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On Bergman operators of exponential type

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To the memory of our colleague and dear friend Stefan Bergman

Abstract. Bergman integral operators for representing solutions of linear partial differential equations provide a way of applying function theoretic methods and results to the study of various general properties of those solutions. The rate of success of this approach depends on the choice and construction of suitable kernels. The present paper is concerned with this basic problem.

We first consider transformations for simplifying the differential equation for the kernel. Then we give a simplified proof of the criterion for kernels of class E (defined in Section 4) whose exponents are polynomials of even degree in the variable of integration. This also entails a characterization of minimal kernels and a representation of these kernels in terms of finitely many Bessel functions. Finally, it is shown that solutions obtained by operators of class E with a minimal kernel satisfy ordinary linear differential equations of second order, whose coefficients are given explicitly.

1. Introduction. Bergman operators are linear integral operators which transform analytic functions into solutions of linear partial differential equations. They serve as a translation principle for applying methods and results of complex analysis to general linear partial differential equations. For instance, there are various theorems which characterize general properties of analytic functions related to the domain of holomorphy, type and location of singularities, behavior near singularities, growth in general and distribution of values, and the coefficient problem for various representations. From these theorems one can obtain results on solutions of given partial differential equations by means of Bergman or other integral operators.

Bergman operators were also suggested by boundary value problems in compressible fluid flow (cf. [9], [11], [12]). A comparison of Bergman's classic [1] with more recent publications, such as [5] or [10], shows that during the past two decades this function theoretic approach to partial differential equations has become a large field of its own, and various types of integral operators have been introduced and applied to practical problems. Since the method provides easy access to large classes of solutions and their general properties, it seems very suitable in connection with the so-called *indirect method*, or *inverse method*, in which one disregards boundary conditions but obtains classes of solutions satisfying certain additional conditions imposed by geometrical, physical or other properties of a field. Cf. P. F. Neményi [13], where applications of the method to flow and elasticity problems are discussed.

The success of the integral operator method depends to a large extent on the simplicity of the operators used. It follows that, from a practical point of view, the determination and investigation of suitable kernels for those operators is a basic problem of the whole theory. The present paper is devoted to this problem in the case of second-order equations in two independent variables and, in particular, to operators with kernels such that one obtains global solutions. Note that, in general, Bergman's method of determining kernels is a method of undetermined coefficient functions in which one depends on convergence proofs modeled after Cauchy's classical work.

2. Concepts and notations. We consider equations of the form

$$\Delta\psi + \alpha(x, y)\psi_x + \beta(x, y)\psi_y + \gamma(x, y)\psi = 0 \tag{2.1}$$

whose coefficients are real-analytic functions of the real variables x, y on some domain $\mathcal{D} \subset \boldsymbol{R}^2$. We now assume x, y to be independent *complex* variables. Then we may continue those coefficients analytically into the complex domain. To simplify the resulting equation, we set $z = x+iy$, $z^* = x-iy$. Note that $z^* = \bar{z}$ (the conjugate) if and only if x and y are real. Furthermore, we may eliminate one of the two first partial derivatives in the usual fashion, say, the derivative with respect to z. Then we obtain the equation

$$Lu = u_{zz^*} + b(z, z^*)u_{z^*} + c(z, z^*)u = 0. \tag{2.2}$$

Analyticity of α, β, γ implies that of b and c. More specifically, we assume that $b, c \in C^\omega(\Omega)$, where $\Omega = \Omega_1 \times \Omega_2$, $0 \in \Omega_1$, and Ω_2 is the domain in the z^*-plane which corresponds to Ω_1 under the above transformation. Furthermore, we exclude the *trivial case* $c = 0$, in which (2.2) reduces to an ordinary differential equation for u_{z^*}.

A *Bergman operator* T *for* (2.2) is a linear integral operator defined on the complex vector space $V(\Omega_1)$ of all functions $f \in C^\omega(\Omega_1)$ and

$$T\colon\ V(\Omega_1) \to S_\Omega(L), \tag{2.3}$$

where $S_\Omega(L)$ is the complex vector space of all C^ω-solutions of (2.2) on Ω. A theory of these and similar operators has been developed by S. Bergman [1], I. N. Vekua [14] and others, the main purpose being that mentioned in the Introduction. A Bergman operator T for (2.2) can be defined by

$$(2.4)\qquad (Tf)(z,z^*) = \int_{-1}^{1} g(z,z^*,t)f(\tfrac{1}{2}z(1-t^2))(1-t^2)^{-1/2}\,dt.$$

From now on, the term *Bergman operator for* (2.2) will be used exclusively for T as defined by (2.3), (2.4). We assume t to be real, without loss of generality. g is called the *kernel* or *generating function* of T. We call g briefly a *Bergman kernel for* (2.2). Conditions for g to be a Bergman kernel are obtained by substituting $u = Tf$ given by (2.4) into (2.2). This yields (cf. S. Bergman [1])

THEOREM 2.1. *In* (2.2) *let* $b, c \in C^\omega(\Omega)$, *where* $\Omega = \Omega_1 \times \Omega_2$ *and*

$$\Omega_1 = \{z \,|\, |z| < \varrho\}, \quad \Omega_2 = \{z^* \,|\, |z^*| < \varrho\}, \quad \varrho > 0 \textit{ fixed}.$$

Furthermore, let g *be a solution of*

$$(2.5)\qquad Mg = (1-t^2)g_{z^*t} - t^{-1}g_{z^*} + 2ztLg = 0$$

on $\Omega \times I$, *where* $I = (-1, 1)$, *such that*

$$(2.6)\qquad (1-t^2)^{1/2}g_{z^*} \to 0 \quad \textit{as } t \to \pm 1 \quad (\textit{uniformly on } \Omega), \qquad g_{z^*}/tz \in C^0(\Omega \times I).$$

Then

$$Tf \in S_\Omega(L) \qquad (f \in C^\omega(\Omega_1)).$$

3. Transformations of (2.5) and kernels. In this section we discuss some transformations of equation (2.5) for Bergman kernels, starting with a motivation as follows.

The applicability of Bergman operators depends largely on the determination of simple kernels. This task still involves many open problems, despite of the general framework provided by Bergman's theory and the fact that a number of equations (2.2) have been treated successfully in that respect. We call g a Bergman kernel *of finite form* if g is a sum of finitely many terms, as opposed to an infinite series obtained by Bergman's method of solving (2.5). It turns out that useful classes of kernels can be derived by assuming functions g of a specific finite form involving finitely many unspecified coefficient functions. The latter have to be determined recursively from a system of second-order non-linear partial differential equations obtained by substituting g into (2.5). Since g is of finite form, such a kernel – if it exists – will be particularly useful for constructing solutions in the large.

It is clear that the assumption of a specific form of g imposes conditions on the form of the coefficients b and c in (2.2). Hence in each case it is essential to show that the class of equations (2.2) admitting operators with kernels of that form is sufficiently large and includes equations of practical interest. Furthermore, the explicit determination of those coefficient functions in g will be complicated in many cases. However, it may be possible to facilitate this task by first transforming (2.5) before substituting g. We list briefly some transformations which entail simplifications of (2.5).

We can eliminate the first term in (2.5) by using

$$\tau = z(1-t^2) \tag{3.1a}$$

as a new independent variable, instead of t. Then (2.5) becomes simply

$$g_{z^*}+2(\tau-z)Lg = 0. \tag{3.1b}$$

A change of that first term and the factor of Lg can be effected by introducing $\tau = zt^2$ instead of t. Then

$$g_{z^*\tau}-(2\tau)^{-1}g_{z^*}+Lg = 0. \tag{3.2}$$

Slightly more flexibility is gained by setting $\tau = \alpha z^{\beta} t^{\gamma}$.

The second term in (2.5) can be changed by setting

$$g = (1-t^2)^{-k/2}\hat{g}.$$

Then (2.5) implies

$$(1-t^2)\hat{g}_{z^*t}+(kt-t^{-1})\hat{g}_{z^*}+2ztL\hat{g} = 0.$$

Similarly, setting

$$g = z^{k/2}(z^{1/2}t)^{l+1}\hat{g}$$

leads to

$$(1-t^2)\hat{g}_{z^*t}+(kt+lt^{-1})\hat{g}_{z^*}+2ztL\hat{g} = 0 \tag{3.3}$$

so that one can eliminate that second term, or get rid of t^{-1}, etc.

Another transformation of the independent variable is $t = \sin\theta$ and yields

$$(g_{z^*}\cot\theta)_{\theta}+2zLg = 0.$$

Those transformations concerned an independent variable or the dependent variable g. As a third type, we may combine those two types of transformation. An example of practical interest is

$$\tau = z^{1/2}t, \quad g = \tau\hat{g}.$$

From (2.5) we then obtain

$$\hat{g}_{z^*\tau}+2\tau L\hat{g} = 0. \tag{3.4}$$

From a given Bergman kernel for a certain equation we may obtain infinitely many other kernels by

THEOREM 3.1. *Let g be a Bergman kernel for a given equation* (2.2). *Let $g_1 = hg$, where h is a function of*

$$\tau = z(1-t^2)/2 \tag{3.5}$$

such that (2.6) *with g replaced by g_1 holds. Then $Mg_1 = 0$.*

The proof follows by substitution. An obvious application is the simplification of given kernels, including the derivation of *minimal kernels* (kernels of lowest possible degree in the variable of integration).

4. Criterion for operators of exponential type. By definition, *operators of exponential type* are Bergman operators T defined by (2.4) with a kernel of the form

$$g = e^{q}, \quad q(z, z^*, t) = \sum_{\mu=0}^{m} q_\mu(z, z^*) t^\mu. \tag{4.1}$$

These operators were introduced in [7]. It was shown in [8] that for $f(z) = z^n$, $n \in N$, these operators T yield solutions $u = Tf$ which also satisfy linear ordinary differential equations (of order not exceeding $m+1$, independent of n), so that, as an important consequence of this, the Fuchs–Frobenius theory becomes applicable to such classes of solutions. Earlier attempts (q in (4.1) consisting of one or two terms only) are mentioned in [7]. Later work by K. Ecker and H. Florian [2] based upon [7] and operators of Bergman–Whittaker type concerns extensions to certain equations in n independent variables. Cf. also [4].

We say that L is of *class E*, writen $L \in E$, if L in (2.2) is such that solutions of (2.2) can be obtained by using an operator of exponential type. The problem of deriving explicit necessary and sufficient conditions for $L \in E$ was solved in [7] as follows.

THEOREM 4.1. *$L \in E$ if and only if the coefficients in* (2.2) *can be represented in form* (A) *or in form* (B):

Case (A)

$$b(z) = -q_0'(z) - z^{-1} q_2(z),$$
$$c(z, z^*) = -(2z)^{-1} q_1(z, z^*) q_{1,z^*}(z, z^*),$$

with arbitrary analytic $q_0(z)$ and

$$q_1(z, z^*) = z^{1/2}\Big[a_0(z^*) + \sum_{\nu=1}^{[\frac{1}{2}(m-1)]} a_\nu z^\nu\Big], \qquad q_2(z) = \sum_{\nu=1}^{[m/2]} k_\nu z^\nu.$$

Case (B)

$$b(z, z^*) = -q_0'(z) - z^{-1} q_2(z, z^*), \qquad c(z^*) = -(2z)^{-1} q_{2,z^*}(z, z^*),$$

with arbitrary analytic $q_0(z)$ and

$$q_2(z, z^*) = k_1(z^*) z + \sum_{\nu=2}^{[m/2]} k_\nu z^\nu.$$

The other coefficient functions of q in (4.1) *are then given by*

$$q_{2\mu+1}(z) = \frac{(-2)^\mu}{3 \cdot 5 \cdot \ldots \cdot (2\mu+1)} \sum_{\nu=\mu}^{[\frac{1}{2}(m-1)]} \nu(\nu-1) \cdot \ldots \cdot (\nu-\mu+1) a_\nu z^{\nu+\frac{1}{2}},$$

$$\mu = 1, 2, \ldots, [\tfrac{1}{2}(m-1)],$$

in case (A) *and* $q_{2\mu+1} = 0$ *in case* (B) *and, in both cases,*

$$q_{2\mu}(z) = -\frac{(-2)^\mu}{2\cdot 4\cdot\ldots\cdot 2\mu}\sum_{\nu=\mu}^{[m/2]}(\nu-1)(\nu-2)\cdot\ldots\cdot(\nu-\mu+1)k_\nu z^\nu,$$

$$\mu = 2, 3, \ldots, [m/2].$$

5. Class $\hat{E}$. Minimal kernel. The proof of the criterion in Section 4 as given in [7] is rather complicated and long. We show that the use of the transformation (3.1) yields a much easier access to exponential operators with an exponent which is an *even* function of t. We proceed as follows. Substituting $g = e^q$ into (3.1b), we obtain

$$\tfrac{1}{2}q_{z^*} + (\tau - z)(q_{zz^*} + q_z q_{z^*} + bq_{z^*} + c) = 0. \tag{5.1}$$

We now assume q in the form

$$q(z, z^*, \tau) = \sum_{\mu=0}^{m} p_\mu(z, z^*)\tau^\mu. \tag{5.2}$$

Note that this can be converted to the form of q in (4.1) with $2m$ instead of m, and conversely. Substitution into (5.1) gives

$$\tfrac{1}{2}\sum_{\mu=0}^{m} p_{\mu,z^*}\tau^\mu + (\tau - z)\Big[\sum_{\mu=0}^{2m} K_\mu \tau^\mu + c\Big] = 0, \tag{5.3}$$

where

$$K_\mu = p_{\mu,zz^*} + \Lambda_\mu + bp_{\mu,z^*}, \qquad \mu = 0, \ldots, 2m, \tag{5.4}$$

and

$$\Lambda_\mu = \sum_{\lambda=0}^{\mu} p_{\lambda,z}\, p_{\mu-\lambda,z^*}, \qquad \mu = 0, \ldots, 2m,$$

with the understanding that $p_j = 0$ if $j < 0$ or $j > m$. In (5.3) the coefficient of each occurring power of τ must vanish. This yields a system of $2m+1$ non-linear second-order partial differential equations for determining $p_0, \ldots, p_m$ in terms of b and c as well as possible forms of b and c such that $L \in E$ with q being of even degree in t. Let $\{\mu\}$ denote the equation obtained by equating the coefficient of τ^μ to zero. Equations $\{m+2\}, \ldots, \{2m\}$ are of a similar form; indeed, $\{\mu\}$ is

$$\Lambda_{\mu-1} = z\Lambda_\mu, \qquad \mu = m+2, \ldots, 2m.$$

Since $\Lambda_{2m} = 0$ by $\{2m+1\}$, we obtain $\Lambda_\mu = 0$, $\mu = m+1, \ldots, 2m-1$, successively from $\{2m\}, \ldots, \{m+2\}$, in this order. Next, $K_m = 0$ from $\{m+1\}$. Equations $\{2\}, \ldots, \{m\}$ are of a similar form; indeed, $\{\mu\}$ is

$$\tfrac{1}{2}p_{\mu,z^*} - zK_\mu + K_{\mu-1} = 0, \qquad \mu = 2, \ldots, m.$$

We show that

$$p_{m,z^*} = 0. \tag{5.5a}$$

Suppose not. Then $p_{\mu,z} = 0$ from $\{\mu+m+1\}$, $\mu = 1, \ldots, m$, successively in descending order, as well as $p_{0,z}+b = 0$ from $\{m+1\}$ and finally $p_{m,z^*} = 0$ from $\{m\}$, a contradiction. This proves (5.5a). Similarly,

$$p_{\mu,z^*} = 0, \quad \mu = 2, \ldots, m-1, \tag{5.5b}$$

successively in descending order and indirectly by deriving a contradiction from $\{\mu\}$. From (5.5) and $\{\mu+1\}$ we obtain $K_\mu = 0$, $\mu = 1, \ldots, m-1$. Equation $\{0\}$ is

$$\tfrac{1}{2}p_{0,z^*} - z(K_0+c) = 0$$

and implies that $p_{0,z^*} \neq 0$ since otherwise $c = 0$, the trivial case which we excluded (cf. Section 2); this can be seen from (5.3) and (5.4). Hence from $\{0\}$ we also have

$$K_0 + c = p_{0,z^*}/2z \neq 0.$$

Equation $\{1\}$ is

$$\tfrac{1}{2}p_{1,z^*} - zK_1 + K_0 + c = 0.$$

By $\{0\}$, it entails

$$p_{1,z^*} = -2(K_0+c) \neq 0.$$

From this and (5.5), using $\{\mu+m-1\}$, we see that $p_{\mu,z} = 0$, $\mu = 1, \ldots, m$, successively in descending order. We now use $K_1 = 0$, $p_{1,z} = 0$, $p_{1,z^*} \neq 0$ and $\{2\}$ to conclude that

$$p_{0,z} + b = 0. \tag{5.6}$$

From $\{0\}$ and $\{1\}$ with $K_1 = 0$ and (5.6), writing $p_1 = p(z^*)$, we finally have

$$b(z, z^*) = -p_{0,z} = \varphi(z) + p(z^*) \quad (\varphi \text{ arbitrary}),$$

$$c(z^*) = \tfrac{1}{2}p'(z^*),$$

so that

$$p_0(z, z^*) = -\int_{z_0}^{z} \varphi(\tilde{z})\,d\tilde{z} - zp(z^*), \tag{5.7}$$

$$p_1(z^*) = p(z^*) = 2\int_{z_0^*}^{z^*} c(\tilde{z}^*)\,d\tilde{z}^*,$$

$p_2, \ldots, p_m$ constant (arbitrary).

We say that L is of *class* $\hat{E}$ if $L \in E$ and m in (4.1) is even. In this section, $L \in \hat{E}$, as can be seen from (3.1b). Hence we can summarize our result as follows.

THEOREM 5.1. *$L \in \hat{E}$ if and only if b and c can be represented as shown in (5.7). The coefficients of (5.2) are then those given in (5.7).*

Call g a *minimal kernel* for (2.2) with $L \in \hat{E}$ if g is the kernel of an operator of exponential type for (2.2) and there does not exist such an operator for (2.2) with a kernel of lower degree in t. Then the last line in (5.7) implies

COROLLARY 5.2. *A minimal kernel for $L \in \hat{E}$ is of second degree in t.*

6. Equations with $b = 0$. If $b = 0$ in (2.2), then in Theorem 4.1,

$$q_2(z) = -zq_0'(z).$$

Hence in this case, q_2 cannot depend on z^*, so that in case (B) of Theorem 4.1 we would now obtain $c = 0$, the trivial case which is excluded for reasons given in Section 2. Taking into account the form of q_1 in Theorem 4.1 and using Theorem 5.1, we thus have the simple

THEOREM 6.1. *Let $b = 0$ in (2.2). Then $L \in E$ if and only if c can be represented in the form*

$$c(z, z^*) = -\tfrac{1}{2}\Big[a_0(z^*) + \sum_{\nu=1}^{[\frac{1}{2}(m-1)]} a_\nu z^\nu\Big] a_0'(z^*). \tag{6.1}$$

In this case, q_1 has the form given in Theorem 4.1 and

$$q_{2\mu+1}(z) = \frac{(-4)^\mu (\mu!)^2}{(2\mu+1)!} \sum_{\nu=\mu}^{[\frac{1}{2}(m-1)]} \binom{\nu}{\mu} a_\nu z^{\nu+\frac{1}{2}}, \qquad \mu = 1, \dots, [\tfrac{1}{2}(m-1)].$$

Furthermore, in this case all operators of exponential type have kernels of the form

$$g = g_0 e^p,$$

where

$$g_0(z, z^*, t) = \exp \sum_{\mu=0}^{[\frac{1}{2}(m-1)]} q_{2\mu+1}(z, z^*) t^{2\mu+1} \tag{6.2}$$

and p is any polynomial in $\tau = z(1-t^2)$ with constant coefficients.

For example, in the case of the Helmholtz equation

$$\Delta\psi + \psi = 0$$

or

$$u_{zz^*} + \tfrac{1}{4}u = 0$$

we obtain by a simple calculation

$$g_0(z, z^*, t) = \exp(i\sqrt{zz^*}\, t), \tag{6.3}$$

and other kernels for operators of exponential type are as indicated in the theorem.

7. A representation theorem involving minimal kernels. There are several principles for introducing integral operators used in the function theoretic approach to partial differential equations. These include substitution of an integral (as in Bergman's proof of Theorem 2.1), integration of solutions with respect to a parameter (for examples, see [5]), conversion of a differential operator (inverse of the method in [6]) and the use of integral representations of special functions. For instance, as an illustration of the last method, the operator for the Helmholtz equation with kernel (6.3) could be obtained from the familiar representation (cf. [3], p. 14)

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \exp i(z\sin\theta - n\theta)\,d\theta$$

of the Bessel function of the first kind if one chooses

$$f(z) = z^n \quad (n\in N). \tag{7.1}$$

Indeed, from (2.4) we then obtain

$$u(z, z^*) = \sqrt{\pi}\,\Gamma(n+\tfrac{1}{2})\left(\frac{z}{z^*}\right)^{n/2} J_n(\sqrt{zz^*}),$$

as was noted in [7]. It is remarkable that the minimal kernel characterized in Corollary 5.2 also leads to solutions of a more general class of equations (2.2) which can be represented in terms of finitely many Bessel functions as follows.

THEOREM 7.1. *Equation* (2.2) *with coefficients of the form*

$$b(z, z^*) = \varphi(z) + p(z^*), \qquad c(z^*) = \tfrac{1}{2}p'(z^*) \tag{7.2}$$

and holomorphic in a neighbourhood of the origin has solutions of the form

$$u(z, z^*) = z^n e^{s(z,z^*)} \sum_{k=0}^{n}\sum_{j=0}^{[k/2]} \gamma_{kj} J_{k-2j}(-izp(z^*)/2), \tag{7.3a}$$

where

$$s(z, z^*) = -\int_{z_0}^{z} \varphi(\tilde{z})\,d\tilde{z} + \tfrac{1}{2}zp(z^*), \tag{7.3b}$$

$$\gamma_{kj} = (-1)^{[k/2]-j}\, i^{k^2}\pi 2^{-k-2n}(2-\delta_{2j,k})\binom{n}{k}\binom{k}{j} \tag{7.3c}$$

and δ_{jk} is the Kronecker symbol. u can be obtained from (2.4) *with the minimal kernel*

$$g = e^q, \quad q(z, z^*, t) = p_0(z, z^*) + p(z^*)\tau = q_0(z) + q_2(z, z^*)t^2 \tag{7.4}$$

and $f(z)$ given by (7.1). *Here, $\tau = z(1-t^2)$ and $p_{0,z}(z, z^*) = -b(z, z^*)$.*

Proof. It follows from Theorem 5.1 that (2.2) with coefficients (7.2) admits an exponential operator with a kernel of the form (7.4). Substituting (7.4) and (7.1) into (2.4), we first obtain

$$u(z, z^*) = 2^{1-n} z^n e^{q_0(z)} \int_0^1 e^{q_2(z,z^*)t^2} (1-t^2)^{n-1/2}\, dt$$

or, setting $t = \sin\theta$ and $\lambda = iq_2/2$,

$$u(z, z^*) = 2^{1-n} z^n e^{q_0(z)+i\lambda(z,z^*)} \int_0^{\pi/2} e^{i\lambda\cos 2\theta} \cos^{2n}\theta\, d\theta.$$

Using

$$\cos^{2n}\theta = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \cos^k 2\theta,$$

we thus obtain

$$u(z, z^*) = \varkappa(z, z^*) \sum_{k=0}^{n} \binom{n}{k} (h_k^{(1)} + ih_k^{(2)}), \tag{7.5}$$

where

$$\varkappa(z, z^*) = 4^{-n} z^n \exp\left[q_0(z) + i\lambda(z, z^*)\right],$$

$$h_k^{(1)} = \int_0^{\pi} \cos(\lambda\cos\Phi) \cos^k \Phi\, d\Phi,$$

$$h_k^{(2)} = \int_0^{\pi} \sin(\lambda\cos\Phi) \cos^k \Phi\, d\Phi,$$

$k = 0, \ldots, n$. Since the integrands are symmetric with respect to $\pi/2$, it follows that

$$h_{2k+1}^{(1)} = 0, \quad k = 0, 1, \ldots, [\tfrac{1}{2}(n-1)],$$

$$h_{2k}^{(2)} = 0, \quad k = 0, 1, \ldots, [n/2].$$

For the remaining integrals, using

$$\cos^{2k}\Phi = 2^{-2k} \sum_{j=0}^{k} \binom{2k}{j} (2-\delta_{kj}) \cos(2k-2j)\Phi,$$

$$\cos^{2k+1}\Phi = 2^{-2k} \sum_{j=0}^{k} \binom{2k+1}{j} \cos(2k-2j+1)\Phi$$

and (cf. [15], p. 21)

$$J_n(z) = \frac{2}{\pi}(-1)^{n/2} \int_0^{\pi/2} \cos n\Phi \cos(z\cos\Phi)\, d\Phi \qquad (n \text{ even}),$$

$$J_n(z) = \frac{2}{\pi}(-1)^{(n-1)/2} \int_0^{\pi/2} \cos n\Phi \sin(z\cos\Phi)\, d\Phi \qquad (n \text{ odd}),$$

we obtain

$$h_{2k}^{(1)} = 2^{-2k}\pi \sum_{j=0}^{k} (-1)^{k-j} \binom{2k}{j} (2-\delta_{kj}) J_{2k-2j}(\lambda),$$

where $k = 0, 1, \ldots, [n/2]$, as well as

$$h^{(2)}_{2k+1} = 2^{-2k}\pi \sum_{j=0}^{k} (-1)^{j-k}\binom{2k+1}{j} J_{2k-2j+1}(\lambda),$$

where $k = 0, 1, \ldots, [(n-1)/2]$. Substitution of these expressions into (7.5) yields (7.3), and the theorem is proved.

We finally consider a relation to ordinary differential equations. It was mentioned in Section 4 that if $L \in E$, then solutions $u = Tf$ of (2.2) with f given by (7.1) satisfy a homogeneous linear ordinary differential equation of order not exceeding $m+1$ (independent of n) with $x = (z+z^*)/2$ as the independent variable. Hence for the minimal kernel in Theorem 7.1 this order cannot exceed three. It is interesting that for this kernel this order even reduces to two. Indeed, we shall prove the following

THEOREM 7.2. *Solutions $u = Tf$ of* (2.2) *with $f(z) = z^n$, $n \in N$, and T an exponential operator whose kernel is minimal satisfy a second-order ordinary linear differential equation*

$$N_2 v = v'' + a_1(x, y)v' + a_0(x, y)v = 0, \tag{7.6}$$

where $v(x, y) = u(z, z^*)$, $x = (z+z^*)/2$, $y = (z-z^*)/2i =$ const, and primes denote derivatives with respect to x. (The coefficients of this equation are given explicitly in the proof.)

Proof. Since the minimal kernel in Theorem 7.1 is of second degree in t, the solution u in the present theorem satisfies a third-order equation of the form

$$N_3 v = a_3 v''' + a_2 v'' + a_1 v' + a_0 v = 0 \tag{7.7}$$

(with variable coefficients); this was shown in [8]. We prove that we can choose $a_3 = 0$. Under the present assumptions the integrand in (2.4) is

$$w(x, y, t) = \tilde{w}(z, z^*, t) = \left(\frac{z}{2}\right)^n (1-t^2)^{n-1/2} \exp\left(q_0(z) + q_2(z, z^*)t^2\right). \tag{7.8}$$

It suffices to show that w satisfies an equation

$$N_3 w = \frac{d}{dt}\left[(1-t^2)w \sum_{k=0}^{3} \beta_k(x, y)t^k\right] \tag{7.9}$$

with $a_3 = 0$ and $a_2 = 1$, from which we can then obtain (7.6) by integration over t from -1 to 1. Since w is even in t, so is $N_3 w$. Hence $\beta_0 = \beta_2 = 0$. We now substitute (7.8) into (7.9), divide by w and compare the coefficients of t^{2k}, $k = 0, 1, 2, 3$. This yields four equations in six unknown functions $\beta_1, \beta_3, a_0, a_1, a_2, a_3$. Taking $a_3 = 0$ and $a_2 = 1$, we obtain $\beta_3 = 0$ from the last of these equations, whereas the others take the form

$\beta_1 = a_0 + a_1 r_1 + a_2(r_1^2 + r_1'), \quad \beta_1 r_3 = a_1 r_2 + 2r_1 r_2 + r_{2,z} + r_{2,z^*}, \quad -2q_2\beta_1 = r_2^2,$

where $r_1(z) = q_0'(z) + n/z$, $r_2 = q_{2,z} + q_{2,z^*}$, $r_3 = 2(q_2 - n - 1)$. From this system we obtain the formulas

$$a_0 = -r_2^2/2q_2 + r_1 r_2 r_3/2q_2 + r_1^2 - r_1' + r_1(r_{2,z} + r_{2,z^*})/r_2,$$
$$a_1 = -r_2 r_3/2q_2 - 2r_1 - (r_{2,z} + r_{2,z^*})/r_2$$

for the coefficients of the differential equation (7.6) as well as a simple expression for β_1, which is of no specific interest. This completes the proof.

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