mathbf{R}\}$ is a cosine family of unbounded normal operators in a Hilbert space H. Assume that the set $D_1(\mathscr{C})$ is dense in H. Assume furthermore that $D^{\infty}[\mathscr{C}(s)] \subset D[\mathscr{C}(t)]$ for $0 < |s| \leqslant |t|$. Then the second generator $\mathscr{C}^{\cdot}$ is essentially normal and if we denote by E the spectral* *measure of its closure we have*

$$\mathscr{C}(t) = \int_{C} \operatorname{coh}(t\lambda^{1/2}) E(d\lambda).$$

Proof. First of all we want to show that $\mathscr{D}_1 = D_1(\mathscr{C})$ is a core for each operator $\mathscr{C}(t)$, $t \in \boldsymbol{R}$. We start with a general remark. Namely that for any $t \in \boldsymbol{R}$, $x \in D(\mathscr{C})$ there is $\|\mathscr{C}(t)^n x\| \leqslant \max\limits_{0 \leqslant k \leqslant n} \|\mathscr{C}(kt)x\|$. This can be proved by induction if only one noticed that

$$\|\mathscr{C}(t)^{n+1} x\| = \|\mathscr{C}(t)^n \mathscr{C}(t) x\| \leqslant \max_{0 \leqslant k \leqslant n} \|\mathscr{C}(kt)\mathscr{C}(t)x\|$$
$$\leqslant \max_{0 \leqslant k \leqslant n} \tfrac{1}{2}\big(\|\mathscr{C}((k+1)t)x\| + \|\mathscr{C}((k-1)t)x\|\big) \leqslant \max_{0 \leqslant k \leqslant n+1} \|\mathscr{C}(kt)x\|.$$

If now $x \in \mathscr{D}_1$ and $\|\mathscr{C}(t)x\| \leqslant M\exp(\omega|t|)$, then from the above it follows that there exist constants c, $B > 0$ such that $\|\mathscr{C}(t)^n x\| \leqslant cB^n$, that is that x is a *bounded vector* for $\mathscr{C}(t)$. By the assumption the set $\mathscr{D}_1$ is dense in H. As it consists of bounded vectors for $\mathscr{C}(t)$ and is invariant under $\mathscr{C}(t)$, it would be a core for each $\mathscr{C}(t)$, $t \in \boldsymbol{R}$, if only we can show it is invariant under $\mathscr{C}(t)^*$, $t \in \boldsymbol{R}$ (see [8] Theorem 1).

In order to do that it is enough to prove that $\mathscr{D} = D(\mathscr{C})$ is invariant under $\mathscr{C}(t)^*$, $t \in \boldsymbol{R}$, as then we will have $\mathscr{C} \subset \mathscr{L}^*(\mathscr{D})$ and the operators from the family $\mathscr{C}$ will doubly commute on $\mathscr{D}$ (Theorem 7); hence, $\|\mathscr{C}(s)\mathscr{C}(t)^* x\| = \|\mathscr{C}(t)^* \mathscr{C}(s) x\| = \|\mathscr{C}(t)\mathscr{C}(s)x\|$. For $x \in D_1$, the last will be no greater than $M\exp[\omega(|t|+|s|)]$.

But the domain $\mathscr{D}$ will be invariant under $\mathscr{C}(t)^*$ if we know that for any $x \in \mathscr{D}$ one has $\mathscr{C}(t)^* x \in D[\mathscr{C}(s)]$ and $\mathscr{C}(s)\mathscr{C}(t)^* x = \mathscr{C}(t)^* \mathscr{C}(s) x$, any $s \in \boldsymbol{R}$. Indeed, the d'Alembert equation for $\mathscr{C}(t)^* x$ follows then easily from the d'Alembert equation for x, and the continuity condition can be deduced from the normality of our cosine function, as in the proof of Theorem 7, independently of other assumptions.

Observe that $\mathscr{D} \subset D^\infty[\mathscr{C}(p)]$ for any $p \in \boldsymbol{R}$. Therefore, from our assumption about the domains follows that for any $x \in \mathscr{D}$, $p, q \in \boldsymbol{R}$, $|p| \leqslant |q|$, one has $\mathscr{C}(p)^{*m} x \in D[\mathscr{C}(q)] = D[\mathscr{C}(q)^*]$, $m \in \boldsymbol{N}$. On the other hand, taking the $*$-adjoint of the d'Alembert equation, we see that $2\mathscr{C}(q)^* \mathscr{C}(p)^* x = \mathscr{C}(q+p)^* x + \mathscr{C}(q-p)^* x$ for any x belonging to the domains of all these operators. Now it can easily be proved by induction that for any $k \in \boldsymbol{N}$, $u \in \boldsymbol{R}$, $u \neq 0$, $x \in D^\infty[\mathscr{C}(u)]$, there exists a real polynomial w depending only on k such that

$$\mathscr{C}(ku)^* x = \mathscr{C}(-ku)^* x = w[\mathscr{C}(u)^*]x.$$

Fix $x \in \mathscr{D}$, $t, s \in \boldsymbol{R}$, $s \neq 0$ and let $\{t_n\}$ be a sequence tending to t, each t_n being of the form ks/n with a certain $k \in \boldsymbol{Z}$. Then there exist polynomials

w_n, v_n such that

$$\mathscr{C}(t_n)^* x = w_n[\mathscr{C}(s/n)^*]x \quad \text{and} \quad \mathscr{C}(s)z = v_n[\mathscr{C}(s/n)]z, \quad z \in \mathscr{D}.$$

Put $W_n = w_n[\mathscr{C}(s/n)]$, $V_n = v_n[\mathscr{C}(s/n)]$. The operators W_n and V_n doubly commute on $\mathscr{D}$, $\mathscr{C}(s/n)$ being normal, and $\mathscr{D} \subset D^\infty[\mathscr{C}(s/n)]$. We also know that $W_n^* x$ equals $\mathscr{C}(t_n)^* x$ and belongs to the domain of $\mathscr{C}(s)$, because $W_n^* x \in D^\infty[\mathscr{C}(s/n)]$. One now see that

$$\begin{aligned}\|\mathscr{C}(s)\mathscr{C}(t_n)^* x\| &= \|\mathscr{C}(s)^* \mathscr{C}(t_n)^* x\| = \|V_n^* W_n^* x\| = \|V_n W_n x\| \\ &= \|W_n V_n x\| = \|\mathscr{C}(t_n)\mathscr{C}(s)x\| \to \|\mathscr{C}(t)\mathscr{C}(s)x\| = \|\mathscr{C}(t)^* \mathscr{C}(s)x\|.\end{aligned}$$

On the other hand, for any $y \in \mathscr{D}$

$$\begin{aligned}\langle \mathscr{C}(s)\mathscr{C}(t_n)^* x, y\rangle &= \langle \mathscr{C}(t_n)^* x, \mathscr{C}(s)^* y\rangle = \langle W_n^* x, V_n^* y\rangle = \langle W_n^* V_n x, y\rangle \\ &= \langle W_n^* \mathscr{C}(s)x, y\rangle = \langle \mathscr{C}(t_n)^* \mathscr{C}(s)x, y\rangle \\ &\qquad \to \langle \mathscr{C}(t)^* \mathscr{C}(s)x, y\rangle.\end{aligned}$$

Therefore, $\mathscr{C}(s)\mathscr{C}(t_n)^* x \to \mathscr{C}(t)^* \mathscr{C}(s)x$, $n \to \infty$, in the norm topology. But $\mathscr{C}(s)$ being normal is a closed operator and $\mathscr{C}(t_n)^* x \to \mathscr{C}(t)^* x$. Hence, $\mathscr{C}(t)^* x \in D[\mathscr{C}(s)]$ and $\mathscr{C}(s)\mathscr{C}(t)^* x = \mathscr{C}(t)^* \mathscr{C}(s)x$. We have proved that the domain $\mathscr{D}$ is invariant for the operators $\mathscr{C}(t)$, $t \in \boldsymbol{R}$.

Take now $\tilde{\mathscr{C}}(t)$ equal to the restriction $\mathscr{C}(t)|\mathscr{D}$. If follows that the family $\tilde{\mathscr{C}}$ is contained in $\mathscr{L}^*(\mathscr{D})$. But $\tilde{\mathscr{C}}$ is a closed family consisting of formally normal operators and $D_2(\tilde{\mathscr{C}}) \supset D_1(\mathscr{C})$ is dense in H. We are in the position to apply Theorem 11. The second generator $\tilde{\mathscr{C}}^{\because}$ is then essentially normal and the spectral representation holds. As the difference between $\mathscr{C}$ and $\tilde{\mathscr{C}}$ is purely formal from the point of view of their cosine properties there must be $\mathscr{C}^{\because} = \tilde{\mathscr{C}}^{\because}$ and $\mathscr{C}^{\because}$ is essentially normal. The other conclusion is that

$$\mathscr{C}(t)|\mathscr{D}_1 = \Big(\int_{\boldsymbol{C}} \operatorname{coh}(t\lambda^{1/2})E(d\lambda)\Big)\Big|\mathscr{D}_1.$$

But we have proved that $\mathscr{D}_1$ is a core for the normal operator $\mathscr{C}(t)$, hence taking the closures of both sides of this equality we obtain $\mathscr{C}(t) = \int_{\boldsymbol{C}} \operatorname{coh}(t\lambda^{1/2})E(d\lambda)$.

Theorem 12 is wholly proved.

7. Final remarks. The representation we obtained for unbounded normal cosine families was also found by Maltese [3], but under much stronger assumptions. Namely the normal operators constituting the cosine family are supposed to have commuting spectral measures. The other assumption is that the intersection of the domains should be determined by a countable subfamily. In the world of unbounded operators the gap between the pointwise commutativity and the strong commutativity is very wide and a lot of papers are devoted to bridge it. Hence our different approach and the usefulness of the analytic vectors.

We, on the other hand, needed an additional assumption that the set $D_1(\mathscr{C})$ or $D_2(\mathscr{C})$ was dense. But when the spectral representation holds it is a necessary condition. Indeed, if $\mathscr{C}(t) = \int_C \operatorname{coh}(t\lambda^{1/2}) E(d\lambda)$ and x is a bounded vector for $A = \int_C \lambda E(d\lambda)$, $\|A^n x\| \leqslant cB^n$ with certain $c, B > 0$, then

$$\|\mathscr{C}(t)x\| \leqslant c\sum B^n t^{2n}/(2n)! \leqslant 2c\exp(B^{1/2}|t|)$$

which implies $x \in D_1(\mathscr{C})$. Therefore the set $D_1(\mathscr{C})$ as containing the bounded vectors for A is dense in H.

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