COLLOQUIUM MATHEMATICUM

VOL. LII 1987 FASC. 2

RECENT DEVELOPMENTS IN THE THEORY OF HAAR SERIES

BY

WILLIAM R. WADE (KNOXVILLE, TENNESSEE)

I. INTRODUCTION

This article surveys recent results on Haar series. To avoid duplication of material appearing in Golubov [1970], a decision was made to concentrate on the decade 1971–1981. References to earlier work will be made when necessary to relate what is herein reported to that which preceded it. Discussion of how these recent results affect the general theory of orthogonal series has been left to those more qualified for this task (e.g., Ul'janov [1972], Olevskiĭ [1975], and Bočkarev [1972] and [1978]).

In addition to this introductory section, the present article contains four sections: II. Convergence of Haar–Fourier series; III. Approximation by Haar series; IV. Haar–Fourier coefficients; and V. Uniqueness. These sections have been divided further into consecutively numbered subsections, each dealing with a particular facet of the subject indicated in the main section, and each carrying a descriptive title to help the reader quickly find those parts in which he is interested. In each subsection one-dimensional results are discussed first and results on multiple Haar series will come next. For notational convenience, most multiple Haar series results will be cited for double Haar series only.

This article ends with a nearly complete listing of all papers on Haar series published during the decade 1971–1981. This listing is ordered alphabetically and then chronologically. In the course of our narrative these papers will be cited, as above, by author and by year.

The following notation and symbolism will be used throughout this article. Let $\chi_0, \chi_1, \ldots$ represent the *Haar system*, i.e., set $\chi_0 \equiv 1$, and if $k = 2^m + p$, where $0 \leqslant p < 2^m$, then

$$\chi_k(x) = \begin{cases} \sqrt{2^m} & \text{if } p/2^m < x < (p+\frac{1}{2})/2^m, \\ -\sqrt{2^m} & \text{if } (p+\frac{1}{2})/2^m < x < (p+1)/2^m, \\ \sqrt{2^m}/2 & \text{if } x = p/2^m, \\ -\sqrt{2^m}/2 & \text{if } x = (p+1)/2^m, \\ 0 & \text{otherwise.} \end{cases}$$

The space of functions f such that $|f|^p$ is integrable over the *unit interval* $[0, 1]$ or the *unit square* $Q \equiv [0, 1]\times[0, 1]$ will be denoted by L^p, $1 \leqslant p < \infty$. The space of functions essentially bounded on $[0, 1]$ or on Q will be denoted by L^∞, and the space of functions continuous on $[0, 1]$ or on Q will be denoted by $\mathscr{C}$. The context will make it clear whether $[0, 1]$ or Q is being used.

By a *Haar series* we mean a formal series of type

$$S = \sum_{k=0}^{\infty} a_k \chi_k,$$

where $a_0, a_1, \ldots$ are real numbers. The partial sums of the Haar series S are defined by

$$S_n \equiv \sum_{k=0}^{n-1} a_k \chi_k, \quad n = 1, 2, \ldots$$

The *rectangular sums* of a double Haar series

$$S = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{k,j} \chi_k \otimes \chi_j$$

are defined by

$$S_{m,n} = \sum_{k=0}^{m-1} \sum_{j=0}^{n-1} a_{k,j} \chi_k \otimes \chi_j, \quad m = 1, 2, \ldots,$$

whereas the *spherical partial sums* of S are defined by

$$S_R = \sum\sum \{a_{k,j} \chi_k \otimes \chi_j \colon \sqrt{k^2 + j^2} \leqslant R\} \quad \text{for } 0 \leqslant R < \infty.$$

Given a functionn f integrable on $[0, 1]$, denote its *Haar–Fourier coefficients* by

$$a_k(f) \equiv \int_0^1 f(t) \chi_k(t)\, dt, \quad k = 0, 1, \ldots,$$

and its *Haar–Fourier series* by

$$S[f] = \sum_{k=0}^{\infty} a_k(f) \chi_k.$$

The *modulus of continuity* of f is

$$\omega(f, \delta) \equiv \sup \{|f(t+h) - f(t)| \colon 0 \leqslant t \leqslant 1-h,\ 0 \leqslant h \leqslant \delta\}$$

and the *L^p-modulus of continuity* of f is

$$\omega_p(f, \delta) \equiv \sup \{(\int_0^{1-h} |f(t+h) - f(t)|^p\, dt)^{1/p} \colon 0 \leqslant h \leqslant \delta\}$$

for $0 \leqslant \delta \leqslant 1$ and $1 \leqslant p < \infty$. Given an increasing function ω defined on $[0, 1]$ such that $\omega(0) = 0$, and $\omega(\delta+\eta) \leqslant \omega(\delta)+\omega(\eta)$ for $0 \leqslant \delta,\ \eta \leqslant \delta+\eta \leqslant 1$, the *Hardy spaces* associated with ω are defined by

$$H_\omega = \{f \in \mathscr{C}:\ \omega(f, \delta) = O(\omega(\delta)) \text{ as } \delta \to 0\}$$

and

$$H_\omega^p = \{f \in L^p:\ \omega_p(f, \delta) = O(\omega(\delta)) \text{ as } \delta \to 0\}.$$

Thus, when $\omega(\delta) = \delta^\alpha$, H_ω is identical with $\operatorname{Lip}\alpha$, and H_ω^p is identical with $\operatorname{Lip}(\alpha, L^p)$.

Given a function f integrable on the unit cube Q, denote its *double Haar–Fourier coefficients* by

$$a_{k,j}(f) = \iint_Q f(t, u)\chi_k(t)\chi_j(u)\,dt\,du, \qquad k, j = 0, 1, \ldots,$$

and its *double Haar–Fourier series* by

$$S[f] = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{k,j}(f)\chi_k \otimes \chi_j.$$

Finally, given a sequence $p_1, p_2, \ldots$ of primes, let $X(p_n)$ denote the Haar-like system introduced by Vilenkin [1947]. Recall that when $p_n = 2$, the system $X(p_n)$ is exactly the Haar system. In general, $X(p_n)$ is said to be of *bounded type* if

$$\limsup_{n\to\infty} p_n < \infty.$$

Few results have been obtained for this system and even fewer will be cited. By and large, those cited are ones which give new information about the classical Haar system not those which generalize what was already known.

II. CONVERGENCE OF HAAR–FOURIER SERIES

1. Pointwise convergence. In the first article on Haar series, Haar [1910] showed that if $f \in L^1$, then $S_n[f]$ converges a.e. to f as $n \to \infty$. This classical result has now been shown to be best possible in the following sense. Prohorenko [1971] proved that given a set $E \subseteq [0, 1]$ of Lebesgue measure zero there is a function $f \in L^p$, $1 \leqslant p < \infty$, such that $S_n[f]$ diverges on E. For the case $p = \infty$, Prohorenko found it necessary to strengthen the condition on E from "of measure zero" to "at most countable". It is still not known whether given an arbitrary set E of measure zero there exists a bounded function whose Haar–Fourier series diverges on E.

Prohorenko's result has been sharpened by Lunina (see Subsection 8 below) and generalized to the systems $X(p_n)$ of bounded type by Zotikov [1973]. Zotikov also examined $X(p_n)$ for unbounded $\{p_n\}$, obtaining conditions sufficient for $X(p_n)$-Fourier series of L^p-functions, $p>1$, to converge a.e. Čaĭdze [1972] has obtained some estimates of the growth of rectangular sums of Haar series whose coefficients do not belong to l^2.

Oskolkov [1977] has estimated rates of pointwise convergence of Haar–Fourier series by looking at Steklov means. A consequence of his work is that if ψ is a positive, monotone decreasing function on (0, 1] and if

$$\int_0^1 \frac{dx}{x\psi(x)} < \infty,$$

then $f \in \mathrm{Lip}(\alpha, L^p)$ for $0<\alpha<1$ and $1 \leqslant p < \infty$ implies that

$$f(x) - S_n(f, x) = o\left(n^{-\alpha}\psi^{1/p}(1/n)\right) \quad \text{as } n \to \infty$$

holds for a.e. $x \in [0, 1]$.

Haar's original result does not hold for multiple Haar series unless the square partial sums $S_{n,n}$ are used. Indeed, Dzagnidze [1964] showed that there exist functions $f \in L^1$ such that $S_{m,n}[f]$ diverges a.e. However, if $f \in L\log^+ L$, i.e., if

$$\iint_Q |f(t, u)| \log^+ |f(t, u)|\, dt\, du < \infty,$$

then $S_{m,n}[f]$ converges to f a.e. as $m, n \to \infty$. Moreover, this result still holds for $f \in L^1$ if as $m, n \to \infty$ they remain in some fixed proper cone, i.e., both m/n and n/m are not greater than a fixed $\lambda \geqslant 1$.

The situation does not change for spherical partial sums. Kemhadze [1977b] showed that if $f \in L^1$ and if $S_{m,n}[f]$ converges a.e. as $m, n \to \infty$, then $S_R[f]$ also converges a.e. as $R \to \infty$. It follows that if $f \in L\log^+ L$, then $S_R[f]$ converges a.e. to f as $R \to \infty$. For the multiple Haar series case, Kemhadze proved that $f \in L\log^{N-1} L$ is sufficient for a.e. convergence of the spherical partial sums; here N is the dimension of the domain of f. Kemhadze [1977a] also announced that given $\varepsilon > 0$ there exist a function $f \in L^1$ and a set E of Lebesgue measure greater than $1-\varepsilon$ such that $S_R[f]$ diverges on E as $R \to \infty$.

Concerning rearrangements of double Haar series, Kemhadze [1975] has identified a 1-1 transformation ω of the integral lattice points onto themselves such that if $f \in L^1$, then the spherical sums

$$\sum_{(k^2+j^2)^{1/2} \leqslant R} a_{\omega(k,j)}(f)\, \chi_{\omega(k,j)}$$

converge a.e. to f as $R \to \infty$. It is not known whether this result holds for rectangular sums of rearrangements. **(P 1312)**

2. Absolute convergence. Ul'janov [1951] established that for $f \in H^p_\omega$, $1<p<\infty$, criteria for a.e. absolute convergence of $S[f]$ and convergence of the series

$$\sum_{k=0}^{\infty} \int_0^1 |a_k(f)\chi_k(t)|\,dt$$

are identical. Bočkarev [1972] showed that this equivalence does not hold when $p = 1$. Indeed, he proved that $S[f]$ is a.e. absolutely convergent for all $f \in H_\omega^1$ when $\omega(\delta) = 1/(\log(1/\delta))^{1/2+\varepsilon}$ for some $\varepsilon > 0$. Addressing the possible extension to $\varepsilon = 0$, he constructs a continuous function φ satisfying

$$\omega(\varphi, \delta) = O\big(1/(\log(1/\delta))^{1/2}\big), \quad 0 < \delta \leqslant \delta_0 < 1,$$

whose Haar–Fourier series can be arranged so as to diverge a.e.

Recall that Haar–Fourier series of absolutely continuous functions converge absolutely at dyadic rationals, but may not at dyadic irrationals. In fact, given any dyadic irrational ξ there exists an absolutely continuous function f whose Haar–Fourier series does not converge absolutely at ξ (McLaughlin [1969]). Hristov [1973] strengthened this result by showing that if

$$\sum_{n=1}^{\infty} \omega(1/n)/n = \infty, \tag{1}$$

then given a dyadic irrational ξ there exists an absolutely continuous function $f \in H_\omega$ whose Haar–Fourier series does not converge absolutely at ξ. The idea for using (1) comes from Ul'janov [1967] who showed that under this condition, given any real number t $(0 \leqslant t \leqslant 1)$, there exists an $f \in H_\omega$ whose Haar–Fourier series does not converge absolutely at t.

Several results concerning absolute convergence of $X(p_n)$-Fourier series can be found in Zotikov [1974]. These contain, as corollaries, earlier work by Ul'janov and McLaughlin on Haar–Fourier series.

For absolute convergence of double Haar–Fourier series see the last paragraph of Subsection 10.

3. Absolute summability. Although Leindler [1961] obtained necessary and sufficient conditions for a Haar series to be $|C, \alpha|$-summable a.e., $\alpha > -1$, the Haar–Fourier series case had not been investigated separately until Gaĭmnazarov [1975] showed that the Haar–Fourier series of an $f \in L^1$ is $|C, \alpha|$-summable a.e., $\alpha > -1$, when

$$\sum_{n=1}^{\infty} (1/n)^{1+\alpha} \inf_{c_k \text{ real}} \Big\|f - \sum_{k=0}^{n} c_k \chi_k\Big\|_1 < \infty.$$

It follows that if

$$\sum_{n=1}^{\infty} \omega_1(f, 1/n)/n^{1+\alpha} < \infty \tag{2}$$

for some $\alpha \in (-1, 0)$, then the Haar–Fourier series of f is $|C, \alpha|$-summable a.e. Whether this result holds for $\alpha = 0$ is not known. In connection with this, it is interesting to note that Ul'janov [1970] has shown that if (2) holds for $\alpha = 0$, then $f \in L\log^+ L$; hence the Haar–Fourier series of f converges unconditionally in L^1-norm (Balašov [1971]).

4. Adjustment of functions on small sets to enhance convergence of Haar–Fourier series. One of Menšov's celebrated results is that given an a.e. finite-valued measurable function f and an $\varepsilon > 0$ there exists a continuous function $\tilde{f}$ which coincides with f off a set of measure not greater than ε and such that the (trigonometric) Fourier series of $\tilde{f}$ converges uniformly on $[0, 2\pi]$.

The Haar series analogue of Menšov's result is trivial since $S[\tilde{f}]$ always converges uniformly when $\tilde{f}$ is continuous. Arutunjan [1966] has shown that the adjustment from f to $\tilde{f}$ can be made so that $S[\tilde{f}]$ converges absolutely a.e. as well.

Since absolute convergence of a Haar–Fourier series is not equivalent to

$$\sum_{k=0}^{\infty} |a_k(f)| < \infty, \tag{3}$$

one can ask whether an adjustment resulting in (3) can be made. The answer is no. In fact, Fridljand [1973] showed that there is a continuous function $\tilde{f}$ such that given any $f \in L^2$ which coincides with $\tilde{f}$ on a set of positive measure it is the case that

$$\sum_{k=0}^{\infty} |a_k(f)|^{\gamma} = \infty \qquad \text{for } 1 \leqslant \gamma < 2.$$

He also announced other versions of this result for $\tilde{f}$ belonging to H_ω or to $\mathrm{Lip}\,\alpha$, $0 \leqslant \alpha \leqslant 1/2$.

III. APPROXIMATION BY HAAR SERIES

5. Haar series with gaps. As observed in the previous section, the Haar series analogue of Menšov's adjustment theorem is trivial. This problem comes alive again if we pass to subseries. Let $0 \leqslant m_1 < m_2 < \dots$ and consider the system $X \equiv \{\chi_{m_i}\}_{i=1}^{\infty}$. Price [1970] showed that a continuous function f can be adjusted on a set of arbitrarily small measure to produce a function $\tilde{f}$ whose X-Fourier series converges both uniformly and absolutely if and only if X is total in measure (i.e., if and only if given a measurable φ there is a sequence of polynomials in X which converges in measure to φ). Later [1972] he showed that the system X can be total in measure but of density zero.

Whether the system X can be used to approximate L^2-functions has turned out to be closely associated with the size of the set

$$A = \{t \in [0, 1]\colon \chi_{m_i}(t) \neq 0 \text{ for infinitely many } i\}.$$

F. A. Talaljan [1972] showed that $m(A) = 1$ is both necessary and sufficient for the existence of a function $f \in L^2$ whose X-Fourier series diverges a.e. On the other hand, Gamlen and Gaudet [1973] proved that if $m(A) > 0$, then the closed linear span of X in L^p is isomorphic to L^p, $1 < p < \infty$, but if $m(A) = 0$, then the closed linear span of X in L^p is isomorphic to l^p, $1 < p < \infty$. It is not known whether Talaljan's result holds for $p \neq 2$ and it is an open question what the closed linear span of X in L^p is when $p = 1$ or $p = \infty$. **(P 1313)**

Recall that a system $\{f_1, f_2, \ldots\}$ is a *basis* in a Banach space B if given $f \in B$ there is a unique series $\sum_{n=1}^{\infty} a_n f_n$ which converges to f in the norm of B. Kazarjan [1978] asked whether there exist bounded functions φ for which

$$\varphi X \equiv \{\varphi\chi_{m_1}, \varphi\chi_{m_2}, \ldots\}$$

is a basis in L^p, $1 \leqslant p < \infty$. The answer seems to depend on the size of the gaps in X. He showed that if $m_i = N + i$, $i \geqslant 1$, for some fixed integer N, then his question has an affirmative answer. He also obtained an affirmative answer for certain cases where X has infinite gaps. A characterization of the allowable gaps has not yet been established.

6. The Haar system as a basis. In addition to the results in the two previous subsections, we have the following to report.

Krotov [1978] investigated the basis problem for the spaces H_ω^p. He found that a necessary and sufficient condition that the Haar system be a basis for H_ω^p, $1 \leqslant p < \infty$, is that there exist a constant C depending only on p and ω such that

$$\delta^{-1/p}\omega(\delta) \leqslant C\eta^{-1/p}\omega(\eta), \quad 0 < \eta \leqslant \delta \leqslant 1. \tag{4}$$

Recall that two bases $\{f_1, f_2, \ldots\}$ and $\{g_1, g_2, \ldots\}$ in a Banach space B are *equivalent* if given coefficients $a_1, a_2, \ldots$ the series $\sum_{n=1}^{\infty} a_n f_n$ and $\sum_{n=1}^{\infty} a_n g_n$ are equiconvergent. Ciesielski, Simon, and Sjölin [1977] proved that the Haar and Franklin systems are equivalent bases in L^p, $1 < p < \infty$. Sjölin [1977] showed that this equivalence does not hold for $p = 1$. In a related result, Ciesielski and Kwapień [1979] showed that the Haar shift operator is not bounded in L^1.

7. The Haar system as an unconditional basis. Marcinkiewicz [1937] proved that Haar–Fourier series of L^p-functions converge unconditionally in L^p, $1 < p < \infty$. During the sixties a push was made to characterize those functions $f \in L^1$ whose Haar–Fourier series converge unconditionally in L^1-norm. Balašov [1971] came close when he established that a sufficient condition for $S[f]$ to converge unconditionally in L^1 is that $f \in L\log^+ L$. Recall (Garsia [1973]) that an $f \in L^1$ is said to belong to *dyadic* H^1 if $\sup\{|S_n[f]|: n \geqslant 0\}$ is integrable on $[0, 1]$. Olevskiĭ [1975], p. 76, observed that the Burkholder–Davis inequality for martingales and known work by Orlicz on unconditional convergence can be used to see that $S[f]$ converges unconditionally in L^1 if and only if f belongs to *dyadic* H^1. The connection with Balašov's theorem is simply this: a non-negative f belongs to *dyadic* H^1 if and only if $f \in L\log^+ L$.

Gapoškin [1974] used martingale techniques to offer a simple, direct proof of the Marcinkiewicz theorem cited in the first paragraph of this subsection. He established, as an intermediate result, that the operator

$$Lf = \sum_{k=0}^{\infty} \eta_k a_k(f) \chi_k$$

is of weak type (1, 1) for any choice of $\eta_k = \pm 1$.

Krotov [1978] proved that a necessary and sufficient condition for the Haar system to be an unconditional basis in H^p_ω, $1 < p < \infty$, is that there exist a constant C depending only on p and ω such that (4) and the inequality

$$\sum_{k=0}^{n-1} 2^{k/p} \omega(2^{-k}) \leqslant C \cdot 2^{n/p} \omega(2^{-n})$$

hold for $n = 1, 2, \ldots$ The case where $p = 1$ remains unexplored.

No work has been done using gap Haar series for unconditional bases. However, even when considering quasi-bases the Haar system is not unconditional in L^1. Shirey [1973] proved that, given a set E of positive Lebesgue measure, the quasi-basis for $L^1(E)$ obtained by restricting the Haar system to the set E is a conditional quasi-basis.

Tkebučava [1973] showed, under rather technical conditions on a convex continuous function g and an $f \in L^1$, that any rearrangement $T[f]$ of the Haar–Fourier series of f satisfies

$$\lim_{n \to \infty} \int_0^1 g\big(f(t) - T_n(f, t)\big)\, dt = 0.$$

He also gives some information about unconditional convergence of Haar–Fourier series in the spaces $L\log^p L$, $p > 0$.

Krancberg [1974] has shown that if there exists an $f \in L^1$ such that a rearrangement $T[f]$ diverges on a set E, then there is a continuous function φ which satisfies the equality

$$\sup_{n \geqslant 1} |T_n(\varphi, t)| = \infty \qquad \text{for a.e. } t \in E.$$

Tkebučava [1979] has examined unconditional convergence of multiple Haar series in certain separable non-reflexive Orlicz spaces L_{Φ}^{*}. His conclusion is that, under certain complicated conditions on Φ (condition $(*)$), the multiple Haar system fails to be an unconditional basis in the norm of L_{Φ}^{*}.

8. Convergence of Haar series. A set E is said to be a $\tilde{\mathscr{G}}_{\delta}$*-set* if it is the countable intersection of nested sets $\tilde{G}_1 \supset \tilde{G}_2 \supset \ldots$, where each $\tilde{G}_k$ is itself a countable union of intervals whose end-points are included only if those end-points are dyadic rationals. Lunina [1976] surprised us all by showing that such sets characterized sets of unbounded divergence for Haar series: given a set E, there exists a Haar series S which satisfies

$$E = \{t\colon \limsup_{n\to\infty} |S_n(t)| = \infty\}$$

if and only if E is a $\tilde{\mathscr{G}}_{\delta}$-set. She went on to obtain the following sharpening of a result of Prohorenko (see Subsection 1). If E is a $\tilde{\mathscr{G}}_{\delta}$-set of measure zero, then there exists a function $f \in L^p$, $1 \leqslant p < \infty$, such that $S[f]$ converges off E but diverges unboundedly on E.

Pal and Schipp [1972] showed that if S is a Haar series whose partial sums are non-negative on $[0, 1]$, then S_n converges a.e. as $n \to \infty$, and the first integral of S is a Ciesielski series which converges off a countable set. Moreover, such an S need not be a Haar–Fourier series. Thus the Steinhaus conjecture is false for Haar series. Parallel results were obtained for Haar series which satisfy $\|S_n\|_1 = O(1)$ as $n \to \infty$.

Confirming the ever widening gap between pointwise convergence and norm convergence, Davtjan and Talaljan [1975] proved that there exist a.e. divergent Haar series S whose coefficients satisfy $a_n = O(1/\sqrt{n})$, as $n \to \infty$, such that given $\varepsilon > 0$ there is a set E of Lebesgue measure at least $1-\varepsilon$ for which S_n converges in the $L^2(E)$-norm as $n \to \infty$.

It is well known that corresponding to each a.e. finite measurable function f there is a Haar series which converges a.e. to f (Talaljan [1960]). A martingale proof of this result has been given by Davtjan [1976].

9. Summability of Haar series. The result of Talaljan cited in the previous paragraph cannot be extended to arbitrary measurable functions. Indeed, no Haar series can diverge to $+\infty$ on a set of positive measure (Talaljan and Arutunjan [1965]).

Šaginjan looked into the possibility that replacing convergence by summability might remedy this malady. For Abel summability, he showed [1974b] that given any measurable set E there is a Haar series summable a.e. to $+\infty$ on E. However, for Cesàro summability, the original situation prevailed. No Haar series is $(C, 1)$-summable to $+\infty$ on a set of positive measure (Šaginjan [1974a]). The main building block of his proof is the rather unexpected fact that, given a measurable E, a Haar series S whose partial Cesàro sums satisfy the inequality

$$\liminf_{n\to\infty} \sigma_n(S, x) > -\infty \quad \text{for } x \in E$$

actually converges to a finite measurable function a.e. on E. The same result holds for all (C, α)-sums, $\alpha > -1$, and all (H, k)-sums as well (Šaginjan [1973]). It follows that all (C, α)-methods of summability, $\alpha > -1$, for Haar series are equivalent at any given point.

Pogosjan [1980] announced results concerning divergence of the series

$$T = \sum_{k=0}^{\infty} \gamma_k \chi_k^* / \sqrt{k+1},$$

where $\gamma_k = 0, \pm 1$, and $\chi_0^*, \chi_1^*, \ldots$ is any rearrangement of the Haar system. It seems that for a large class of regular methods of summability one can choose the numbers γ_k so that T is a.e. summable to $+\infty$. A consequence is that if $\{n_i\}_{i=1}^{\infty}$ is a subsequence of natural numbers which satisfies the equality

$$\limsup_{i\to\infty} n_{i+1}/n_i = \infty,$$

then one can choose the numbers γ_k so that $T_{n_i} \to \infty$ a.e. as $i \to \infty$. On the other hand, if n_{i+1}/n_i is bounded for all i, then T_{n_i} cannot diverge to $+\infty$ on a set of positive measure.

A sequence of coefficients $a_0, a_1, \ldots$ is said to be *block-order decreasing* if there exists a constant $C \geqslant 1$ such that

$$\max_{2^n \leqslant k < 2^{n+1}} |a_k| \leqslant C \min_{2^{n-1} \leqslant k < 2^n} |a_k|, \qquad n = 1, 2, \ldots$$

Such coefficients were first considered by Ul'janov [1963] as a generalization of monotone decreasing coefficients. Ebralidze [1976] considers a summability method

$$\tau_N = \sum_{k=0}^{\infty} R_{N,k}\, a_k \chi_k,$$

where the sequence $\{R_{N,k}\}$ is block-order decreasing in the variable k uniformly in N, and converges to 1 for all fixed k as $N \to \infty$. He shows that given a set E of the second category, if either

$$\limsup_{N\to\infty} \tau_N(x) < \infty \quad \text{for } x \in E$$

or

$$\liminf_{N\to\infty} \tau_N(x) > -\infty \quad \text{for } x \in E$$

and if $\{a_k\}_{k=0}^{\infty}$ is itself block-order decreasing, then

$$\sum_{k=1}^{\infty} |a_k|/\sqrt{k} < \infty.$$

It follows that if

$$\sum_{k=1}^{\infty} |a_k|/\sqrt{k} = \infty,$$

then τ_N is unbounded on any interval in [0, 1]. Since this method of summation includes convergence of Haar series (use $R_{N,k} \equiv 1$), both results cited above hold for S_N in place of τ_N.

IV. HAAR–FOURIER COEFFICIENTS

10. Absolute convergence of series of Haar–Fourier coefficients. Ciesielski and Musielak [1959] were first to identify conditions sufficient for convergence of the series

$$\sum_{k=1}^{\infty} |a_k(f)|^{\gamma} \tag{5}$$

and

$$\sum_{k=1}^{\infty} k^{\alpha} |a_k(f)|. \tag{6}$$

For example, (5) converges for a fixed $\gamma < 1$ if

$$\sum_{k=1}^{\infty} k^{-\gamma/2} \omega_1(f, 1/k) < \infty$$

and (6) converges if

$$\sum_{k=1}^{\infty} k^{\alpha - 1/2} \omega_1(f, 1/k) < \infty.$$

When $\gamma = 1$, (5) converges if $f \in \operatorname{Lip}\alpha$, $0 < \alpha \leqslant 1$, or if f is of bounded p-variation, $1 \leqslant p < 2$.

The problem of determining exact conditions for convergence of (5) and (6) was almost closed by the extensive work of Ul'janov and of Golubov done in the sixties. Many of their results are best possible and can be extended no further (see Golubov [1970] for an exposition). A unified treatment of this problem was published by McLaughlin [1973] who simultaneously considered the trigonometric, Walsh, Haar, and Franklin systems. His results also extend those of Ul'janov and Golubov to the systems $X(p_n)$ of bounded type.

One of Golubov's theorems is that if f is of bounded p-variation, $p \geqslant 1$, then (5) converges for $\gamma > 2p/(2+p)$ and (6) converges for $\alpha < 1/p - 1/2$. In the limiting case where $\gamma = 2p(2+p)$ and $\alpha = 1/p - 1/2$, he found a function $f \in \mathrm{Lip}(1/p)$ for which both (5) and (6) diverge. Čanturija [1979] (announced in [1974]) has generalized this theorem and improved the counterexample from $\mathrm{Lip}(1/p)$ to H_ω, where $\omega(\delta) = (\log(1/\delta))^{-1/2-1/p}$.

If f is absolutely continuous and non-constant, then (5) converges when $\gamma > 2/3$ and diverges when $\gamma \leqslant 2/3$. In fact (see p. 21 of Bočkarev [1978]), if f is absolutely continuous, then

$$\lim_{n\to\infty} 2^{n(3\gamma/2-1)} \sum_{k=1}^{2^n} |a_{2^n+k}(f)|^\gamma = 2^{-2\gamma} \int_0^1 |f'(t)|\, dt.$$

It follows that $\sqrt{2^n} \sum_{k=1}^{2^n} |a_{2^n+k}(f)|$ converges to $\frac{1}{4}\mathrm{Var} f$ as $n \to \infty$ and that

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} |a_{2^n+k}(f)|^{2/3} = 2^{-4/3} \int_0^1 |f'(t)|^{2/3}\, dt$$

both hold for any absolutely continuous f.

Ul'janov [1978] investigated the Haar series analogue of Levy's theorem. Let A_1 denote those functions f which satisfy (5) for $\gamma = 1$. Ul'janov [1978] shows that if φ is bounded, then a necessary and sufficient condition for $\varphi(A_1) \subset A_1$ to hold is that $\varphi \in \mathrm{Lip}\, 1$. It follows that the class A_1 is closed under products and that $|f|^\alpha \in A_1$, $\alpha \geqslant 1$, when $f \in A_1$. In the case where φ is measurable but not Lip 1, Ul'janov [1978] shows that there is a continuous function f $(0 \leqslant f \leqslant 1)$ such that $f \in A_1$ but $\varphi(f) \notin A_1$.

Matveev [1974] investigated convergence of the series

$$\sum_{k=1}^{\infty} \sqrt{2^k} \left| \sum_{j=2^k}^{2^{k+1}-1} a_j(f) \right| \tag{7}$$

for continuous f. Let $\Omega(f, \delta)$ denote the modulus of continuity associated with the second difference, i.e.,

$$\Omega(f, \delta) = \sup\left\{ \left| f(t_1) - 2f\left(\frac{t_1+t_2}{2}\right) + f(t_2) \right| : t_1, t_2 \in [0, 1], |t_1 - t_2| \leqslant \delta \right\}.$$

Matveev [1974] shows that if

$$\sum_{k=1}^{\infty} \Omega(f, 1/k) < \infty$$

for a periodic continuous f, then (7) converges. However, given $\varepsilon > 0$ there exists a periodic continuous f such that

$$\sum_{k=1}^{\infty} k^{-\varepsilon} \Omega(f, 1/k) < \infty$$

but such that

$$\sum_{k=1}^{\infty} \sqrt{2^k} \sum_{j=2^k}^{2^{k+1}-1} a_j(f)$$

diverges.

For adjustment of functions to result in divergence of (5) see Subsection 4. For results on convergence of (6) for $\alpha = -1/2$, where $a_k(f)$ has been replaced by any monotone decreasing sequence of real numbers, see Subsection 9.

Several results cited above go over to double Haar series. Kraczkowski [1977] showed that if f is of bounded p-variation, then

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |a_{k,j}(f)|^{\gamma} < \infty \qquad \text{for } \gamma > 2p/(p+2)$$

and

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (kj)^{\alpha} |a_{k,j}(f)| < \infty \qquad \text{for } \alpha < 1/p - 1/2.$$

Given a function f continuous on Q and two real numbers $\delta_1, \delta_2 > 0$, we denote by $\omega(f; \delta_1, \delta_2)$ the supremum, over $|h_i| \leqslant \delta_i$ and $(x_1, x_2) \in Q$, of the following three expressions:

$$|f(x_1+h_1, x_2) - f(x_1, x_2)|, \qquad |f(x_1, x_2+h_2) - f(x_1, x_2)|,$$

$$|f(x_1+h_1, x_2+h_2) - f(x_1, x_2+h_2) + f(x_1+h_1, x_2) - f(x_1, x_2)|.$$

A parallel situation defines $\omega_p(f; \delta_1, \delta_2)$ for $f \in L^p(Q)$, $1 \leqslant p < \infty$. The spaces $\mathrm{Lip}(\alpha_1, \alpha_2)$ (respectively, $\mathrm{Lip}(\alpha_1, \alpha_2; p)$) consist of those functions f whose moduli of continuity satisfy $\omega(f; \delta_1, \delta_2) = O(\delta_1^{\alpha_1} \delta_2^{\delta_2})$ (respectively, $\omega_p(f; \delta_1, \delta_2) = O(\delta_1^{\alpha_1} \delta_2^{\alpha_2})$). Szelmeczka [1974] proved that if f belongs to $\mathrm{Lip}(\alpha_1, \alpha_2; p)$, $\alpha_i > 0$, if $\beta_1, \beta_2 \geqslant 0$ and

$$\alpha_i > \frac{\beta_i + 1}{p} - \frac{1}{2}, \qquad i = 1, 2,$$

then

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} k^{\beta_1} j^{\beta_2} |a_{k,j}(f)|^p < \infty.$$

Kraczkowski [1978] has obtained conditions on $\omega(f)$, $\omega_p(f)$ and the variation of f sufficient to conclude that the series

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} k^{\beta_1} j^{\beta_2} |a_{k,j}(f)|^{\gamma} |\chi_k \otimes \chi_j|^{\lambda}$$

is convergent for various choices of $\beta_i \geq 0$, $\lambda \geq 0$, $\gamma > 0$. Included as corollaries of his work are the following. Suppose that $f \in \mathrm{Lip}(\alpha_1, \alpha_2; p)$ for some p $(1 \leq p < \infty)$. If $\alpha_1, \alpha_2 > 0$, then the double Haar–Fourier series $S[f]$ converges absolutely on the unit square Q. But if $\alpha_1, \alpha_2 > 1/2$, then

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |a_{k,j}(f)| < \infty .$$

For other conditions sufficient to conclude that this series is finite, see Gaĭmnazarov [1971].

11. Growth of Haar–Fourier coefficients. Ul'janov [1964b] showed that if f is of bounded variation v on $[0, 1]$, then

$$\sum_{k=2^n}^{2^{n+1}-1} |a_k(f)| \tag{8}$$

is not greater than $3v/\sqrt{2^{n+2}}$, $n = 1, 2, \ldots$ Horoško [1972] showed that

$$\Big| \sum_{k=2^n}^{2^{n+1}-1} a_k(f) \Big| \tag{9}$$

is not greater than $v/\sqrt{2^{n+4}}$, $n = 1, 2, \ldots$, and also estimated (8) and (9) when $f \in H_\omega$. He showed that (8) is dominated by

$$2^{(3n-2)/2} \int_0^{2^{-n}} \omega(t)\, dt$$

and that (9) is dominated by

$$2^{3n/2} \int_0^{2^{-(n+1)}} \omega(t)\, dt .$$

The constants in all these inequalities are best possible.

Golubov [1964] (1) proved that a continuous function f belongs to $\mathrm{Lip}\,\alpha$ for some α $(0 < \alpha \leq 1)$ if and only if $a_k(f) = O(k^{-1/2-\alpha})$ as $k \to \infty$. Krotov [1973], [1975] generalized this result as follows. If ω is a modulus of continuity which satisfies

$$\sum_{k=n}^{\infty} \omega(2^{-k}) = O\big(\omega(2^{-n})\big) \quad \text{as } n \to \infty, \tag{10}$$

then f belongs to H_ω if and only if

$$a_k(f) = O\big(k^{-1/2}\omega(1/k)\big). \tag{11}$$

(1) The results mentioned in this paragraph were essentially proved by Z. Ciesielski in: *On Haar functions and on the Schauder basis of the space* $C\langle 0, 1\rangle$, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., 7 (1959), p. 227–232. [Note of the Editors]

The original results stated in [1973] were for $\omega(\delta) = O(\delta^\alpha)$ in place of (10). However, there is an error in the proof of Lemma 1 of [1973] and, in fact, that lemma is false.

A different type of generalization of Golubov's characterization of Lip α may be found in Splettstösser and Wagner [1977]. Here the idea that Lip α has some connection with differentiability is reinforced.

Concerning double Haar–Fourier coefficients, the inequalities of Horoško mentioned in the first paragraph of this subsection have been established in two dimensions by Kraczkowski [1977]. Moreover, Skvorcov [1973a] obtained the following analogue of the Riemann–Lebesgue lemma: If S is a double Haar series whose rectangular sums $S_{m,n}$ converge, as $m, n \to \infty$, at all points of a cross $(\{\alpha\} \times [0, 1]) \cup ([0, 1] \times \{\beta\})$, where α, β are dyadic irrationals, then $a_{k,j}\chi_k(\alpha)\chi_j(\beta) \to 0$ as $k+j \to \infty$.

12. Monotone Haar–Fourier coefficients. Krotov [1973] showed that if the Haar–Fourier coefficients of some continuous f were monotonically decreasing to zero, then $f \in \mathrm{Lip}\, 1$.

Golubov [1964] had shown that in order for the Haar–Fourier coefficients of a continuously differentiable f to be monotone decreasing it is necessary and sufficient that

(12) f' be non-decreasing and non-negative and $2^{-3/2} \leqslant f'(t)/f'(x) \leqslant 2^{3/2}$ for $x, t \in [0, 1]$.

Moreover, these bounds are best possible. Krotov [1974] showed that the Haar–Fourier coefficients of a continuous f are monotone decreasing if and only if there exists a countable set Z such that f' exists and is continuous off Z with jumps of the first kind on Z, and such that (12) holds off Z.

Ebralidze [1979] found that if f is continuously differentiable but not constant, then $\{a_k(f)\}_{k=2}^{\infty}$ is block-order monotone (see Subsection 9) if and only if $\max_{x\in[0,1]} |f'(x)| \leqslant C \min_{x\in[0,1]} |f'(x)|$ $(C \geqslant 1)$. An n-dimensional version of this result using partial derivatives was also announced.

13. Conditions sufficient to conclude that f is constant. Golubov [1964] was the first to notice that Haar–Fourier coefficients of non-constant smooth functions cannot decay too rapidly: if f is continuous and if $a_k(f) = o(k^{-3/2})$ as $k \to \infty$, then f is constant. A (Lip α)-version of this result was obtained by Krotov ([1973], [1975]), who showed that if ω satisfies (10) and if f is continuous and satisfies the condition

$$\lim_{\delta\to 0} \omega(f, \delta)/\omega(\delta) > 0,$$

then $a_k(f) = o\big(k^{-1/2}\omega(1/k)\big)$, as $k \to \infty$, implies that f is constant. (In view of (11), this result is sharp in the sense of order.) He also showed that this result does not hold for arbitrary $f \in H_\omega$.

Bočkarev [1969] has a nice result along these lines. If f is continuous and if

$$\sum_{k=1}^{\infty} \sqrt{k}\,|a_k(f)| < \infty,$$

then f is constant. Cagareĭšvili [1971] announced that this result holds for functions which are Darboux continuous as well. He has also shown [1976] that if $f \in \operatorname{Lip}\alpha$, $0 < \alpha \leqslant 1$, and if

$$\sum_{k=1}^{\infty} |a_k(f)|^{2\alpha/(1+2\alpha)} < \infty,$$

then f is constant. The exponent $2\alpha/(1+2\alpha)$ is exact and a multiple Haar series analogue of this result is also reported. Finally, if f is continuous on the square and if both series

$$\sum_{k=1}^{\infty} \sqrt{k}\,|a_{k,j}(f)| \quad \text{and} \quad \sum_{j=1}^{\infty} \sqrt{j}\,|a_{k,j}(f)|$$

converge for $j = 1, 2, \ldots$ and $k = 1, 2, \ldots$, then f is constant (Cagareĭšvili [1970]).

For some rather technical conditions sufficient to conclude that f is constant see [1979] by the author.

14. Multipliers. A sequence $\lambda = \{\lambda_k\}_{k=0}^{\infty}$ is said to be a *multiplier* from a space of functions A to another space B if given any $f \in A$ there is an $f_\lambda \in B$ such that

$$S[f_\lambda] \equiv \sum_{k=0}^{\infty} \lambda_k a_k(f) \chi_k. \tag{13}$$

The collection of multipliers from A to B is denoted by (A, B).

The first result on multipliers for Haar series was obtained by Marcinkiewicz [1937], who showed that $(L^p, L^p) = l^\infty$ for $1 < p < \infty$. The cases $p = 1$ and $p = \infty$ remain unexplored, although Krotov [1977] showed that $(\mathscr{C}, \mathscr{C}) = \{\lambda\colon \lambda_k \text{ is constant for } k \geqslant 2\}$.

For any sequence $\xi_0, \xi_1, \ldots$ of real numbers denote by $l^\infty(\xi_k)$ those sequences $\{\lambda_k\}$ which satisfy $\lambda_k = O(\xi_k)$ as $k \to \infty$; thus $l^\infty(1) = l^\infty$. For the remainder of this paragraph, let H_α^p denote the space H_ω^p where $\omega(\delta) = \delta^\alpha$, $\delta \in [0, 1]$, with $0 < \alpha \leqslant 1$ and $1 \leqslant p \leqslant \infty$. Golubov [1972] proved that any bounded sequence is a multiplier from H_α^p to H_α^p for $1 \leqslant p < \infty$ and $0 < \alpha < 1/p$. Krotov [1977] obtained the converse of this result, thereby showing in this case that $l^\infty = (H_\alpha^p, H_\alpha^p)$. In fact, he proved that

$$(H_\alpha^p, H_\alpha^q) = l^\infty\{k^{\alpha-\beta+1/q-1/p}\} \tag{14}$$

holds for $1 \leqslant p \leqslant q < \infty$ and $\beta q < 1$. A complete characterization for

(H_α^p, H_α^q) in the case $p > q$ is not yet known, although Krotov includes many partial results for these multipliers as well as for (H_α^p, L^q).

V. UNIQUENESS

15. Uniqueness of convergent Haar series. Suppose that $\varphi_0, \varphi_1, \ldots$ is a sequence of finite-valued functions defined on some set F and let $a_0, a_1, \ldots$ be a sequence of real numbers. By the symbol $a_k = o(\varphi_k(t))$ as $k \to \infty$ (where t is some point in F) we mean that

$$\lim_{j\to\infty} a_{k_j}/\varphi_{k_j}(t) = 0,$$

where $k_1, k_2, \ldots$ are those indices l which satisfy $\varphi_l(t) \neq 0$.

It is well known that, unless some kind of growth condition is imposed, uniqueness does not hold for Haar series which are allowed to diverge at one or more points. The most widely used such growth condition was identified by Arutunjan and Talaljan [1964]. They showed that if S is a Haar series whose coefficients satisfy the condition

$$a_k = o(\chi_k(t)) \text{ as } k \to \infty \quad \text{for every } t \in [0, 1] \tag{15}$$

and if f is a finite-valued, (Lebesgue) integrable function such that

$$\lim_{j\to\infty} S_{n_j}(x) = f(x) \tag{16}$$

for all but countably many $x \in [0, 1]$, where $\{n_j\}_{j=1}^{\infty}$ is any subsequence of positive integers, then S is the Haar–Fourier series of f. (Mušegjan [1970] generalized this result to the $X(p_n)$-systems.)

When the full sequence of partial sums is used in place of (16), the result of Arutunjan and Talaljan can be specialized in several directions. Arutunjan [1966] showed that if S is a Haar series, whose coefficients satisfy (15), which converges (off some countable set E) to a finite-valued Perron integrable function f, then S is the Perron Haar–Fourier series of f. The growth condition (15) cannot be relaxed at a single point where S diverges (see Theorem 4 of McLaughlin and Price [1969]) but can be relaxed at any dyadic irrational where S converges (see [1975] by the author).

Mušegjan [1971] proved that if S is a Haar series which satisfies (15) and if some rearrangement of S converges, off a countable set, to a finite-valued bounded function f, then S is the Haar–Fourier series of f. It follows that if S and T are Haar series whose coefficients satisfy (15) and if the same rearrangement of S and T converges, off a countable set (to some finite-valued function g), then S and T are the same series. Mušegjan [1978] showed that this corollary need not hold if S is rearranged differently than T. In fact, in a beautiful but complicated construction he finds two distinct

Haar series S and T whose coefficients satisfy the condition $a_k = o(1/\sqrt{k})$ as $k \to \infty$ (stronger than (15)) and a finite-valued function g such that certain rearrangements of S and T converge everywhere to g. It is still not known whether Mušegjan's [1971] result holds for some unbounded f.

If one assumes that f belongs to *dyadic* H^1 (see Subsection 7), then the following result can be proved. If S is a Haar series which satisfies the inequality

$$\int_0^1 \Big(\sum_{k=1}^{\infty} [S_{2^k} - S_{2^{k-1}}]^2\Big)^{1/2} < \infty$$

and if the 2^N-th partial sums of S converge, off a countable set, to f, then S is the Haar–Fourier series of f (see [1980] by the author). The main benefit here is that the pointwise condition (15) has been replaced with an integrated growth condition which might be easier to use in applications.

For double Haar series, condition (15) has been generalized in three different ways:

$$a_{k,j} = o\big(\chi_k(t)\,\chi_j(u)\big) \text{ as } k+j \to \infty \quad \text{for } (t, u) \in Q, \tag{17}$$

$$a_{k,j} = o\big(\chi_k(t)\,\chi_j(u)\big) \text{ as } k, j \to \infty \quad \text{for } (t, u) \in Q, \tag{18}$$

and the iterated conditions

$$a_{k,j} = o\big(\chi_k(t)\,\chi_j(u)\big) \text{ as } k \to \infty \quad \text{for } j \geqslant 0, (t, u) \in Q, \tag{19}$$

$$a_{k,j} = o\big(\chi_k(t)\,\chi_j(u)\big) \text{ as } j \to \infty \quad \text{for } k \geqslant 0, (t, u) \in Q. \tag{20}$$

Obviously, (17) contains (18) and (19)–(20) but (18) and (19)–(20) are incomparable.

Skvorcov [1973b] proved that if the coefficients of a double Haar series satisfy (17) and if the rectangular sums $S_{n,m}$ converge as $n, m \to \infty$, for all but countably many points in the unit square Q, to a finite-valued, Perron integrable function f, then S is the double Perron Haar–Fourier series of f. Ebralidze [1973] announced a parallel result for Lebesgue integrable f and everywhere convergent Haar series whose coefficients satisfy (19) and (20).

Movsisjan [1976] obtained uniqueness for iterated sums of double Haar series. Specifically, he proved that if $\{t_m\}_{m=1}^{\infty}$ is a countable subset of [0, 1], if for each integer $j \geqslant 1$ the series

$$\varphi_j(t) = \sum_{k=0}^{\infty} a_{k,j}\,\chi_k(t) \tag{21}$$

converges for $t \neq t_m$, $m = 1, 2, \ldots$, if

$$\sum_{j=0}^{\infty} \varphi_j(t)\,\chi_j(u) \tag{22}$$

converges to zero off the vertical lines $\{t_m\} \times [0, 1]$, $m = 1, 2, \ldots$, and if (19) holds, then $a_{k,j} \equiv 0$ for $k, j \geqslant 0$. On the other hand, convergence of (22) in the absence of convergence of (21) is not sufficient to conclude that $a_{k,j} \equiv 0$ for $k, j \geqslant 0$, even under a stronger growth condition than (19) or (20). It is not known whether uniqueness holds for iterated sums which converge to integrable functions. **(P 1314)**

Uniqueness also holds for spherical sums. Skvorcov [1981] proved that if S is a double Haar series whose coefficients satisfy (19) and (20), if S_R converges to f as $R \to \infty$, except perhaps on countably many crosses $(\{t_m\} \times [0, 1]) \cup ([0, 1] \times \{u_m\})$, where f is a finite-valued, Perron integrable function, then S is the Perron Haar–Fourier series of f. This theorem was generalized by Movsisjan [1977] who had obtained uniqueness for bounded f.

Movsisjan [1974] has also studied uniqueness for subsequences of the rectangular partial sums. He proved that if S is a double Haar series satisfying (19) and (20), and if S_{n_j, m_k} converges as $j, k \to \infty$, except perhaps on a countable subset of Q, to a finite-valued integrable f, then S is the double Haar–Fourier series of f.

16. Uniqueness of summable Haar series. Skvorcov [1971] first considered uniqueness for summable Haar series. Let $\tau_n(S)$ denote the n-th partial sum of a Haar series S with respect to some Toeplitz method of summation with positive matrix. He showed that if

$$\liminf_{n \to \infty} \tau_n(S) \leqslant f \leqslant \limsup_{n \to \infty} \tau_n(S)$$

on $[0, 1]$, where f is a finite-valued, Perron integrable function, and if (15) is satisfied by the coefficients of S, then S is the Perron Haar–Fourier series of f. When the method of summation also has a positive finite-row matrix, he proved the following for rearrangements of Haar series. If S is any rearrangement of a Haar series whose coefficients satisfy (15) and if $\tau_n(S) \to 0$ as $n \to \infty$, then S is the zero series.

The author [1971], [1973] has studied uniqueness of $(C, 1)$-summable Haar series. For example, if $\sigma_{n_j}(S)$ converges in measure, as $j \to \infty$, to an integrable function f, if the coefficients of S satisfy (15), and if

$$\limsup_{j \to \infty} |\sigma_{n_j}(S, x)| < \infty$$

for all but countably many $x \in [0, 1]$, then S is the Haar–Fourier series of f.

No work has been done on uniqueness of summable double Haar series.

17. Sets of uniqueness. A set $E \subseteq [0, 1]$ is called a *set of uniqueness (for Haar series)* if the only Haar series S which converges to zero off E is the zero series. It is known that the empty set is a set of uniqueness for Haar series, but that no non-empty set is (see McLaughlin and Price [1969]). Thus sets of uniqueness are studied for various classes of Haar series, segregated according to growth conditions on their coefficients.

As reported in Subsection 15, any countable set is a set of uniqueness for Haar series whose coefficients satisfy (15). Mušegjan [1967] showed that for Borel sets this result is best possible. Indeed, he proved that, given a set E, a necessary and sufficient condition that E be a set of uniqueness for the class of Haar series satisfying (15) is that E contain no non-empty perfect sets. In a latter paper [1969] he constructed perfect non-empty sets of uniqueness for the class of Haar series whose coefficients satisfy the condition

$$|a_{k_j}/\chi_{k_j}(t)| < \varphi_{k_j}(t) \quad \text{for all } t \in [0, 1],$$

where $k_1 < k_2 < \ldots$ are those indices l for which $\chi_l(t) \neq 0$, and $\varphi_1, \varphi_2, \ldots$ is a fixed sequence of positive measurable functions which converges everywhere to zero on [0, 1].

The author [1977] has examined sets of uniqueness for Haar series satisfying certain growth conditions $G(p)$, $-\infty < p < \infty$, which in the case $p = 2$ are order equivalent to but slightly stronger than (15). From Mušegjan's work it follows that Borel sets of uniqueness for the condition $G(2)$ are countable, and that there are non-empty perfect sets of uniqueness for the condition $G(p)$ when $p < 2$. The author showed that a countable union of closed sets of uniqueness for the condition $G(p)$, $p \geqslant 0$, is again a set of uniqueness for $G(p)$. He also proved that sets of uniqueness for $G(0)$ can have positive measure.

Skvorcov [1973] first systematically studied sets of uniqueness for multiple Haar series. As in the one-dimensional case, the only set of uniqueness for Haar series is the empty set, and some growth condition must be used to get non-trivial sets of uniqueness. He proved that, given a set $E \subseteq Q$, a necessary and sufficient condition for E to be a set of uniqueness for the class of Haar series which satisfies (17) is that E contain no perfect set which intersects the diagonal of Q. He also announced that, for condition (18), E is a set of uniqueness if and only if E contains no non-empty perfect subset. It follows that a Borel set E is a set of uniqueness for Haar series which satisfy (18) if and only if E is countable.

18. Null series. A Haar series which converges to zero a.e. but not everywhere is called a *null series*. Clearly, many of the results cited in Subsections 15 and 17 are concerned with null series. In this section we report two specific results due to Skvorcov.

In [1977] he considers how slowly the coefficients of a null series can diminish to zero. He shows that given any positive monotone decreasing sequence ω_k which satisfies the condition

$$\sum_{k=1}^{\infty} \omega_k^2 = \infty$$

there exists a null series $\sum_{k=0}^{\infty} a_k \chi_k$ whose coefficients satisfy $0 \leqslant a_k \leqslant \omega_k$ for $k \geqslant 0$.

Finally, in [1980] he constructs a non-zero double Haar series whose coefficients satisfy (17) such that given any $(x, y) \in Q$ there exist sequences $\{n_j\}_{j=1}^{\infty}$ and $\{m_j\}_{j=1}^{\infty}$, depending on (x, y), for which

$$\lim_{j \to \infty} S_{n_j, m_j}(x, y) = 0.$$

Thus unlike the one-dimensional case, uniqueness does not hold for double Haar series which converge through different subsequences as one moves from point to point.

BIBLIOGRAPHY

F. G. Arutunjan

[1966] *On series based on Haar's system*, Dokl. Akad. Nauk Armjan. SSR 42, p. 134–140.

F. G. Arutunjan and A. A. Talaljan

[1964] *Uniqueness of series with respect to the Haar and Walsh system*, Izv. Akad. Nauk SSSR Ser. Mat. 28, p. 1391–1408.

L. Balašov

[1971] *Series in the Haar system*, Mat. Zametki 10, p. 369–374.

S. V. Bočkarev

[1969] *On the coefficients of series with respect to the Haar system*, Mat. Sb. 80, p. 97–116.

[1971] *Absolute convergence of Fourier series with respect to complete orthonormal systems of functions*, ibidem 85, p. 431–439.

[1972] *Absolute convergence of Fourier series with respect to complete orthonormal systems*, Russian Math. Surveys 27, p. 55–82.

[1978] *A method of averaging in the theory of orthogonal series and some problems in the theory of bases*, Trudy Mat. Inst. Steklov., No. 146, 92 pp.

V. Š. Cagareĭšvili

[1970] *Series with respect to the Haar system*, Soobšč. Akad. Nauk Gruzin. SSR 60, p. 37–39.

[1971] *The Fourier coefficients of a continuous function with respect to the Haar system*, ibidem 63, p. 37–39.

[1976] *Fourier–Haar coefficients*, ibidem 81, p. 29–31.

Z. Ciesielski and S. Kwapień

[1979] *Some properties of the Haar, Walsh–Paley, Franklin, and the bounded polygonal orthonormal bases in L^p spaces*, Special issue (dedicated to Władysław Orlicz on the occasion of his seventy-fifth birthday), Comment. Math. Prace Mat. 2, p. 37–42.

Z. Ciesielski and J. Musielak

[1959] *On absolute convergence of Haar series*, Colloq. Math. 7, p. 61–65.

Z. Ciesielski, P. Simon, and P. Sjölin

[1977] *Equivalence of Haar and Franklin bases in L^p spaces*, Studia Math. 60, p. 195–210.

G. A. Čaĭdze

[1972] *Multiple series with respect to the Haar system*, Soobšč. Akad. Nauk Gruzin. SSR 66, p. 541–544.

Z. A. Čanturija

[1974] *The absolute convergence of series of Haar–Fourier coefficients*, Soobšč. Akad. Nauk Gruzin. SSR 75, p. 281–284.

[1979] *On absolute convergence of the series of Haar–Fourier coefficients*, Special issue (dedicated to Władysław Orlicz on the occasion of his seventy-fifth birthday), Comment. Math. Prace Mat. 2, p. 25–35.

R. S. Davtjan

[1976] *Representations of functions of orthogonal series possessing martingale properties*, Mat. Zametki 19, p. 673–680.

R. S. Davtjan and A. A. Talaljan

[1975] *The convergence of series in complete orthonormal systems on sets of positive measure*, Izv. Akad. Nauk Armjan. SSR Ser. Mat. 10, p. 342–355.

O. P. Dzagnidze

[1964] *Representation of measurable functions of two variables by orthogonal series*, Soobšč. Akad. Nauk Gruzin. SSR 34, p. 277–282.

A. D. Ebralidze

[1973] *Uniqueness of multiple series in the Haar system*, Soobšč. Akad. Nauk Gruzin. SSR 70, p. 537–539.

[1976] *Haar and Rademacher series*, ibidem 83, p. 297–300.

[1979] *Fourier–Haar coefficients of continuously differentiable functions*, ibidem 94, p. 29–31.

Ju. S. Fridljand

[1973] *The nonremovable Carleman singularity for Haar's system*, Mat. Zametki 14, p. 799–807.

G. Gaĭmnazarov

[1971] *The absolute convergence of double Haar–Fourier series*, Dokl. Akad. Nauk. Tadžik. SSR 14, p. 3–6.

[1975] *$|C, \alpha|$ summability of Haar series*, ibidem 18, p. 3–6.

G. Gaĭmnazarov and M. F. Timan

[1971] *Best approximation and absolute convergence of Fourier–Haar series*, Dokl. Akad. Nauk SSSR 198, p. 1280–1282.

J. Gamlen and R. Gaudet

[1973] *On subsequences of the Haar system in $L^p[0, 1]$ $(1 < p < \infty)$*, Israel J. Math. 15, p. 404–413.

V. F. Gapoškin

[1974] *The Haar system as an unconditional basis in $L^p[0, 1]$*, Mat. Zametki 15, p. 191–196.

A. M. Garsia

[1973] *Martingale inequalities*, Reading, Massachusetts.

B. I. Golubov

[1964] *Fourier series of continuous functions with respect to the Haar system*, Izv. Akad. Nauk SSSR Ser. Mat. 28, p. 1271–1296.

[1970] *Series with respect to the Haar system*, Itogi Nauki – Serija "Matematika" (Mat. Analiz), p. 109–146.

[1972] *Best approximations of functions in the L^p metric by Haar and Walsh polynomials*, Mat. Sb. (N.S.) 87, p. 254–274.

A. Haar

[1910] *Zur Theorie der orthogonalen Funktionensysteme*, Math. Ann. 69, p. 331–371.

L. Homutenko

[1971] *Exact estimates of Fourier coefficients with respect to the Haar system of functions of bounded variation*, Mat. Zametki 9, p. 355–363.

N. P. Horoško

[1972] *Estimation of the supremum of Fourier coefficients on certain classes of functions with respect to Haar, Rademacher, and Walsh systems*, Teor. Funkciĭ Funkcional. Anal. i Priložen., Vyp. 15, p. 3–12.

N. P. Horoško and V. V. Lipovik

[1975] *A sharp estimate of the approximation of functions of the class* $H_\omega[0, 1]$ *by Fourier-Haar sums in the metric of* L, Studies in Contemporary Problems of Summability and Approximation of Functions and Their Applications, No. 6, p. 89–92, Dniepropetrovsk. Gos. Univ., Dniepropetrovsk.

[1976] *A sharp estimate of the approximation of functions of the class* $H_\omega[0, 1]$ *by Fourier-Haar sums in the* L^2 *metric*, ibidem 157, p. 63–66, Dniepropetrovsk. Gos. Univ., Dniepropetrovsk.

V. H. Hristov

[1973] *The absolute convergence of Haar-Fourier series of absolutely continuous functions*, Vestnik Moskov. Univ. Ser. I. Mat. Meh. 28, p. 36–42.

K. S. Kazarjan

[1978] *The multiplicative complementation of some incomplete bases in* L^p, $1 \leqslant p < \infty$, Anal. Math. 4. p. 37–52.

G. G. Kemhadze

[1975] *A remark on the convergence of spherical partial sums of the Fourier series of functions of class* L^p, $p > 1$, Dokl. Akad. Nauk SSSR 22, p. 277–280.

[1977a] *The divergence of spherical partial sums of multiple Haar-Fourier series*, Soobšč. Akad. Nauk Gruzin. SSR 85, p. 537–539.

[1977b] *The convergence of spherical partial sums of multiple Haar-Fourier series*, Trudy Tbiliss. Mat. Inst. 55, p. 27–39.

A. Kraczkowski

[1977] *On double Haar series of functions of class* V_p, Funct. Approx. Comment. Math. 5, p. 25–30.

[1978] *An application of double moduli variation to double Haar-Fourier series*, ibidem 6, p. 119–133.

A. S. Krancberg

[1974] *Rearrangements of the Haar system*, Mat. Zametki 15, p. 63–73.

V. G. Krotov

[1973] *On series with respect to the Haar system*, Sibirsk. Mat. Ž. 14, p. 111–127.

[1974] *Continuous functions with monotonically decreasing Haar-Fourier coefficients*, ibidem 15, p. 439–444.

[1975] *On series with respect to the Haar system*, ibidem 16, p. 417–418.

[1977] *On the multipliers of Fourier series with respect to the Haar system*, Anal. Math. 3, p. 187–198.

[1978] *Unconditional convergence of Fourier series in the Haar system in the spaces* Λ^p_ω, Mat. Zametki 23, p. 685–695.

M. A. Lunina

[1976] *The set of points of unbounded divergence of series in the Haar system*, Vestnik Moskov. Univ. Ser. I. Mat. Meh. 31, p. 13–20.

J. Marcinkiewicz

[1937] *Quelques théorèmes sur les séries orthogonales*, Ann. Polon. Math. 16, p. 84–96.

V. A. Matveev

[1974] *On Haar-Fourier coefficients of continuous functions*, Izv. Vysš. Učebn. Zaved. Matematika 148, p. 65–71.

J. R. McLaughlin

[1969] *Haar series*, Trans. Amer. Math. Soc. 137, p. 153–176.

[1973] *Absolute convergence of series of Fourier coefficients*, ibidem 184, p. 291–316.

J. R. McLaughlin and J. J. Price

[1969] *Comparison of Haar series with gaps and trigonometric series*, Pacific J. Math. 28, p. 623–627.

M. Mikolás

[1972] *Über die Charakterisierung orthogonaler Systeme von Haarschem Typ*, Publ. Math. Debrecen 19, p. 239–248.

H. O. Movsisjan

[1974] *The uniqueness of double series in the Haar and Walsh systems*, Izv. Akad. Nauk Armjan. SSR Ser. Mat. 9, p. 40–61.

[1976] *The uniqueness of successively convergent double Haar series*, ibidem 11, p. 314–331.

[1977] *The uniqueness of double series in the Haar system that converge with respect to spheres*, Akad. Nauk Armjan. SSR Dokl. 64, p. 137–142.

G. M. Mušegjan

[1967] *On uniqueness sets for the Haar system*, Izv. Akad. Nauk. Armjan. SSR Ser. Mat. 2, p. 350–361.

[1970] *The uniqueness of series for a certain class of orthonormal systems*, ibidem 5, p. 138–153.

[1971] *The uniqueness of series with respect to rearranged Haar systems*, ibidem 6, p. 21–34.

[1978] *The coefficients of everywhere convergent series with respect to the Haar system with permuted terms*, ibidem 13, p. 275–300.

A. M. Olevskiĭ

[1975] *Fourier series with respect to general orthogonal systems*, Berlin.

K. I. Oskolkov

[1977] *Approximation properties of summable functions on sets of full measure*, Mat. Sb. 103 (145), p. 563–589.

L. G. Pal and F. Schipp

[1972] *On the Steinhaus conjecture with respect to the Haar series*, Proc. of the Conf. on Constructive Theory of Functions (Budapest, 1969), p. 343–349, Akademiai Kiadó, Budapest.

M. B. Petrovskaja

[1964] *Uniqueness theorems for series with respect to the Haar system*, Vestnik Moskov. Univ. Ser. I. Mat. Meh. 5, p. 15–28.

N. B. Pogosjan

[1980] *Summability of Haar–Walsh series to* $+\infty$, Uspehi Mat. Nauk 35, p. 219–220.

J. J. Price

[1970] *Haar series and adjustments of functions on small sets*, Illinois J. Math. 14, p. 82–87.

[1972] *Sparse subsets of orthonormal systems*, Proc. Amer. Math. Soc. 35, p. 161–164.

V. Prohorenko

[1971] *Divergent Fourier series with respect to Haar's system*, Izv. Vysš. Učebn. Zaved. Matematika No. 1 (104), p. 62–68.

F. Schipp

[1979] *On a generalization of the Haar system*, Acta Math. Acad. Sci. Hungar. 33, p. 183–188.

J. Shirey

[1973] *Restricting a Schauder basis to a set of positive measure*, Trans. Amer. Math. Soc. 184, p. 61–71.

P. Sjölin

[1977] *The Haar and Franklin systems are not equivalent bases in L^1*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25, p. 1099–1100.

V. A. Skvorcov

[1964] *A Cantor-type theorem for the Haar system*, Vestnik Moskov. Univ. Ser. Mat. Meh. No. 5, p. 12–28.

[1971] *Uniqueness theorems on Haar series for summation methods*, Mat. Zametki 9, p. 449–458.

[1973a] *The coefficients of multiple Haar and Walsh series*, Vestnik Moskov. Univ. Ser. Mat. Meh. 28, p. 77–79.

[1973b] *Sets of uniqueness for multi-dimensional Haar series*, Mat. Zametki 14, p. 789–798.

[1977] *The rate of convergence to zero of coefficients of null-series in the Haar and Walsh system*, Izv. Akad. Nauk SSSR Ser. Mat. 41, p. 703–716.

[1980] *On an example of double Haar series*, Mat. Zametki 28, p. 343–353.

[1981] *On uniqueness for double Haar series with convergent spherical partial sums*, Vestnik Moskov. Univ. Ser. Mat. Meh. No. 1 (106), p. 12–17.

W. Splettstösser and H. J. Wagner

[1977] *Eine dyadische Infinitesimalrechnung für Haar-Funktionen*, Z. Angew. Math. Mech. 57, p. 527–541.

J. Szelmeczka

[1974] *On absolute convergence of multiple Haar and Rademacher series*, Comment. Math. Prace Mat. 17, p. 475–479.

L. A. Šaginjan

[1973] *The summability of series in the Haar system by methods of (C, α) and (H, k)*, Akad. Nauk Armjan. SSR Dokl. 57, p. 206–211.

[1974a] *Summability of series in the Haar system by the $(C, 1)$ method*, Mat. Zametki 15, p. 394–404.

[1974b] *The summability to $+\infty$ by the Abel method of series in the Haar system*, Akad. Nauk Armjan. SSR Dokl. 58, p. 3–9.

A. A. Talaljan

[1960] *The representation of measurable functions by series*, Uspehi Mat. Nauk 15, p. 77–141.

A. A. Talaljan and F. G. Arutunjan

[1965] *Convergence of series with respect to the Haar system to $+\infty$*, Mat. Sb. 66, p. 240–247.

F. A. Talaljan

[1972] *The subsystems of the Haar system*, Akad. Nauk Armjan. SSR Dokl. 55, p. 3–6.

Tian Ping Chen

[1980] *Approximating continuous functions by Haar series*, Acta Math. Sinica 23, p. 226–238.

G. E. Tkebučava

[1973] *Series in the Haar system*, Soobšč. Akad. Nauk Gruzin. SSR 69, p. 277–280.

[1979] *On unconditional convergence of Haar series*, Approx. Theory (VIth Semest. Stefan Banach International Math Ctr., Warsaw, 1975), p. 261–272, Banach Center Publ., 4, Warszawa.

P. L. Ul'janov

[1951] *Some properties of series with respect to the Haar system*, Mat. Zametki 1, p. 225–231.

[1963] *On series with respect to the Haar system*, Dokl. Akad. Nauk SSSR 149, p. 532–534.

[1964a] *Solved and unsolved problems of the theory of trigonometric and orthogonal series*, Uspehi Mat. Nauk. 19, p. 3–69.

[1964b] *Series with respect to the Haar system*, Mat. Sb. 63, p. 356–391.

[1967] *On absolute and uniform convergence of Fourier series*, ibidem 72, p. 193–225.

[1970] *Embedding theorems and relations between best approximations (moduli of continuity) in different metrics*, ibidem 81, p. 104–131.

[1972] *Representations of functions by series and classes $\varphi(L)$*, Russian Math. Surveys 27 (1972), p. 1–54.

[1978] *Absolute convergence of Fourier–Haar series for superpositions of functions*, Anal. Math. 4, p. 225–236.

N. Ya. Vilenkin

[1947] *A class of complete orthonormal systems*, Izv. Akad. Nauk SSSR Ser. Mat. 11, p. 363–400.

W. R. Wade

[1971] *Uniqueness theory of Cesàro summable Haar series*, Duke Math. J. 38, p. 221–227.

[1973] *Uniqueness of Haar series which are $(C, 1)$ summable to Denjoy integrable functions*, Trans. Amer. Math. Soc. 176, p. 489–498.

[1975] *Growth of Haar series on the dyadic rationals and uniqueness*, Proc. Amer. Math. Soc. 50, p. 198–201.

[1977] *Sets of uniqueness for Haar series*, Acta Math. Acad. Sci. Hungar. 30, p. 265–281.

[1979] *Walsh series and growth of functions on nested dyadic intervals*, J. Indian Math. Soc. 43, p. 1–11.

[1980] *Uniqueness of Walsh series which satisfy an averaged growth condition*, SIAM J. Math. Anal. 11, p. 933–937.

I. Wierzbicka

[1973] *The absolute summability of Haar series by some matrix method*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 21, p. 405–412.

S. V. Zotikov

[1973] *The convergence a.e. of Fourier series in systems of Haar type*, Sibirsk. Mat. Ž. 14, p. 760–765.

[1974] *The absolute convergence of series in a system of Haar type*, Izv. Vysš. Učebn. Zaved. Matematika 150, p. 31–43.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TENNESSEE
KNOXVILLE, TENNESSEE

Reçu par la Rédaction le 16.3.1982