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DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

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S. ARMENTROUT

A decomposition of E^3 into straight arcs and singletons

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CONTENTS

1. Introduction *

In this paper we shall give an example of an upper semicontinuous decomposition of E^3 into straight arcs and singletons such that the associated decomposition space is topologically distinct from E^3.

The decomposition described in this paper has a number of interesting properties. It has a Cantor set of nondegenerate elements. There exist two parallel planes P and Q such that each nondegenerate element of the decomposition has one end on P and the other end on Q. In [2] it is shown that the associated decomposition space is locally peripherially spherical, i.e., each of its points has arbitrarily small neighborhoods bounded by 2-spheres. In fact, each point of the decomposition space has arbitrarily small (closed) neighborhoods which are compact absolute retracts and have 2-spheres as their (topological) boundaries. Thus each point of the space has arbitrarily small compact simply connected neighborhoods. The example described in this paper yields an example of a 3-dimensional spheroidal space which is not a sphere ([2]). It follows from a result due to Boals ([7]) that the product of the real line E^1 and the space of the decomposition of this paper is homeomorphic to E^4.

The question of whether each upper semicontinuous decomposition of E^3 into straight arcs and singletons yields E^3 as a decomposition space was stated in [3]. Partial affirmative solutions were given by Bing ([4], Theorem 3), McAuley ([9], Theorem 2), and Sher ([12], Theorem 6). In [6], Sections 6–8, Bing described an example of an upper semicontinuous decomposition of E^3 into straight arcs and singletons and conjectured that the associated decomposition space is topologically distinct from E^3. It is an open question whether Bing's conjecture is correct. McAuley also described ([10]) an example of an upper semicontinuous decomposition of E^3 into straight arcs and singletons, and conjectured that the associated decomposition space is topologically distinct from E^3. It was proved by Cannon ([8]) that the space of McAuley's decomposition is homeomorphic to E^3.

The method of constructing the example of this paper is a slight variation of that used by Bing in constructing the example of [6], Sections 6–8.

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In Section 3, we describe the decomposition G to be studied in this paper. Section 4 gives some notation and terminology to be used in the remainder of the paper. In Section 5, we establish the main result, that the decomposition space associated with G is not homeomorphic to E^3. The proof of this result depends heavily on the main lemma, Lemma 3. The proof of Lemma 3 is long, and is given in sections 6 through 15.

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2. Notation and terminology

The statement that a is a *straight arc* in E^3 means that a is an arc contained in some straight line in E^3. If x and y are two distinct points of E^3, then $\langle xy\rangle$ denotes the straight arc with endpoints x and y.

If G is an upper semicontinuous decomposition of E^3, then E^3/G denotes the associated decomposition space, Pr denotes the projection map from E^3 onto E^3/G, and H_G denotes the union of all the non-degenerate elements of G.

If n is a positive integer and M is an n-manifold with boundary, then BdM and Int$M$ denote the boundary and interior, respectively, of M.

If K is a subset of a topological space, then ClK denotes the closure of K.

3. Description of G

Define $\mu(0)$ to be 2, and for each non-negative integer i, define $\mu(i+1)$ to be $[2+2\mu(i)]$. It is easily shown inductively that for each non-negative integer k, $\mu(k) = 2(2^{k+1}-1)$.

Define n_0 to be 4, and for each non-negative integer m, define n_{m+1} to be $\mu(n_m)$. We note that $n_0 = 4$, $n_1 = 62$, and n_2 exceeds 10^{18}.

Let P and Q be two horizontal planes in E^3 with P above Q.

The statement that Γ is a *special graph* means that there exist two distinct points p_1 and p_2 of P, an even positive integer m greater than 3, and m distinct collinear points of Q such that (1) the lines p_1p_2 and q_1q_2 are skew and (2) $\Gamma = \bigcup_{i=1}^{2} \bigcup_{j=1}^{m} \langle p_iq_j\rangle$. The two points p_1 and p_2 are the *branch points* of the special graph Γ. The *strands* of the special graph Γ are the arcs

$$\langle p_1q_1\rangle \cup \langle p_2q_1\rangle,\ \langle p_1q_2\rangle \cup \langle p_2q_2\rangle, \ldots, \langle p_1q_j\rangle \cup \langle p_2q_j\rangle, \ldots$$
$$\ldots, \langle p_1q_m\rangle \cup \langle p_2q_m\rangle.$$

We are now prepared to describe G. The set H_G is the intersection of sets $T_0, T_1, T_2, \ldots$ where, for each i, T_i is a compact polyhedral 3-manifold with boundary, and for each i, $T_{i+1} \subset \operatorname{Int} T_i$. For each i, T_i is a closed neighborhood of a certain graph Γ_i. We shall now describe Γ_0.

Let p_1 and p_2 be two distinct points of P, and let q_1, q_2, q_3 and q_4 be four distinct collinear points of Q such that the lines p_1p_2 and q_1q_2 are skew. Let Γ_0 denote $\bigcup_{i=1}^{2} \bigcup_{j=1}^{4} \langle p_i q_j \rangle$. See Figure 1. Observe that Γ_0 is a special graph and has exactly n_0 strands. Let T_0 denote a closed polyhedral tubular neighborhood of Γ_0.

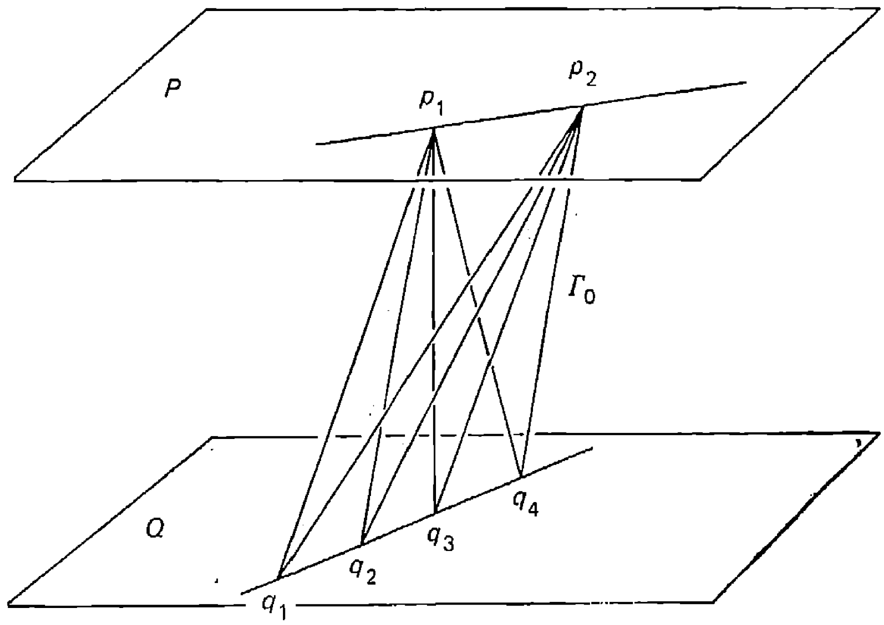

Fig. 1

If $i = 1$ or 2 and $j = 1, 2, 3,$ or 4, we shall replace the interval $\langle p_i q_j \rangle$ by a special graph Γ_{ij} of exactly n_1 strands such that Γ_{ij} lies in $\operatorname{Int} T_0$ and, in fact, in a small neighborhood of $\langle p_i q_j \rangle$. We shall then define Γ_1 to be $\bigcup_{i=1}^{2} \bigcup_{j=1}^{4} \Gamma_{ij}$.

Suppose $i = 1$ or 2 and $j = 1, 2, 3,$ or 4. Γ_{ij} is the union of $2n_1$ straight arcs. We select two distinct points p_{ij1} and p_{ij2} near p_i on P, and n_1 distinct collinear points $q_{ij1}, q_{ij2}, \ldots,$ and q_{ijn_1} near q_j on Q. It is to be true that the lines $p_{ij1}p_{ij2}$ and $q_{ij1}q_{ij2}$ are skew, and additional conditions stated below are to be satisfied. Γ_{ij} is $\bigcup_{k=1}^{2} \bigcup_{l=1}^{n_1} \langle p_{ijk} q_{ijl} \rangle$, and Γ_1 is $\bigcup_{i=1}^{2} \bigcup_{j=1}^{4} \Gamma_{ij}$. See Figure 2.

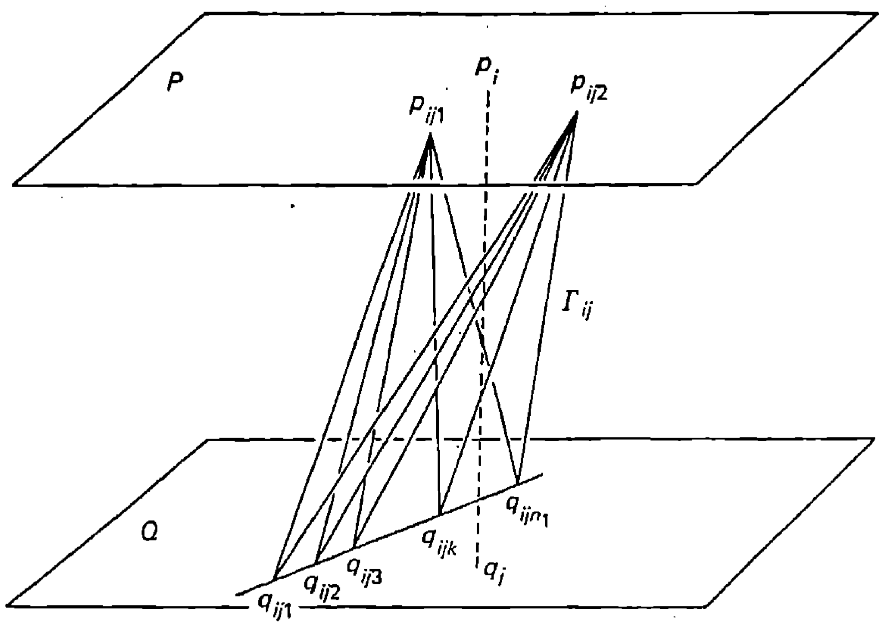
P
p_i
p_{ij1}
p_{ij2}
Γ_{ij}
Q
q_{ijn_1}
q_{ijk}
q_i
q_{ij3}
q_{ij2}
q_{ij1}

Fig. 2

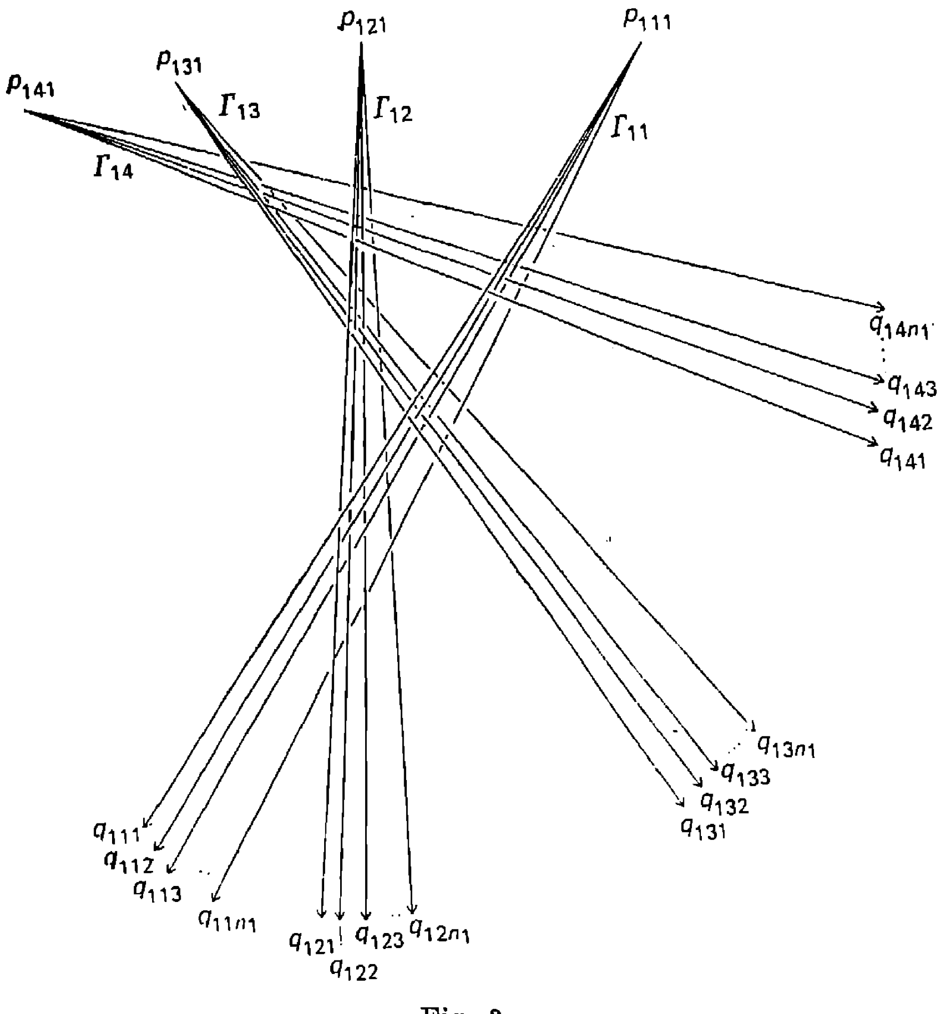
p_{121}
p_{111}
p_{131}
p_{141}
Γ_{13}
Γ_{12}
Γ_{11}
Γ_{14}
q_{14n_1}
q_{143}
q_{142}
q_{141}
q_{13n_1}
q_{133}
q_{132}
q_{131}
q_{111}
q_{112}
q_{113}
q_{11n_1}
q_{121}
q_{122}
q_{123}
q_{12n_1}

Fig. 3

Figure 3 shows how certain of the segments of Γ_1 are entwined. In Figure 3, only half of the segments in each of $\Gamma_{11}, \Gamma_{12}, \Gamma_{13}$, and Γ_{14} are shown.

The entwining is to follow the following pattern:

(1) Suppose $i = 1$ or 2. Then if j and k are $1, 2, \ldots$, or n_1, and $j < k$, each of s and r is 1 or 2, and each of t and u is 1, 2, 3, or 4, the following hold: (a) If $t \leqslant u$, $\langle p_{ijs} q_{ijt} \rangle$ passes over $\langle p_{ikr} q_{iku} \rangle$, and (b) if $u < t$, $\langle p_{ikr} q_{iku} \rangle$ passes over $\langle p_{ijs} q_{ijt} \rangle$.

(2) If $i = 1$ or 2 and $j = 1, 2, \ldots$, or n_1, p_{ij1} and p_{ij2} are so close together that if $k = 1, 2, \ldots$, or n_1, $x = 1$ or 2, $y = 1, 2, \ldots$, or n_1, and $z = 1, 2, \ldots$, or n_1, then $\langle p_{ij1} q_{ijk} \rangle$ passes over $\langle p_{ixy} q_{ixz} \rangle$ if and only if $\langle p_{ij2} q_{ijk} \rangle$ does.

Figure 4 shows how one strand of Γ_{1j}, for $j = 1, 2, 3$, or 4, entwines the n_1 strands of Γ_{2j} near q_j. Only one strand of Γ_{1j} is shown but each other strand of Γ_{1j} entwines the strands of Γ_{2j} in exactly the same way.

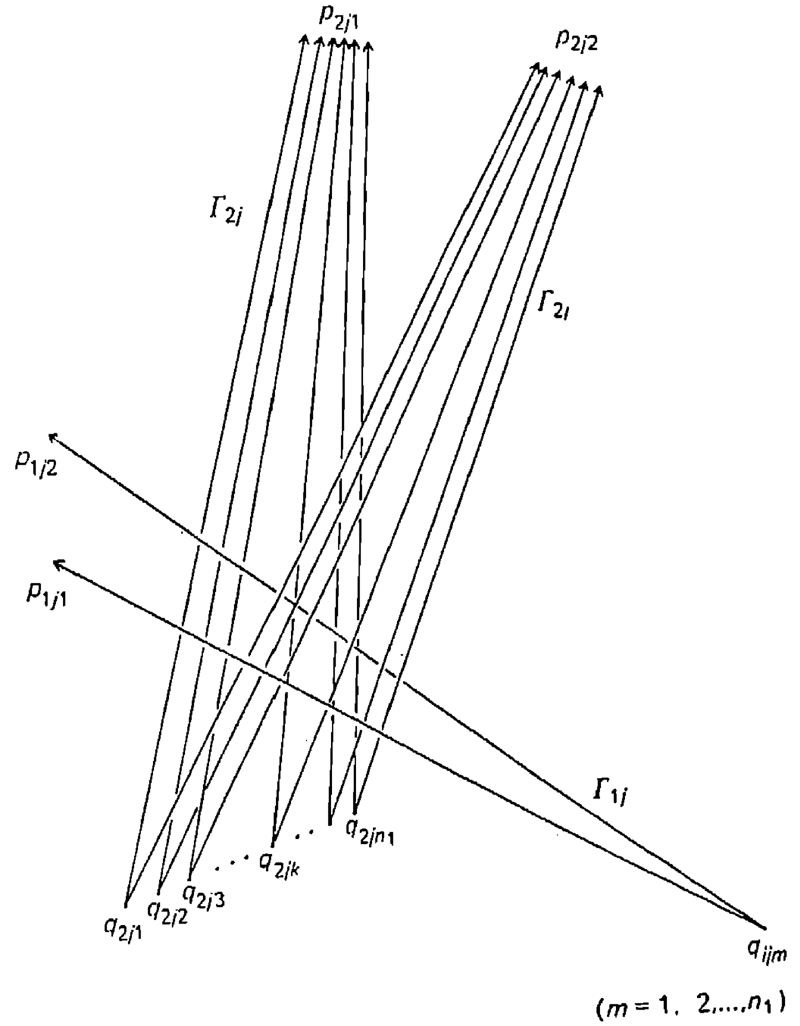

Fig. 4

The existence of a graph Γ_1 having the indicated properties may be proved by a modification of the argument given by Bing on pages 442–443 of [6].

If $i = 1$ or 2 and $j = 1, 2, 3$, or 4, let T_{ij} be a thin closed polyhedral tubular neighborhood of Γ_{ij}. It is to be true that for each such i and j, (1) $T_{ij} \subset \operatorname{Int} T_0$ and (2) if $k = 1$ or 2, $l = 1, 2, 3$, or 4, and $(i, j) \neq (k, l)$, then T_{ij} and T_{kl} are disjoint. Let T_1 be $\bigcup_{i=1}^{2} \bigcup_{j=1}^{4} T_{ij}$. Notice that $T_1 \subset \operatorname{Int} T_0$.

We assume that for each i and j such that $i = 1$ or 2 and $j = 1, 2, 3$, or 4, T_{ij} is constructed so near Γ_{ij} that there exist two disjoint cubes-with-4-handles W_1 and W_2 in $\operatorname{Int} T_0$ such that $\bigcup_{j=1}^{4} T_{1j} \subset \operatorname{Int} W_1$ and $\bigcup_{j=1}^{4} T_{2j} \subset \operatorname{Int} W_2$, and W_1 and W_2 are as indicated in Figure 5.

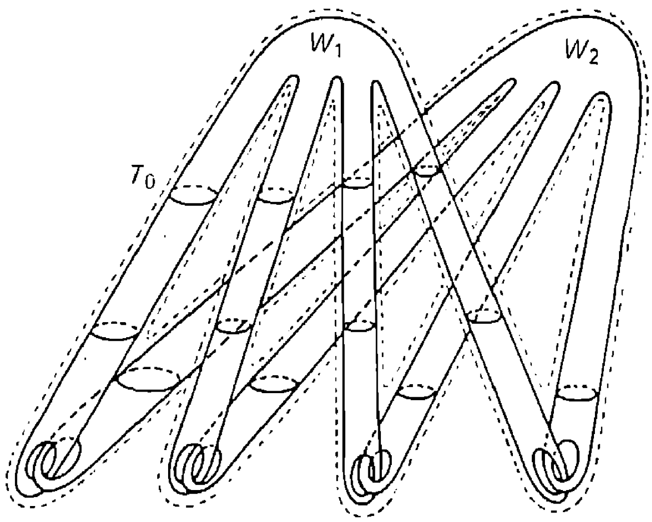

Fig. 5

To see that such sets W_1 and W_2 exist, consider T_0 with a part cut away as shown in Figure 6. Let W_1'' and W_2'' denote the indicated sets. Shrink W_1'' inward slightly to get W_1', and similarly shrink W_2'' inward slightly to get W_2'.

Now for each i and j, Γ_{ij} lies in the union of two triangular discs D_{ij1} and D_{ij2} as shown in Figure 7. Thicken the D_{ij1}'s and D_{ij2}'s slightly, add the thickened D_{1j1}'s and D_{1j2}'s to W_1' and add the thickened D_{2j1}'s and D_{2j2}'s to W_2'. This yields sets W_1 and W_2 having the properties described above.

Suppose $i_1 = 1$ or 2, $j_1 = 1, 2, 3$, or 4, $i_2 = 1$ or 2, and $j_2 = 1, 2, \ldots,$ or n_1. For each $i_1 j_1 i_2 j_2$, we shall replace the interval $\langle p_{i_1 j_1 i_2} q_{i_1 j_1 j_2} \rangle$ by a special graph $\Gamma_{i_1 j_1 i_2 j_2}$ of exactly n_2 strands such that $\Gamma_{i_1 j_1 i_2 j_2}$ lies in $\operatorname{Int} T_{i_1 j_1}$, and in fact near $\langle p_{i_1 j_1 i_2} q_{i_1 j_1 j_2} \rangle$. We construct $\Gamma_{i_1 j_1 i_2 j_2}$ relative

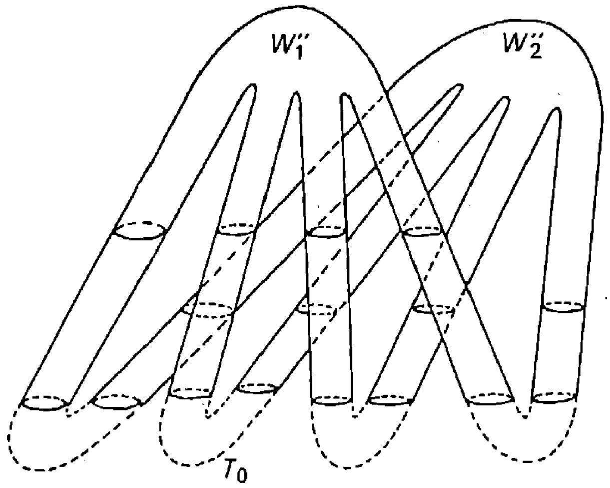

Fig. 6

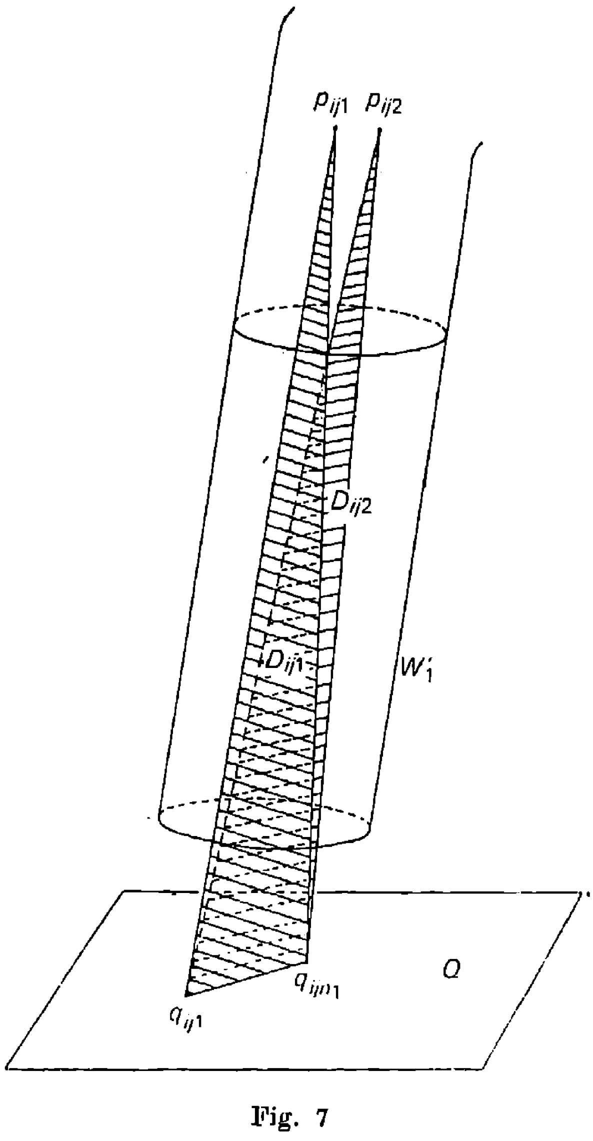

Fig. 7

to $\langle p_{i_1j_1i_2}q_{i_1j_1j_2}\rangle$, $T_{i_1j_1}$, and $\Gamma_{i_1j_1}$ just as $\Gamma_{i_1j_1}$ was constructed relative to $\langle p_{i_1}q_{j_1}\rangle$, T_0, and Γ_0 except that $\Gamma_{i_1j_1i_2j_2}$ has exactly n_2 strands. Then we define Γ_2 to be $\bigcup_{i_1=1}^{2}\bigcup_{j_1=1}^{n_0}\bigcup_{i_2=1}^{2}\bigcup_{j_2=1}^{n_1}\Gamma_{i_1j_1i_2j_2}$.

For each index $i_1j_1i_2j_2$, let $T_{i_1j_1i_2j_2}$ be a thin closed polyhedral tubular neighborhood of $\Gamma_{i_1j_1i_2j_2}$. It is to be true that for each index $i_1j_1i_2j_2$, (1) $T_{i_1j_1i_2j_2}$ lies in $\operatorname{Int}T_{i_1j_1}$ and hence in $\operatorname{Int}T_1$, and (2) if $k_1l_1k_2l_2$ is an index distinct from $i_1j_1i_2j_2$, $T_{i_1j_1i_2j_2}$ and $T_{k_1l_1k_2l_2}$ are disjoint. Let T_2 be $\bigcup_{i_1=1}^{2}\bigcup_{j_1=1}^{n_0}\bigcup_{i_2=1}^{2}\bigcup_{j_2=1}^{n_1}T_{i_1j_1i_2j_2}$. Notice that $T_2\subset\operatorname{Int}T_1$.

We assume that T's described above are constructed so that for each index i_1j_1, there exists two disjoint cubes-with-n_1-handles $W_{i_1j_11}$ and $W_{i_1j_12}$ in $\operatorname{Int}T_{i_1j_1}$ such that $\bigcup_{j=1}^{n_1}T_{i_1j_11j}\subset\operatorname{Int}W_{i_1j_11}$, $\bigcup_{j=1}^{n_1}T_{i_1j_12j}\subset\operatorname{Int}W_{i_1j_12}$, and $W_{i_1j_11}$ and $W_{i_1j_12}$ are as indicated in Figure 8.

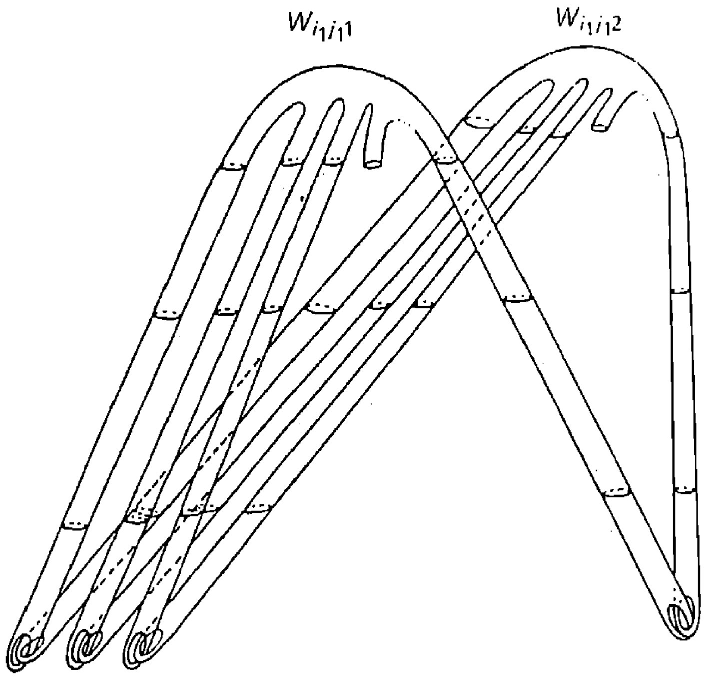

Fig. 8

Let this process be continued. We obtain a sequence $\Gamma_0,\Gamma_1,\Gamma_2,\ldots$ of finite graphs such that for each non-negative integer r, Γ_r is the union of mutually disjoint special graphs of exactly n_r strands. We also obtain a sequence $T_0,T_1,T_2,\ldots$ of compact 3-manifolds with boundary such that for each non-negative integer r, T_r is a thin closed polyhedral tubular neighborhood of Γ_r, and $T_{r+1}\subset\operatorname{Int}T_r$. We require that the T_r's get progressively so thin that each component of $\bigcap_{r=1}^{\infty}T_r$ is a straight arc.

Let G be the decomposition of E^3 such that the nondegenerate elements of G are precisely the components of $\bigcap_{r=1}^{\infty} T_r$. It follows from [11], Chapter V, Theorem 20, that G is upper semicontinuous. Thus G is an upper semicontinuous decomposition of E^3 into straight arcs and singletons.

Further, each nondegenerate element of G is a straight arc with one end on P and the other on Q. G has a Cantor set of nondegenerate elements, or, equivalently, $\Pr[H_G]$ is a Cantor set. Any plane parallel to P and intersecting H_G intersects H_G in a Cantor set.

4. Preliminaries to the main result

We find it convenient to introduce some notation for indexes. 0 is a *stage* 0 index, and is the only stage 0 index. Suppose m is a positive integer. The statement that α is a *stage m index* means that there exist $2m$ integers $i_1, j_1, i_2, j_2, \ldots, i_m$, and j_m such that (1) if $k = 1, 2, \ldots,$ or m, then $i_k = 1$ or 2, and $j_k = 1, 2, \ldots,$ or n_{k-1}, and (2) $\alpha = i_1 j_1 i_2 j_2 \ldots i_m j_m$. The statement that α is an *index* means that for some non-negative integer m, α is a stage m index.

Suppose m is a non-negative integer and α is a stage m index. Suppose $i = 1$ or 2, and $j = 1, 2, \ldots,$ or n_m. If $m = 0$, αi and αij denote the indexes i and ij, respectively. Suppose $m > 0$ and $\alpha = i_1 j_1 i_2 j_2 \ldots i_m j_m$. Then αi denotes the index $i_1 j_1 \ldots i_m j_m i$, and αij denotes the index $i_1 j_1 \ldots i_m j_m ij$.

Suppose A and B are two horizontal planes between P and Q, with A above B, such that each component of $(A \cup B) \cap T_0$ is a disc, and A and B lie as shown in Figure 9.

If β is an arc and k is a non-negative integer, then β has *oscillation* k if and only if there exist k points of $A \cup B$ on β such that no two adjacent ones (on β) belong to the same one of A and B, and (2) if l is an integer greater than k, (1) does not hold for l.

If J is a simple closed curve and k is a non-negative integer, then J has *oscillation* k if and only if (1) J contains an arc of oscillation k and (2) if l is an integer greater than k, J contains no arc of oscillation l. (These definitions differ slightly from those of Bing's, on page 450 of [6]).

Suppose α is an index and h is a homeomorphism from E^3 onto E^3. The statement that γ is a *homotopy-$h[\Gamma_\alpha]$ in $h[T_\alpha]$* means that there exist (1) a homeomorphism f from Γ_α into $h[T_\alpha]$ such that $\gamma = f[\Gamma_\alpha]$, and (2) a homotopy F from $\Gamma_\alpha \times [0, 1]$ into $h[T_\alpha]$ such that $F_0 = h|\Gamma_\alpha$ and $F_1 = f$. If α is an index, then γ is a *homotopy-Γ_α in T_α* if and only if γ is a homotopy-$h[\Gamma_\alpha]$ in $h[T_\alpha]$ where h is the identity map from E^3 onto E^3.

Suppose γ is the image, under some homeomorphism into E^3, of a special graph. Then γ has *Property* X if and only if either (1) some simple closed curve in γ has oscillation at least 6, or (2) there is a θ-curve τ in γ such that each simple closed curve in τ has oscillation at least 4. The definition of Property X is essentially due to Bing ([6], p. 450).

5. The main result

LEMMA 1. *If Γ is any homotopy-Γ_0 in T_0, then Γ has Property* X.

Proof. Suppose Γ is a homotopy-Γ_0 in T_0. We shall prove that each simple closed curve in Γ has oscillation at least 4. It will then follow that Γ has Property X.

Since Γ is a homotopy-Γ_0 in T_0, there is a homotopy F from $\Gamma_0 \times \times [0,1]$ into T_0 such that F_0 is the identity map on Γ_0, F_1 is a homeomorphism, and $F_1[\Gamma_0] = \Gamma$. Suppose J is a simple closed curve in Γ. Then since F_1 is a homeomorphism, there is a simple closed curve J_0 in Γ_0 such that $J = F_1[J_0]$.

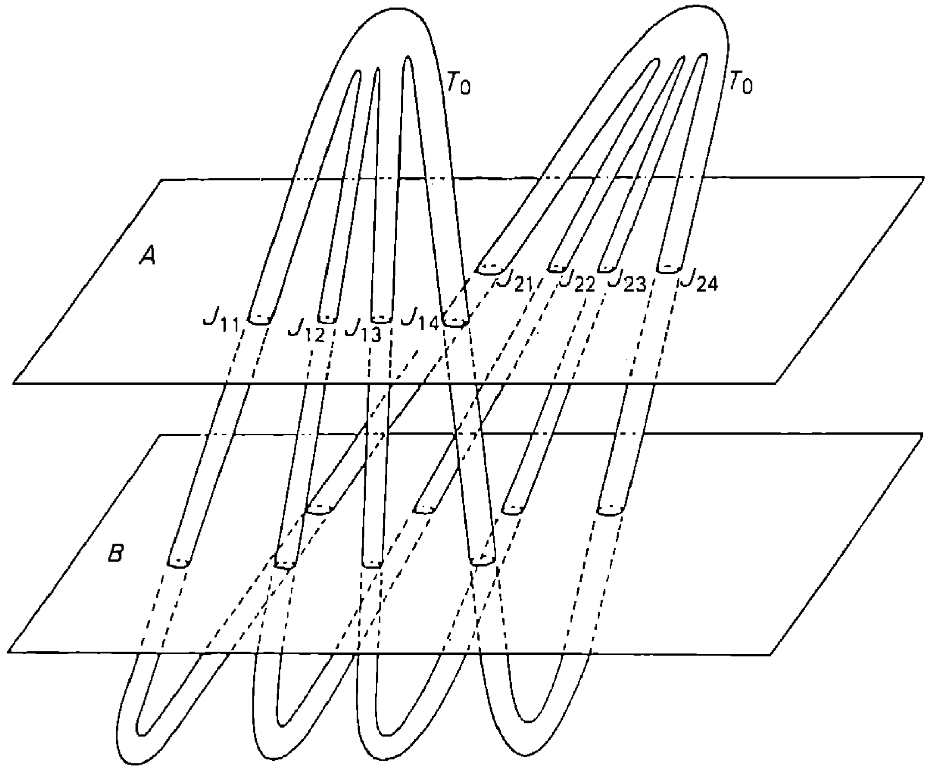

Fig. 9

Let $J_{11}, J_{12}, \ldots,$ and J_{24} denote the simple closed curves indicated in Figure 9, and for each index ij, let J'_{ij} denote a simple closed curve in A obtained by swelling J_{ij} outwards slightly. For each index ij, let Δ_{ij}

denote the disc on A bounded by J'_{ij}; it is to be true that distinct Δ's are disjoint.

There is an integer j such that $j = 1, 2, 3$, or 4, and $\langle p_1 q_j \rangle \cup \langle p_2 q_j \rangle \subset J_0$. Then J_0 links both J'_{1j} and J'_{2j}. Since $J = F_1[J_0]$ and F is into T_0, J links both J'_{1j} and J'_{2j}. Hence J contains a point x of Δ_{1j} and a point y of Δ_{2j}. Now $T_0 \cap B$ separates x from y in T_0, and thus each arc of J from x to y intersects B. It follows easily that J has oscillation at least 4. Hence, Γ has Property X.

LEMMA 2. *If h is any homeomorphism from E^3 onto E^3 such that $h|E^3-\operatorname{Int}T_0$ is the identity on $E^3-\operatorname{Int}T_0$, and Γ is any homotopy-$h[\Gamma_0]$ in $h[T_0]$, then Γ has Property* X.

Proof. There is a homotopy H from $\Gamma_0 \times [0, 1]$ into T_0 such that H_0 is the identity on Γ_0, and H_1 is an embedding of Γ_0 into $\operatorname{Bd}T_0$. (This is easy to see if T_0 is first "flattened".) Let F be the homotopy from $\Gamma_0 \times [0, 1]$ into T_0 defined as follows: (1) If $0 \leqslant t \leqslant 1/2$ and $x \in \Gamma_0$, $F(x, t) = H(x, 2t)$. (2) If $1/2 \leqslant t \leqslant 1$ and $x \in \Gamma_0$, then $F(x, t) = hH(x, 2(1-t))$. Since $h|\operatorname{Bd}T_0$ is the identity on $\operatorname{Bd}T$, it follows that F is well-defined and continuous. Clearly F_0 is the identity on Γ_0, and $H_1 = h|\Gamma_0$. Since $h[T_0] = T_0$, it follows that $h[\Gamma_0]$ is a homotopy-Γ_0 in T_0.

Since Γ is a homotopy-$h[\Gamma_0]$ in $h[T_0]$, it follows easily that Γ is a homotopy-Γ_0 in T_0. Then by Lemma 1, Γ has Property X.

The following is the main lemma of the paper. Its proof is given in sections 6 through 15.

LEMMA 3. *Suppose m is a non-negative integer, a is a stage m index, h is a homeomorphism from E^3 onto E^3, and each homotopy-$h[\Gamma_a]$ in $h[T_a]$ has Property* X. *Then there exist integers i and j such that* (1) $i = 1$ *or* 2, *and $j = 1, 2, \ldots$, or n_m, and* (2) *if Γ is any homotopy-$h[\Gamma_{aij}]$ in $h[T_{aij}]$, then Γ has Property* X.

LEMMA 4. *If h is any homeomorphism from E^3 onto E^3 such that $h|E^3-\operatorname{Int}T_0$ is the identity on $E^3-\operatorname{Int}T_0$, then there is an element g of G such that $h[g]$ intersects both A and B.*

Proof. By Lemma 2, if Λ_0 is any homotopy-$h[\Gamma_0]$ in $h[T_0]$, then Λ_0 has Property X. By induction and Lemma 3, there exists a sequence $i_1, j_1, i_2, j_2, \ldots$ such that for each positive integer k, (1) $i_k = 1$ or 2, and $j_k = 1, 2, \ldots$, or n_{k-1}, and (2) if Λ_k is any homotopy-$h[\Gamma_{i_1 j_1 \ldots i_k j_k}]$ in $h[T_{i_1 j_1 \ldots i_k j_k}]$, then Λ_k has Property X.

For each positive integer k, let a_k denote $i_1 j_1 \ldots i_k j_k$. It follows that for each k, $h[T_{a_k}]$ intersects both A and B. Let g be $\bigcap_{k=1}^{\infty} T_{a_k}$. Then g is an element of G and $h[g]$ intersects both A and B.

THEOREM. *E^3/G is not homeomorphic to E^3.*

Proof. Suppose that E^3/G is homeomorphic to E^3. By [1], Theorem 2, there exists a homeomorphism h from E^3 onto E^3 such that (1) $h|E^3-\operatorname{Int}T_0$ is the identity on $E^3-\operatorname{Int}T_0$, and (2) if $g\epsilon G$, $(\operatorname{diam}h[g])$ is less than the distance between the planes A and B. In particular, if $g\epsilon G$, $h[g]$ does not intersect both A and B. But by Lemma 4, there is an element g_0 of G such that $h[g_0]$ intersects both A and B. This is a contradiction, and the theorem is established.

6. Lemmas on Property X

The remainder of the paper is devoted to a proof of Lemma 3.

Let us recall the hypothesis and conclusion of Lemma 3. We suppose that h is a homeomorphism from E^3 onto E^3, m is a non-negative integer, a is a stage m index, and each homotopy-$h[\Gamma_a]$ of $h[T_a]$ has Property X. Lemma 3 then states that for some i and j such that $i=1$ or 2, and $j=1,2,\ldots,$ or n_m, each homotopy-$h[\Gamma_{aij}]$ in $h[T_{aij}]$ has Property X. Our proof of Lemma 3 will be an indirect one, and we shall assume that for each such i and j there is a homotopy-$h[\Gamma_{aij}]$ γ_{aij} in $h[T_{aij}]$ such that γ_{aij} does not have Property X. The proof of Lemma 3 can be divided into three main steps.

The first step is to construct, in each $h[T_{aij}]$, a simple closed curve J_{aij} having certain properties relative to A and B. Each J_{aij} is constructed from γ_{aij}. This step is carried out in Section 6.

The second step is to use the J's from the first step to construct homotopy centerlines of $h[W_{a1}]$ and $h[W_{a2}]$ with certain properties relative to A and B. This step is complicated, and takes up Sections 7–13.

The third step is to use the homotopy centerlines of $h[W_{a1}]$ and $h[W_{a2}]$ constructed in the second step to construct a homotopy-$h[\Gamma_a]$ Γ of $h[T_a]$ such that Γ does not have Property X. This is done in Section 14. In Section 15, we prove Lemma 3.

Now we describe some assumptions we shall make concerning the construction of G. Suppose m is a non-negative integer and a is a stage m index. If Γ_a is a component of Γ_m, we shall assume that the following hold: (1) The line $p_{a1}p_{a2}$ runs from left front to right back, and p_{a1} is to the left of p_{a2}. (2) The line $q_{a1}q_{a2}$ runs from left front to right back, and the points $q_{a1}, q_{a2}, \ldots,$ and q_{an_m} occur on this line in the order $q_{a1}q_{a2}\ldots q_{an_m}$ with q_{a1} leftmost and q_{an_m} rightmost. See Figure 10.

We shall number the strands of Γ_a. If $k=1,2,\ldots,$ or n_m, then the kth strand of Γ_a is the strand containing q_{ak}. Thus the strands of Γ_a are numbered from left to right.

We have the following lemma. Figure 11 illustrates the hypotheses of the lemma.

LEMMA 5. *Suppose m is a non-negative integer, α is a stage m index, $i = 1$ or 2, and if $j = 1, 2, \ldots,$ or n_m, $k_j = 1$ or 2. Suppose that each of s and t is $1, 2, \ldots,$ or n_{m+1} with $s < t$. For each integer j such that $j = 1, 2, \ldots,$ or n_m, let Λ_j denote the union of $\langle p_{\alpha ijk_j} q_{\alpha ijs}\rangle$, $\langle p_{\alpha ijk_j} q_{\alpha ijt}\rangle$, and $\langle q_{\alpha ijs} q_{\alpha ijt}\rangle$. Then any two distinct ones of $\Lambda_1, \Lambda_2, \ldots,$ and Λ_{n_m} are linked.*

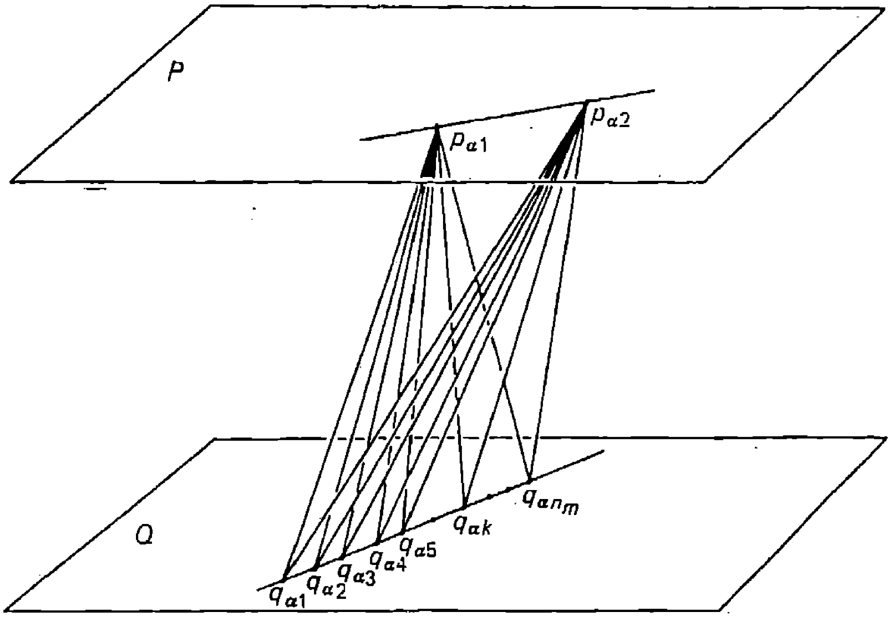

Fig. 10

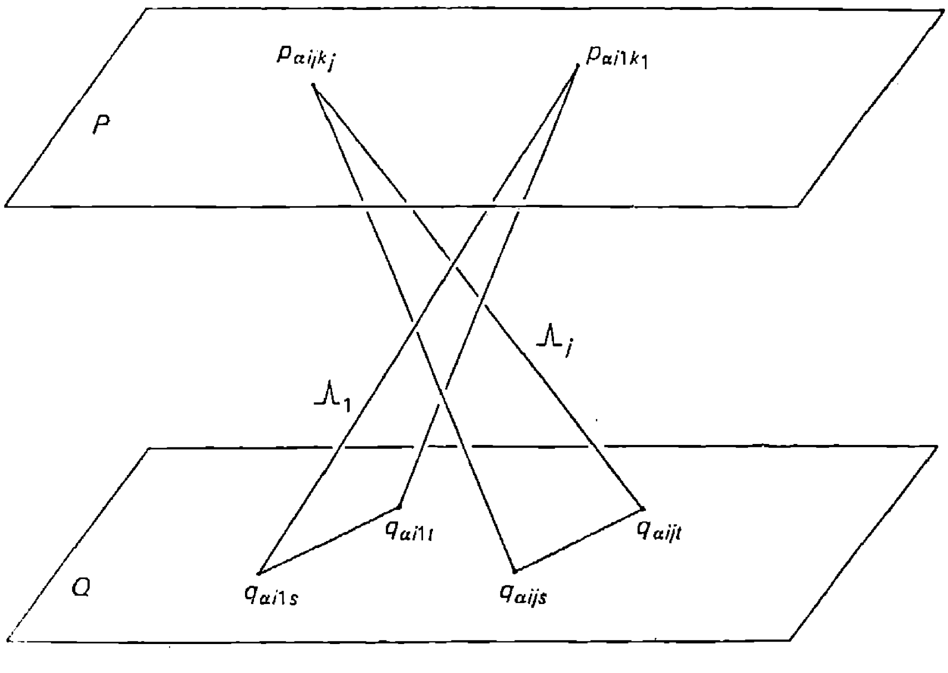

Fig. 11

Proof. This follows directly from the construction of the Γ_{aij}'s as described in Section 3; see the conditions there concerning the entwining of the Γ_{aij}'s near the p's.

For purposes of later application, we remark that the choice of i above corresponds to choosing one of W_{a1} and W_{a2}. It should be emphasized that the linking mentioned is independent of the choices of the k_j's.

Now we shall study embeddings of special graphs. If γ is an image under an embedding h into E^3 of a special graph γ_0, then by a *strand* of γ is meant the image under h of a strand of γ_0. Clearly the strands of γ are independent of the particular embedding chosen.

LEMMA 6. *Suppose γ_0 is a special graph, h is an embedding of γ_0 in E^3, $\gamma = h[\gamma_0]$, and γ does not have Property* X. *Then there exist at most two strands s_1 and s_2 of γ such that if s is any strand of γ distinct from both s_1 and s_2, then s has oscillation at most* 2.

Suppose β is an arc in E^3. Then β is *oriented* provided one of the two traversals of β has been designated positive and the other negative.

Suppose β is an oriented arc in E^3; note that β has finite oscillation. The *pattern* of β is (1) the empty set $\emptyset$ if and only if β has oscillation 0, (2) a k-tuple $(C_1, C_2, \ldots, C_k)$ if and only if β has oscillation k where $k > 0$ and (a) each of $C_1, C_2, \ldots,$ and C_k is either A or B, (b) adjacent ones of $C_1, C_2, \ldots,$ and C_k are different, and (c) C_1 is A if the first point (in the positive direction of β) of $(A \cup B) \cap \beta$ belongs to A, and C_1 is B if this first point belongs to B. In the case of patterns, we shall write $C_1C_2 \ldots C_k$ in place of $(C_1, C_2, \ldots, C_k)$.

Thus the possible patterns are $\emptyset, A, B, AB, BA, ABA, BAB, \ldots$ If the arc β has non-zero oscillation, the number of symbols in the pattern is the oscillation of β.

Suppose γ is the image under an embedding h into E^3 of some special graph γ_0. Suppose p and q are the branch points of γ_0; then $h(p)$ and $h(q)$ are the *branch points* of γ. The statement that γ is *oriented* means that (1) one of $h(p)$ and $h(q)$ is selected as *top* point and the other as *bottom*, and (2) *each* strand of γ is oriented so that positive is from top to bottom. It is important that the strands of γ be oriented consistently. If γ is oriented and s is a strand of γ, the *pattern* of s is the pattern relative to the orientation of s described above.

LEMMA 7. *Suppose k is an even integer, γ is the image under an embedding into E^3 of a special graph, γ has at least k strands, and γ is oriented. If γ does not have Property* X, *then among any family K of exactly k distinct strands of γ, there exist at least $1/2(k-2)$ strands of K such that* (1) *each such strands has oscillation at most* 2, *and* (2) *any two such strands of oscillation* 2 *have the same pattern.*

LEMMA 8. *Suppose a simple closed curve J on E^3 is the union of two arcs β and β' from a point a to a point b, both oriented positively from a to b. Suppose* (1) *each of β and β' has oscillation at most 2, and* (2) *if they both have oscillation 2, they have the same pattern. Then if x and y are any two points of J, there is an arc on J from x to y and of oscillation at most 2.*

Simple examples show the necessity for condition (2) in the hypothesis.

The following lemma is the main result of this section. Its proof reveals a reason for selecting the μ's as we did in Section 3.

LEMMA 9. *Suppose h is a homeomorphism from E^3 onto E^3, m is a non-negative integer, a is a stage m index, $i = 1$ or 2, and if $j = 1, 2, \ldots,$ or n_m, γ_{aij} is an oriented homotopy-$h[\Gamma_{aij}]$ in $h[T_{aij}]$ such that γ_{aij} does not have Property X. Then there exist integers k_{ai} and l_{ai} such that $1 \leqslant k_{ai} < l_{ai} \leqslant n_{m+1}$, and if $j = 1, 2, \ldots,$ or n_m, the strands of γ_{aij} numbered k_{ai} and l_{ai} both have oscillation at most 2, and if they both have oscillation 2, they have the same pattern.*

Proof. Consider γ_{ain_m}; it has exactly n_{m+1} strands. Then since it does not have Property X, by Lemma 7, there exist at least $1/2(n_{m+1}-2)$ strands of γ_{ain_m} such that (1) each such strand has oscillation at most 2 and (2) any two such strands of oscillation 2 have the same pattern. Hence there exist exactly $1/2(n_{m+1}-2)$ strands of γ_{ain_m} with properties (1) and (2) and we select exactly $1/2(n_{m+1}-2)$ such strands. Let K_{n_m} denote the family of strands of γ_{ain_m} selected. Recall that $n_{m+1} = \mu(n_m)$, and hence $1/2(n_{m+1}-2) = 1/2\big(\mu(n)_m-2\big) = \mu(n_m-1)$. Thus K_{n_m} has exactly $\mu(n_m-1)$ elements.

The strands of K_{n_m} have certain numbers as described earlier in this section, and we let K'_{n_m} denote the family of strands of $\gamma_{ai}(n_m-1)$ having the same numbers as the strands of K_{n_m}. By Lemma 7, there exist at least $1/2[\mu(n_m-1)-2]$ strands of K'_{n_m} such that (1) each such strand has oscillation at most 2 and (2) any two strands of oscillation 2 have the same pattern. Hence there exists a family $K_{(n_m-1)}$ of exactly $1/2[\mu(n_m-1)-2]$ strands of K'_{n_m} having properties (1) and (2). Since $1/2[\mu(n_m-1)-2] = \mu(n_m-2)$, $K_{(n_m-1)}$ has exactly $\mu(n_m-2)$ members.

Let $K'_{(n_m-1)}$ denote the family of strands of $\gamma_{ai(n_m-2)}$ having the same numbers as the strands of $K_{(n_m-1)}$. $K'_{(n_m-1)}$ has exactly $\mu(n_m-2)$ elements. We may repeat the argument above and select a family $K_{(n_m-2)}$ of exactly $\mu(n_m-3)$ strands of $K'_{(n_m-1)}$ having properties (1) and (2) described above.

Continue this process. It terminates with the selection of a family K_1 of exactly $\mu(0)$ strands of γ_{ai1} having properties (1) and (2) described above. There results a finite sequence $K_{n_m}, K'_{n_m}, K_{(n_m-1)}, K'_{(n_m-1)}, \ldots$ $\ldots, K_2, K'_2$, and K_1, such that (a) if $j = 1, 2, \ldots,$ or n_m, K_j is a family of strands of γ_{aij}, and if $j > 1$, K'_j is the family consisting of the strands of $\gamma_{ai(j-1)}$ with the same numbers as the strands of K_j.

Now $\mu(0) = 2$, and hence we have selected some two strands of γ_{ai1}. Suppose these strands are numbered k_{ai} and l_{ai} with $k_{ai} < l_{ai}$. It is clear that if $j = 1, 2, \ldots,$ or n_m, the strands of γ_{aij} numbered k_{ai} and l_{ai} belong to K_j. Hence for each such j, the strands of γ_{aij} numbered k_{ai} and l_{ai} both have oscillation at most 2, and if both have oscillation 2, they have the same pattern. This establishes Lemma 9.

7. Construction of certain open sets

We are ready to begin proving the lemmas for the second step in the proof of Lemma 3. These deal mainly with construction of homotopy centerlines for the $h[W_{ai}]$'s. The main result along these lines is Lemma 18, which is established in Sections 12 and 13.

Certain preliminary results are established in Sections 7–11.

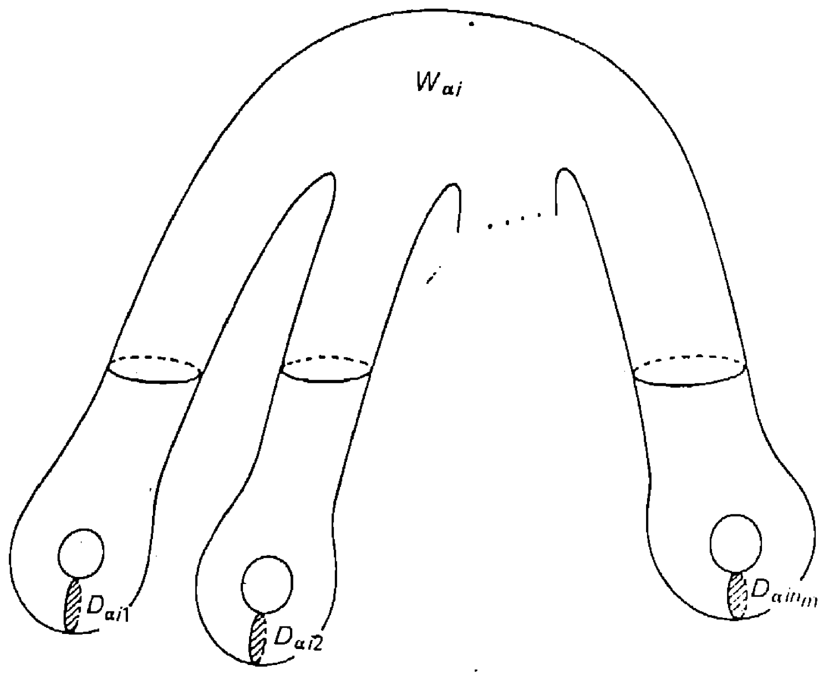

Fig. 12

Suppose that m is a non-negative integer, a is a stage m index, $i = 1$ or 2, and k_{ai} and l_{ai} are integers such that $1 \leqslant k_{ai} < l_{ai} \leqslant n_m$. If $j = 1, 2, \ldots,$ or n_m, let K_{aij} denote the union of strands k_{ai} and l_{ai} of Γ_{aij}; K_{aij} is a simple closed curve in T_{aij}. (The notation chosen does not indicate the dependence of the J's on k_{ai} and l_{ai}. In the situation to be considered, we shall work with a fixed k_{ai} and l_{ai}.)

We shall construct some connected open sets in $h[W_{ai}]$. For this purpose, we consider a covering space. Let Ω_{ai} denote an open set obtained by a very slight enlargement of W_{ai}. Let Ω^*_{ai} denote the universal covering space of Ω_{ai}, and let Φ_{ai} denote the projection mapping from Ω^*_{ai} onto Ω_{ai}. Let W^*_{ai} denote $\Phi^{-1}_{ai}[W_{ai}]$; W^*_{ai} is homeomorphic to the universal covering space of W_{ai}. Note that Ω^*_{ai} is homeomorphic to E^3, and so is Int W^*_{ai}.

Let $D_{ai1}, D_{ai2}, \ldots,$ and D_{ain_m} denote mutually disjoint polyhedral discs in W_{ai} as shown in Figure 12; if $j = 1, 2, \ldots,$ or n_m, $\mathrm{Bd}D_{aij} \subset \mathrm{Bd}W_{ai}$, $\mathrm{Int}D_{aij} \subset \mathrm{Int}\,W_{ai}$, and $(\mathrm{Int}\,W_{ai}) - \bigcup_{j=1}^{n_m} D_{aij}$ is an open 3-cell C_{ai}. Let C^*_{ai} be an open 3-cell covering C_{ai} exactly once.

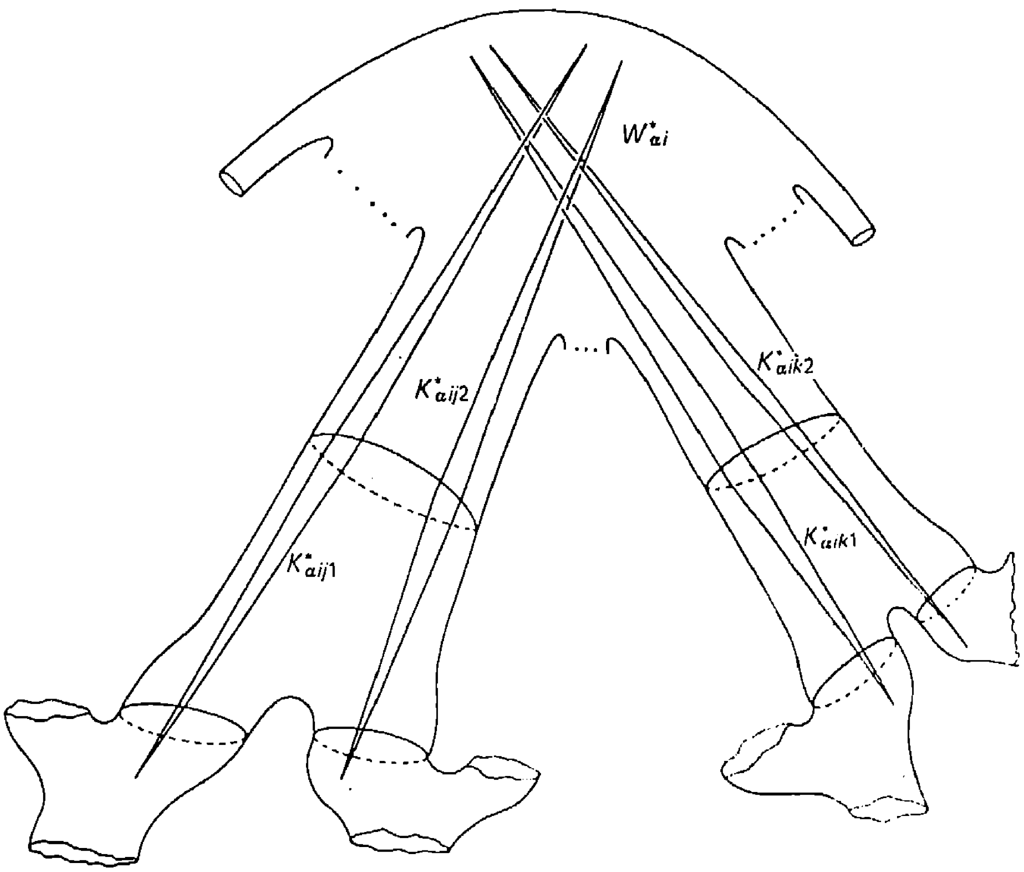

Fig. 13

If $j = 1, 2, \ldots,$ or n_m, let p^*_{aij1} and p^*_{aij2} denote the points of $\Phi^{-1}_{ai}[p_{aij1}]$ and $\Phi^{-1}_{ai}[p_{aij2}]$, respectively, lying in C^*_{ai}. Since K_{aij} is homotopic to 0 in W_{ai}, then K_{aij} lifts to a simple closed curve in W^*_{ai}. Let K^*_{aij1} and K^*_{aij2} denote the copies of K_{aij} in W^*_{ai} that contain p^*_{aij1} and p^*_{aij2}, respectively. If $r = 1$ or 2, K^*_{aijr} covers K_{aij} exactly once. See Figure 13.

The K^*'s have the following important property: Suppose k and l are distinct integers, $1 \leqslant k \leqslant n_m$, $1 \leqslant l \leqslant n_m$, $r_k = 1$ or 2, and $r_l = 1$

or 2. Then $K^*_{aikr_k}$ and $K^*_{ailr_l}$ are linked in Ω^*_{ai}. This follows immediately from Lemma 5. See Figure 13.

If $j = 1, 2, \ldots,$ or n_m, T_{aij} lies in a 3-cell in W_{ai}, so T_{aij} lifts homeomorphically into W^*_{ai}. Let T^*_{aij1} and T^*_{aij2} denote the copies of T_{aij} in W^*_{ai} that contain K^*_{aij1} and K^*_{aij2} respectively.

Now suppose h is a homeomorphism from E^3 onto E^3, and if $j = 1, 2, \ldots,$ or n_m, J_{aij} is a simple closed curve in $h[T_{aij}]$ homotopic in $h[T_{aij}]$ to $h[K_{aij}]$. Let H_{aij} be a homotopy in $h[T_{aij}]$ between $h[K_{aij}]$ and J_{aij}. Since K_{aij} is homotopic to 0 in T_{aij}, it follows that $h^{-1}[J_{aij}]$ is homotopic to 0 in T_{aij}, and hence $h^{-1}[J_{aij}]$ lifts to a simple closed curve J^*_{aij1} in T^*_{aij1}. Now the homotopy $h^{-1} \circ H_{aij}$ lifts to a homotopy in T^*_{aij1} between K^*_{aij1} and J^*_{aij1}. Similarly, $h[^{-1}J_{aij}]$ lifts to a simple closed curve J^*_{aij2} in T^*_{aij2}, homotopic in T^*_{aij2} to K^*_{aij2}.

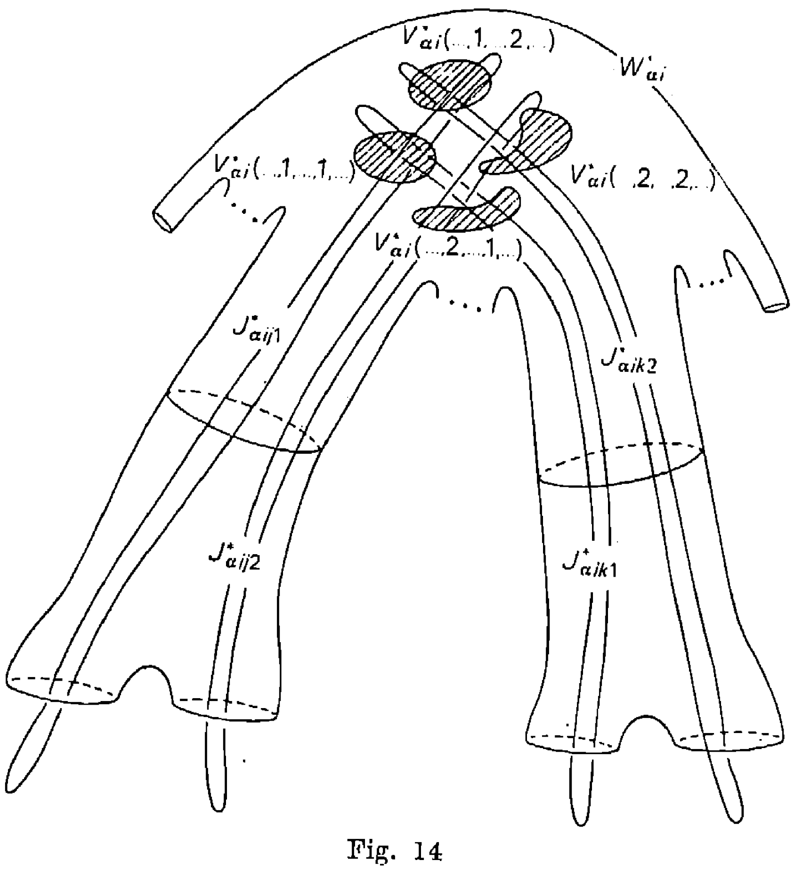

Fig. 14

Suppose k and l are distinct integers, $1 \leqslant k \leqslant n_m$, $1 \leqslant l \leqslant n_m$, $r_k = 1$ or 2, and $r_l = 1$ or 2. Then (1) $T^*_{aikr_k}$ and $T^*_{ailr_l}$ are disjoint, (2) $J^*_{aikr_k}$ and $K^*_{aikr_k}$ are homotopic in $T^*_{aikr_k}$ and $J^*_{ailr_l}$ and $K^*_{ailr_l}$ are homotopic in $T^*_{ailr_l}$, and (3) $K^*_{aikr_k}$ and $K^*_{ailr_l}$ are linked in Ω^*_{ai}. Hence $J^*_{aikr_k}$ and $J^*_{ailr_l}$ are linked in Ω^*_{ai}.

Since $\operatorname{Int} W^*_{ai}$ is an open 3-cell, there is a 3-cell E^*_{ai} in $\operatorname{Int} W^*_{ai}$ such that $\operatorname{Int} E^*_{ai}$ contains $\bigcup_{j=1}^{n_m} (T^*_{aij1} \cup T^*_{aij2})$; hence $\operatorname{Int} E^*_{ai}$ contains each J^*_{aij1} and J^*_{aij2}.

Let R be the set of all n_m-tuples of 1's and 2's. If $r \in R$ and $j = 1, 2, \ldots,$ or n_m, then r_j will denote the jth co-ordinate of r.

Corresponding to each element r of R, we shall construct a connected open set $V^*_{ai}(r)$ in Ω^*_{ai}. If k and l are distinct integers, $1 \leqslant k \leqslant n_m$, and $1 \leqslant l \leqslant n_m$, $J^*_{aikr_k}$ and $J^*_{ailr_l}$ are linked in Ω^*_{ai}. Further, $\Phi^{-1}_{ai}[h^{-1}[A \cup B]]$ is a 2-manifold embedded as a closed set in Ω^*_{ai}. Hence by Theorems 3 and 5 of [5], there is a connected open set $V^*_{ai}(r)$ in $(\operatorname{Int} E^*_{ai}) - \Phi^{-1}_{ai}[h^{-1}[A \cup B]]$ such that if $j = 1, 2, \ldots,$ or n_m, $V^*_{ai}(r)$ intersects $J^*_{aijr_j}$. See Figure 14.

If $r \in R$, let $V_{ai}(r)$ denote $h\Phi_{ai}[V^*_{ai}(r)]$. Then $V_{ai}(r)$ is a connected open set in $h[W_{ai}]$, $V_{ai}(r)$ is disjoint from $A \cup B$, and if $j = 1, 2, \ldots,$ or n_m, $V_{ai}(r)$ intersects J_{aij}.

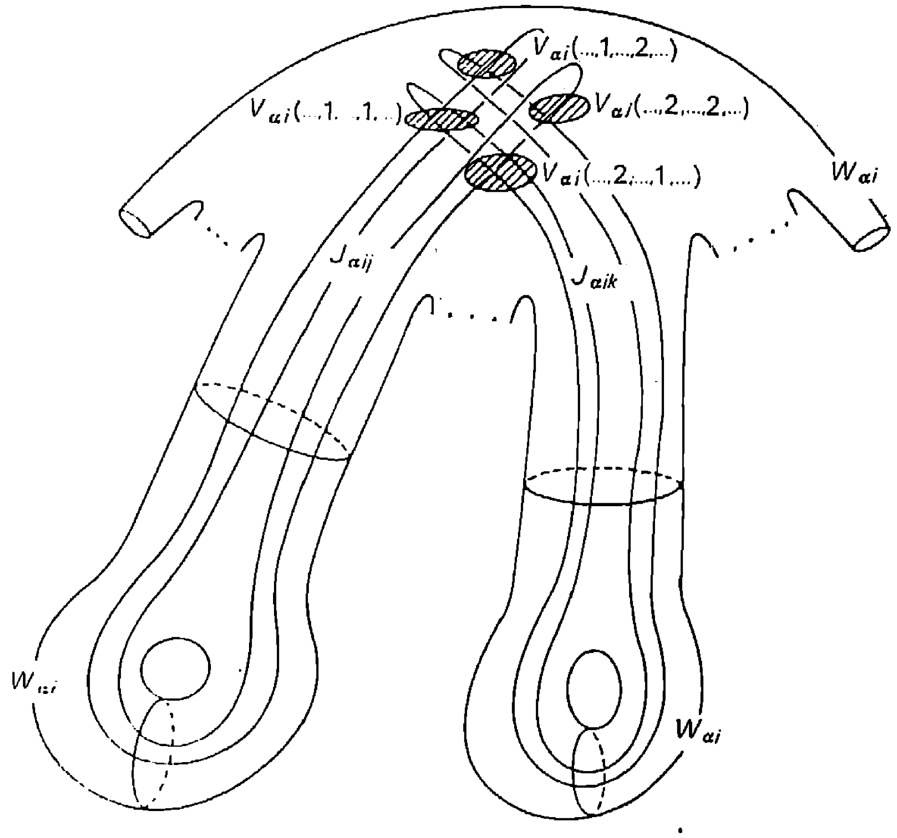

Fig. 15

It may help the reader in the sequel to think of the V_{ai}'s in the following manner: We regard each J_{aij} as having two upper "loops", a "left" one and a "right" one. If $r \in R$ and $r_j = 1$, then we regard $V_{ai}(r)$ as hitting the left upper loop of J_{aij}, and if $r_j = 2$, we regard $V_{ai}(r)$ as hitting the right upper loop of J_{aij}. See Figure 15.

8. Constructing homotopy centerlines

Suppose m is a non-negative integer, a is a stage m index, and $i = 1$ or 2. Then there is a bouquet of circles L_0 such that W_{ai} can be regarded as a tubular neighborhood of L_0. Each such L_0 is a *centerline* of W_{ai}. The point common to the circles of L_0 is the *centerpoint* of L_0. By a *loop* of L_0 is meant a simple closed curve of L_0.

Suppose that h is a homeomorphism from E^3 onto E^3. A subset L of $h[W_{ai}]$ is a *homotopy-centerline* of $h[W_{ai}]$ if and only if there exist (1) an embedding g of a centerline L_0 of W_{ai} into $h[W_{ai}]$ such that $g[L_0] = L$ and (2) a homotopy H from $L_0 \times [0, 1]$ into $h[W_{ai}]$ such that $H_0 = h|L_0$ and $H_1 = g$. The image $g(p)$ under g of the centerpoint p of L_0 is the *centerpoint* of L. By a *loop* of L is meant the image under g of a loop of L_0.

In this and the next five sections, we shall show how to construct homotopy-centerlines of $h[W_{ai}]$ having certain properties relative to A and B. In these sections, we shall assume the hypotheses stated below.

THE HYPOTHESIS OF SECTION 8. Suppose that h is a homeomorphism from E^3 onto E^3, m is a non-negative integer, a is a stage m index, and $i = 1$ or 2. Suppose also that k_{ai} and l_{ai} are integers such that (1) $1 \leqslant k_{ai} < l_{ai} \leqslant n_{m+1}$ and (2) if $j = 1, 2, \ldots,$ or n_m, K_{aij} denotes the union of the k_{ai}th and l_{ai}th strands of Γ_{aij}. Suppose that if $j = 1, 2, \ldots,$ or n_m, J_{aij} is a simple closed curve in $h[T_{aij}]$ homotopic in $h[T_{aij}]$ to $h[K_{aij}]$ and such that if x and y are any two points of J_{aij}, there is an arc on J_{aij} from x to y and of oscillation at most 2.

Throughout Sections 8–13, R will have the same meaning as in Section 7. Relative to the J_{aij}'s specified as above, there exists, for each r of R, a connected open set $V_{ai}(r)$ having properties described in Section 7. We retain the notation of Section 7.

Suppose that J is a simple closed curve in E^3 and p is a point of J not on $A \cup B$. If n is a non-negative integer, then the *p-based oscillation of* J is n if and only if for any two points x and y of J distinct from p and such that the arc xpy of J misses $A \cup B$, the arc xy of J not containing p has oscillation n.

LEMMA 10. *If J is a simple closed curve in E^3, p is a point of J, and p is not on $A \cup B$, then J has p-based oscillation at most 2 if and only if for any point x of $J - \{p\}$, there is an arc on J from x to p and of oscillation at most one.*

The statement that $h[W_{ai}]$ has *property* I means that there is a homotopy centerline L of $h[W_{ai}]$ such that if p is the centerpoint of L, each loop of L has p-based oscillation at most 2.

Suppose r and s are elements of R, and β is an arc in $h[W_{ai}]$. Then β is *admissible from $V_{ai}(r)$ to $V_{ai}(s)$* if and only if there exists an arc β^*

in W^*_{ai} such that (1) one endpoint of β^* is in $V^*_{ai}(r)$ and the other is in $V^*_{ai}(s)$, and (2) $\beta = h\Phi_{ai}[\beta^*]$.

Suppose x is 1 or 2. Then $\hat{x}$ denotes the one of 1 and 2 distinct from x. Suppose $r \in R$, and suppose $r = (r_1, r_2, \ldots, r_{n_m})$. Then $\hat{r}$ denotes $(\hat{r}_1, \hat{r}_2, \ldots \ldots, \hat{r}_{n_m})$.

LEMMA 11. *Suppose that $r \in R$ and there is an arc β in $h[W_{ai}]$ admissible from $V_{ai}(r)$ to $V_{ai}(\hat{r})$ and having oscillation* 0. *Then $h[W_{ai}]$ has Property* I.

Proof. Let β^* be an arc in W^*_{ai} such that one endpoint x^* of β^* lies in $V^*_{ai}(r)$, the other y^* is in $V^*_{ai}(\hat{r})$, and $\beta = h\Phi_{ai}[\beta^*]$. If $j = 1, 2, \ldots,$ or n_m, let x^*_j be a point of $J^*_{aijr_j} \cap V^*_{ai}(r)$, let β^*_j denote an arc in $V^*_{ai}(r)$ from x^* to x^*_j, let y^*_j denote a point of $J^*_{aij\hat{r}_j} \cap V^*_{ai}(\hat{r})$, and let δ^*_j denote an arc in $V^*_{ai}(\hat{r})$ from y^* to y^*_j. We shall assume that for each such j, $x^*_j \neq y^*_j$. Let x and y denote $h\Phi_{ai}(x^*)$ and $h\Phi_{ai}(y^*)$, respectively, and for each j, let x_i, y_i, β_i, and δ_i denote $h\Phi_{ai}(x^*_j)$, $h\Phi_{ai}(y^*_j)$, $h\Phi_{ai}(\beta^*_j)$, and $h\Phi_{ai}[\delta^*_j]$.

If $j = 1, 2, \ldots,$ or n_m, there is an arc λ_j on J_{aij} from x_j to y_j and of oscillation at most 2. For each j, let L_j denote $\lambda_j \cup \beta \cup \beta_j \cup \delta_j$. We may assume that for each j, L_j is a simple closed curve and if $j \neq k$, $L_j \cap L_k = \{x\}$. Let L denote $\bigcup_{j=1}^{n_m} L_j$. It can be proved that L is a homotopy centerline of $h[W_{ai}]$ with centerpoint x.

Suppose $j = 1, 2, \ldots,$ or n_m. Then β is disjoint from $A \cup B$ by hypothesis. Further, $\beta_j \subset V_{ai}(r)$, $\delta_j \subset V_{ai}(\hat{r})$, and $V_{ai}(r)$ and $V_{ai}(\hat{r})$ both miss $A \cup B$. Since λ_j has oscillation at most 2, it is clear that L_j may be assumed to have x-based oscillation at most 2. The following proposition completes the proof of Lemma 11.

PROPOSITION 1. *L is a homotopy centerline of $h[W_{ai}]$.*

LEMMA 12. *Suppose that $r \in R$ and* (1) *there is an arc β_1 in $h[W_{ai}]$ admissible from $V_{ai}(r)$ to $V_{ai}(\hat{r})$ and having pattern A, and* (2) *there is an arc β_2 in $h[W_{ai}]$ admissible from $V_{ai}(r)$ to $V_{ai}(\hat{r})$ and having pattern B. Then $h[W_{ai}]$ has Property* I.

Proof. There is an arc β^*_1 in W^*_{ai} with one endpoint x^* in $V^*_{ai}(r)$, the other endpoint y^* in $V^*_{ai}(\hat{r})$, and such that $h\Phi_{ai}[\beta^*_1] = \beta_1$. Thus β^*_1 is disjoint from $\Phi^{-1}_{ai}h^{-1}[B]$. Similarly, there is an arc β^*_2 in W^*_{ai} with one endpoint u^* in $V^*_{ai}(r)$, the other endpoint v^* in $V^*_{ai}(\hat{r})$, and such that β^*_2 is disjoint from $\Phi^{-1}_{ai}h^{-1}[A]$. There exist an arc x^*u^* in $V^*_{ai}(r)$ from x^* to u^*, and an arc y^*v^* in $V^*_{ai}(\hat{r})$ from y^* to v^*. Then if λ^* denotes the path $x^*u^* \cup \beta^*_2 \cup v^*y^*$, λ^* is disjoint from $\Phi^{-1}_{ai}h^{-1}[A]$ and has endpoints x^* and y^*. Since W^*_{ai} is simply connected, λ^* is homotopic in W^*_{ai} to β^*_1 with fixed endpoints. By Theorem 6 of [6], there is a path β^* in W^*_{ai} from x^* to y^* and disjoint from $\Phi^{-1}_{ai}h^{-1}[A \cup B]$; we may assume $\Phi_{ai}[\beta^*]$ is an arc β. Clearly β is an arc in $h[W_{ai}]$ admissible from $V_{ai}(r)$ to $V_{ai}(\hat{r})$ and of oscillation 0. Thus Lemma 12 follows from Lemma 11.

LEMMA 13. *Suppose $r \in R$ and there is an arc β in $h[W_{ai}]$ admissible from $V_{ai}(r)$ to $V_{ai}(\hat{r})$ and having oscillation one. Then $h[W_{ai}]$ has Property* I.

Proof. If $j = 1, 2, \ldots,$ or n_m, let p_j and q_j denote points obtained as follows: There is a point p_j^* of $V_{ai}^*(r) \cap J_{aijr_j}^*$, and let p_j be $\Phi_{ai}(p_j^*)$. There is a point q_j^* of $V_{ai}^*(\hat{r}) \cap J_{aij\hat{r}_j}^*$, and let q_j be $\Phi_{ai}(q_j^*)$. For each j, orient each of the two arcs on J_{aij} from p_j to q_j so that the positive direction is from p_j to q_j. Let $k_1, k_2, \ldots,$ and k_t denote those values of j such that each arc of J_{aij} from p_j to q_j has pattern AB.

PROPOSITION 2. *If j is neither $k_1, k_2, \ldots,$ nor k_t, either* (1) *there is an arc of J_{aij} from p_j to q_j and of oscillation at most one, or* (2) *each arc of J_{aij} from p_j to q_j has pattern BA.*

We return to the proof of Lemma 13. Let s be the n_m-tuple defined as follows: (1) If j is neither $k_1, k_2, \ldots,$ nor k_t, let $s_j = r_j$; (2) if $j = k_1, k_2, \ldots,$ or k_t, let s_j be $\hat{r}_j$. For each j, let y_j^* be a point of $V_{ai}^*(s) \cap \cap J_{aijs_j}^*$, and let y_j denote $\Phi_{ai}(y_j^*)$.

Suppose β has pattern A. We consider several cases.

Case 1. There exist k_e and j_0 such that (1) j_0 is neither $k_1, k_2, \ldots,$ nor k_t, (2) some arc $p_{j_0}y_{j_0}$ on J_{aij_0} has pattern Ø or B, and (3) some arc $q_{k_e}y_{r_e}$ on J_{aik_e} has pattern Ø or B.

In this case, let λ_0 be an arc on J_{aij_0} from p_{j_0} to y_{j_0} of pattern Ø or B, and let λ_e be an arc on J_{aik_e} from p_{k_e} to y_{k_e} of pattern Ø or B. Let λ_0^* and λ_e^* be the arcs on $J_{aij_0r_{j_0}}^*$ and $J_{aik_e\hat{r}_{k_e}}^*$, respectively, that cover λ_0 and λ_e, respectively. Note that λ_0^* has endpoints $p_{j_e}^*$ and $y_{j_0}^*$, and λ_e^* has endpoints $p_{k_e}^*$ and $y_{k_e}^*$. There is an arc λ^* in $V_{ai}^*(s)$ joining $y_{j_0}^*$ and $y_{k_e}^*$. Then let μ denote $\Phi_{ai}[\lambda_0^* \cup \lambda^* \cup \lambda_e^*]$; we assume μ is an arc. Clearly μ is admissible from $V_{ai}(r)$ to $V_{ai}(\hat{r})$, and μ has pattern Ø or B. Then either Lemma 11 or Lemma 12 applies, and $h[W_{ai}]$ has Property I.

Case 2. There exists j_0 such that (1) j_0 is neither $k_1, k_2, \ldots,$ nor k_t and (2) some arc on J_{aij_0} from p_{j_0} to y_{j_0} has pattern Ø or B.

We assume that Case 1 does not hold. Now for each l, each arc of J_{aik_l} from p_{k_l} to q_{k_l} has pattern AB, and hence, in the direction from q_{k_l} to p_{k_l}, has pattern BA. For each l, let $y_{k_l}q_{k_l}$ denote the arc of J_{aik_l} from y_{k_l} to q_{k_l} not containing p_{k_l}, and oriented positively from y_{k_l} to q_{k_l}. It follows that for each l, $y_{k_l}q_{k_l}$ has pattern Ø, B, or AB. If Case 1 does not hold, then for each l, $y_{k_l}q_{k_l}$ has pattern AB. Additionally, if for each l, β_{k_l}' denotes the arc of J_{aik_l} from p_{k_l} to q_{k_l} containing y_{k_l} and β_{k_l} denotes the subarc $p_{k_l}y_{k_l}$ of β_{k_l}', then for each l, β_{k_l} has pattern Ø or A.

Let λ_0 be an arc of J_{aij_0} from p_{j_0} to y_{j_0} with pattern B. Let λ_0^* be an arc in $J_{aij_0r_{j_0}}^*$ covering λ_0; λ_0^* has endpoints $p_{j_0}^*$ and $y_{j_0}^*$. If $l = 1, 2, \ldots,$

or t, let $\lambda^*_{k_l}$ be an arc in $V^*_{ai}(r)$ from $p^*_{j_0}$ to $p^*_{k_l}$, and let $\mu^*_{k_l}$ be an arc in $V^*_{ai}(s)$ from $y^*_{j_0}$ to $y^*_{k_l}$. For each l, let L_{k_l} denote $\beta_{k_l} \cup h\Phi_{a_i}[\lambda^*_{k_l} \cup \mu^*_{k_l} \cup \lambda^*_0]$.

Suppose j is neither $k_1, k_2, \ldots,$ nor k_t. Let β_j denote an arc of J_{aij} from p_j to q_j. Let λ^*_j be an arc in $V^*_{ai}(r)$ from $p^*_{j_0}$ to p^*_j, and let μ^*_j be an arc in $V^*_{ai}(\hat{r})$ from $q^*_{j_0}$ to q^*_j. There is an arc β^* in W^*_{ai} such that one endpoint of β^* is in $V^*_{ai}(r)$, the other endpoint of β^* is in $V^*_{ai}(\hat{r})$, and $h\Phi_{ai}[\beta^*] = \beta$. Let σ^* be an arc in $V^*_{ai}(r)$ from one endpoint of β^* to $p^*_{j_0}$, and let τ^* be an arc in $V^*_{ai}(\hat{r})$ from the other endpoint of β^* to $q^*_{j_0}$. For each such j, let L_j denote $B_j \cup h\Phi_{ai}[\sigma^* \cup \lambda^*_j \cup \mu^*_j \cup \tau^* \cup \beta^*]$.

Let p denote $h\Phi_{ai}[p^*_{j_0}]$. We may assume that if $j = 1, 2, \ldots,$ or n_m, L_j is a simple closed curve and if j and k are distinct, $L_j \cap L_k = \{p\}$ Let L denote $\bigcup_{j=1}^{n_m} = L_j$. By an argument similar to that for Proposition 1, .t may be proved that L is a homotopy centerline of $h[W_{ai}]$.

If $l = 1, 2, \ldots,$ or t, the p-based oscillation of L_{k_l} is at most 2 since β_{k_l} has pattern Ø or A, and λ_0 has pattern B. Suppose j is neither $k_1, k_2, \ldots,$ nor k_t. If β_j has oscillation 2, then by Proposition 2, it has pattern BA (from p_j to q_j). Since β has pattern A, it follows that the p-based oscillation of L_j is at most 2. Thus in this case, $h[W_{ai}]$ has Property I.

Case 3. Suppose Case 2 does not hold and $\{k_1, k_2, \ldots, k_t\}$ is not empty. There is an arc β^* in W^*_{ai} with one endpoint p^* in $V^*_{ai}(r)$, the other endpoint q^* in $V^*_{ai}(\hat{r})$, and such that $h\Phi_{ai}[\beta^*] = \beta$. Let q denote $h\Phi_{ai}(q^*)$.

Suppose $l = 1, 2, \ldots,$ or t. Let β_{k_l} be an arc on J_{aik_l} from p_{kl} to q_{k_l}; β_{k_l} has pattern AB. Let $\sigma^*_{k_l}$ be an arc in $V^*_{ai}(r)$ from p^* to $p^*_{k_l}$, and let $\tau^*_{k_l}$ be an arc in $V^*_{ai}(\hat{r})$ from q^* to $q^*_{k_l}$. Let L_{k_l} denote $\beta \cup \beta_{k_l} \cup h\Phi_{ai}[\sigma^*_{k_l} \cup \tau^*_{k_l}]$. We may assume that L_{k_l} is a simple closed curve. Note that the q-based oscillation of L_{k_l} is 2.

Suppose j is neither $k_1, k_2, \ldots,$ nor k_t. First suppose that one of the arcs of J_{aij} from p_j to q_j has oscillation at most one. Let β_j be one such arc. Let σ^*_j be an arc in $V^*_{ai}(r)$ from p^* to p^*_j, and let τ^*_j be an arc in $V^*_{ai}(\hat{r})$ from q^* to q^*_j. Let L_j denote $\beta_j \cup \beta \cup h\Phi_{ai}[\sigma^*_j \cup \tau^*_j]$. We may assume that L_j is a simple closed curve. Note that the q-based oscillation of L_j is at most 2.

Suppose now that neither arc of J_{aij} from p_j to q_j has oscillation at most one. By Proposition 2, each arc of J_{aij} from p_j to q_j has pattern BA. Let β'_j denote the arc of J_{aij} from p_j to q_j containing y_j, let $p_j y_j$ denote the subarc of β'_j from p_j to y_j, and let β_j denote the subarc of β'_j from y_j to q_j. Now $p_j y_j$ has pattern (in the order from p_j to y_j) BA. This is true because its possible patterns are Ø, B and BA, and since Case 2 does not hold, $p_j y_j$ has pattern BA. Hence β_j has pattern A or Ø.

Now consider the arc λ_1 of J_{aik_1} from y_{k_1} to q_{k_1}, positively oriented from y_{k_1} to q_{k_1}. Since each arc of J_{aik_1} from p_{k_1} to q_{k_1} has pattern AB, then λ_1 has oscillation at most 2, and if it has oscillation 2, it has pattern AB.

Let σ_j^* denote an arc in $V_{ai}^*(s)$ from $y_{k_1}^*$ to y_j^*, let τ_j^* denote an arc in $V_{ai}^*(\hat{r})$ from $q_{k_1}^*$ to q^*, and let ϱ_j^* denote an arc in $V_{ai}^*(\hat{r})$ from q^* to q_j^*. Let L_j denote $\beta_j \cup \lambda_1 \cup h\Phi_{ai}[\sigma_j^* \cup \tau_j^* \cup \varrho_j^*]$. We may assume that L_j is a simple closed curve. It is easily checked that the q-based oscillation of L_j is at most 2.

Let L denote $\bigcup_{j=1}^{n_m} L_j$; we may assume that if j and k are distinct, $L_j \cap L_k = \{q\}$. By an argument similar to that for Proposition 1, it may be shown that L is a homotopy centerline for $h[W_{ai}]$. Hence in this case $h[W_{ai}]$ has Property I.

Case 4. $\{k_1, k_2, \ldots, k_t\}$ is empty.

By Proposition 2, for each j, either (1) some arc of J_{aij} from p_j to q_j has oscillation at most one or (2) each arc of J_{aij} from p_j to q_j has pattern BA. Suppose $j = 1, 2, \ldots,$ or n_m. If (1) holds relative to J_{aij}, let β_j be an arc of J_{aij} from p_j to q_j and of oscillation at most one. Otherwise, let β_j be an arc of J_{aij} from p_j to q_j.

For each j, let σ_j^* be an arc in $V_{ai}^*(r)$ from p^* to p_j^* and let τ_j^* be an arc in $V_{ai}^*(\hat{r})$ from q^* to q_j^*; here we use the notation of the previous case. Let L_j denote $\beta_j \cup \beta \cup h\Phi_{ai}[\sigma_j^* \cup \tau_j^*]$. We may assume that L_j is a simple closed curve. It is clear that L_j has p-based oscillation at most 2.

Let L denote $\bigcup_{j=1}^{n_m} L_j$. We may assume that if k and l are distinct, $L_j \cap L_k = \{p\}$. By an argument similar to that for Proposition 1, it can be shown that L is a homotopy centerline of $h[W_{ai}]$. Hence in this case $h[W_{ai}]$ has Property I.

This establishes Lemma 13 in case β has pattern A. A similar argument holds in case β has pattern B. Thus Lemma 13 is established.

The following lemma summarizes the results of this section.

LEMMA 14. *Suppose $r \in R$ and there is an arc β in $h[W_{ai}]$ admissible from $V_{ai}(r)$ to $V_{ai}(\hat{r})$ and having oscillation at most one. The $h[W_{ai}]$ has Property* I.

9. Patterns on the sides of Δ

THE HYPOTHESIS OF SECTION 9. Suppose $r \in R$, $s \in R$, and s is neither r nor $\hat{r}$.

Suppose that Δ_0 is a square disc with top a_0, right side b_0, bottom c_0, and left side d_0. Let p_0, y_0, q_0, and x_0 denote the corner points $a_0 \cap d_0$, $a_0 \cap b_0$, $b_0 \cap c_0$, and $c_0 \cap d_0$, respectively.

Suppose that f is a continuous function from Δ_0 into W^*_{ai} such that $f(p_0)\in V^*_{ai}(r), f(y_0)\in V^*_{ai}(\hat{s}), f(q_0)\in V^*_{ai}(\hat{r})$, and $f(x_0)\in V^*_{ai}(s)$. Let p^*, y^*, q^*, and x^* denote $f(p_0), f(y_0), f(q_0)$, and $f(x_0)$ respectively. Let Δ^* denote $f[\Delta_0]$, and let a^*, b^*, c^*, and d^* denote $f[a_0], f[b_0], f[c_0]$, and $f[d_0]$, respectively. We shall suppose that $h\Phi_{ai}f|\mathrm{Bd}\,\Delta_0$ is a homeomorphism from $\mathrm{Bd}\,\Delta_0$ into $h[W_{ai}]$.

Let Δ denote $h\Phi_{ai}[\Delta^*]$, let a, b, c, and d denote $h\Phi_{ai}[a^*]$, $h\Phi_{ai}[b^*]$, $h\Phi_{ai}[c^*]$, and $h\Phi_{ai}[d^*]$, respectively, and let p, y, q, and x denote $a\cap d$, $a\cap b$, $b\cap c$, and $c\cap d$, respectively. Then $p\in V_{ai}(r)$, $y\in V_{ai}(\hat{s})$, $q\in V_{ai}(\hat{r})$, and $x\in V_{ai}(s)$. The arcs a, b, c, and d are oriented so that the positive directions are from p to y, y to q, q to x, and x to p, respectively. This concludes the hypothesis of Section 9.

We think of Δ as a square disc with top a, right side b, bottom c, and left side d. Accordingly, we speak of "adjacent" and "opposite" sides of Δ. Further, following these arcs in their positive orientations amounts to moving around this square disc in a clockwise sense.

LEMMA 15. *Assume the hypotheses of Sections* 8 *and* 9. *Then if any one of the following cases holds,* $h[W_{ai}]$ *has Property* I:

1. *There exist some adjacent two of* a, b, c, *and* d *such that* (i) *each has oscillation at most one and* (ii) *if both have oscillation one, they have the same pattern.*

2. a, b, c, *and* d *have patterns* A, B, A, B, *respectively, or* B, A, B, A, *respectively.*

3. *Some three of* a, b, c, *and* d *have oscillation one and the fourth has oscillation* 2.

4. *Some adjacent two of* a, b, c, *and* d *have oscillation one and the remaining two have oscillation* 2.

Proof. Case 1. In this case, it is easily proved that there is an arc in $h[W_{ai}]$ admissible either from $V_{ai}(r)$ to $V_{ai}(\hat{r})$, or from $V_{ai}(s)$ to $V_{ai}(\hat{s})$, and of oscillation at most one. Then Lemma 14 applies.

Case 2. By Theorem 7 of [5], there is an arc λ^* in W^*_{ai}, either from $V^*_{ai}(\hat{r})$, or from $V^*_{ai}(s)$ to $V^*_{ai}(\hat{s})$, such that if $\lambda = h\Phi_{ai}[\lambda^*]$, λ is an arc and λ has oscillation at most one. Hence Lemma 14 applies.

Case 3. We suppose that no previous case applies.

Subcase 3a. a, b, c, and d have patterns AB, A, B, A, respectively.
With the aid of Theorem 7 of [5], it follows that there is an arc in $h[W_{ai}]$ admissible either from $V_{ai}(r)$ to $V_{ai}(\hat{r})$, or from $V_{ai}(s)$ to $V_{ai}(\hat{s})$, and of oscillation at most one. Thus Lemma 14 applies.
The remaining subcases are similar.

Case 4. Suppose no previous case holds.

Let A_0 and B_0 denote $f^{-1}\Phi_{ai}^{-1}h^{-1}[A \cap \Delta]$ and $f^{-1}\Phi_{ai}^{-1}h^{-1}[B \cap \Delta]$, respectively.

Subcase 4*a*. *a*, *b*, *c*, and *d* have pattern *A*, *B*, *AB*, *AB*.

Suppose some component of A_0 intersects both a_0 and c_0. It would follow that there is an arc in $h[W_{ai}]$ admissible from $V_{ai}(r)$ to $V_{ai}(\hat{r})$ and missing B. Thus Lemma 14 applies. A similar argument applies in case some component of B_0 hits both b_0 and d_0.

Suppose some component of A_0 hits both a_0 and d_0. Then there is an arc in $h[W_{ai}]$ admissible from $V_{ai}(s)$ to $V_{ai}(\hat{s})$ and missing A_0. Thus Lemma 14 applies. A similar argument applies if some component of B_0 hits both b_0 and c_0.

Thus, suppose no one of the preceding holds. Then no component of A_0 hits both a_0 and $b_0 \cup c_0 \cup d_0$, and no component of B_0 hits both b_0 and $a_0 \cup c_0 \cup d_0$. Then there is an arc in Δ from p to q and missing $A \cup B$. It follows that Lemma 14 applies.

Subcase 4b. *a*, *b*, *c*, and *d* have patterns *A*, *B*, *AB*, *BA*, respectively.

It follows that there is an arc in Δ_0 joining a pair of opposite corners and hitting at most one of A_0 and B_0. Hence Lemma 14 applies.

10. The AB-condition

Suppose the hypothesis and notation of Sections 8 and 9.

We shall say that Δ satisfies the AB-*hypothesis* if and only if (1) each of some opposite pair of *a*, *b*, *c*, and *d* has oscillation 2, and (2) no one of *a*, *b*, *c*, and *d* has oscillation more than 2.

Recall that $h\Phi_{ai}f|\mathrm{Bd}\,\Delta_0$ is a homeomorphism from $\mathrm{Bd}\,\Delta_0$ onto $\mathrm{Bd}\,\Delta$. Let A_0 and B_0 denote $f^{-1}\Phi_{ai}^{-1}h^{-1}[A \cap \Delta]$ and $f^{-1}\Phi_{ai}^{-1}h^{-1}[B \cap \Delta]$, respectively. Note that the "pattern" of a_0, for example (oriented positively from p_0 to y_0) relative to A_0 and B_0, is the same as the pattern of *a*.

Suppose now that Δ satisfies the AB-hypothesis. Then Δ satisfies the AB-*condition relative to a and c* if and only if (1) both *a* and *c* have oscillation 2, and (2) there exist points u_0 and v_0 of a_0 and c_0, respectively, and an arc u_0v_0 in Δ_0 from u_0 to v_0 such that (i) each of p_0u_0 and u_0y_0 intersects at most one of A_0 and B_0, (ii) each of x_0v_0 and v_0q_0 intersects at most one of A_0 and B_0, and (iii) the arc u_0v_0 misses $A_0 \cup B_0$.

If Δ satisfies the AB-condition relative to *a* and *c*, then, with the notation above, let *u*, *v*, and *uv* denote $h\Phi_{ai}f(u_0)$, $h\Phi_{ai}f(v_0)$, and $h\Phi_{ai}f[u_0v_0]$. We may assume that *uv* is an arc, and throughout the remainder of the paper shall do so.

The statement that Δ satisfies the AB-*condition relative to b and d* is defined analogously. Δ satisfies the AB-*condition* if and only if for some two opposite sides, Δ satisfies the AB-condition relative to those two sides.

LEMMA 16. *Suppose that Δ satisfies the* AB-*hypothesis. Then either* (1) $h[W_{ai}]$ *has Property* I *or* (2) Δ *satisfies the* AB-*condition.*

Proof. Let $\theta_1, \theta_2, \ldots,$ and θ_8 denote arcs on $\mathrm{Bd}\,\Delta_0$, such that (1) $\theta_1, \theta_2, \ldots,$ and θ_8 partition $\mathrm{Bd}\,\Delta_0$, and occur on $\mathrm{Bd}\,\Delta_0$ in the clockwise order $\theta_1, \theta_2, \ldots, \theta_8$, (2) $a_0 = \theta_1 \cup \theta_2$, $b_0 = \theta_3 \cup \theta_4$, $c_0 = \theta_5 \cup \theta_6$, and $d_0 = \theta_7 \cup \theta_8$, and (3) each of $\theta_1, \theta_2, \ldots,$ and θ_8 intersects at most one of A_0 and B_0.

Throughout the proof, by a *diagonal arc* we shall mean an arc in Δ_0 either from p_0 to q_0 or from x_0 to y_0. Note that if there exists a diagonal arc missing one of A_0 and B_0, then Lemma 14 implies that $h[W_{ai}]$ has Property I.

We consider first the case where a and c have oscillation 2. For convenience, we suppose a has pattern AB.

Consider the arcs $\theta_0 \cup \theta_1$ and $\theta_4 \cup \theta_5$. If some component of A_0 hits both these arcs, there is a diagonal arc missing B_0, and hence Lemma 14 applies. Similar results hold relative to B_0, and relative to the arcs $\theta_2 \cup \theta_3$ and $\theta_6 \cup \theta_7$. Hence, throughout the remainder of the proof, we suppose that no such components of $A_0 \cup B_0$ exist.

If no component of $A_0 \cup B_0$ hits both $\theta_1 \cup \theta_8 \cup \theta_7 \cup \theta_6$ and $\theta_2 \cup \theta_3 \cup \theta_4 \cup \theta_5$, then clearly Δ satisfies the AB-condition relative to a and c. Hence in what follows, we suppose such a component of $A_0 \cup B_0$ exists.

Let ϱ and ω denote the arcs $\theta_8 \cup \theta_1 \cup \theta_2 \cup \theta_3$ and $\theta_4 \cup \theta_5 \cup \theta_6 \cup \theta_7$, respectively, and let σ and τ denote $\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4$ and $\theta_5 \cup \theta_6 \cup \theta_7 \cup \theta_8$, respectively.

Case 1. Some component of A_0 hits both θ_1 and θ_3.

If, in this case, some component of A_0 hits θ_1 and either θ_6 or θ_7, then it is easy to see that there is a diagonal arc (from x_0 to y_0) missing B_0. Also, if some component of A_0 hits θ_3 and θ_5, there is a diagonal arc (from p_0 to q_0) missing B_0. Thus we suppose no such components of A_0 exist.

Case 1a. Some side of Δ has oscillation at most one.

If b has pattern A, it is clear that there is a diagonal arc missing B_0. Thus suppose b has pattern AB, and that d has oscillation at most one. If d has pattern A or $\emptyset$, there is a diagonal arc missing B_0. Suppose d has pattern B. It follows from results above that no component of A_0 hits both σ and τ. Hence there is a diagonal arc (from p_0 to q_0) missing A_0. This concludes Case 1a.

Case 1b. All sides of Δ have oscillation 2. In this case, it follows that no component of A_0 hits both ϱ and ω.

Now we shall consider the case where some component of B_0 hits both ϱ and ω. Clearly such a component hits θ_8 and, by previous results, θ_6. It then follows as in Case 1a that no component of A_0 hits both σ and τ, and that there is a diagonal arc missing A_0.

Thus we may suppose no component of either A_0 or B_0 hits both ϱ and ω. Then, clearly, Δ satisfies the AB-condition relative to b and d; recall that, in this case, b and d both have oscillation 2.

Case 2. Some component of A_0 hits both θ_8 and θ_3. (We do not, in this case, use the supposition that a has pattern AB).

If b or d has oscillation at most one, it is clear that there is a diagonal arc missing B_0. Suppose both b and d have oscillation 2. Then, clearly, no component of B_0 hits both ϱ and ω. Suppose some component of A_0 hits both ϱ and ω. Then it is easily checked that there is a diagonal arc missing B_0. Thus we may suppose that no component of $A_0 \cup B_0$ hits both ϱ and ω. Then Δ satisfies the AB-condition relative to b and d (which, in this instance, both have oscillation 2).

The remaining cases are similar to those above, and this completes the proof of Lemma 16.

We introduce some notation concerning arcs to be used throughout the remainder of the paper. In notations such as "the arc st" or "the arc swt", it is understood that s and t are the endpoints of the arc. If the arc is oriented, the positive orientation is from s to t.

Suppose J is a simple closed curve and s, w, and t are three distinct points of J. By the arc swt of J is meant the arc of J from s to t containing w. By the arc $s\hat{w}t$ of J is meant the arc of J from s to t **not** containing w.

11. Types of simple closed curves

Suppose the hypothesis of Section 8 holds. In the proof of Lemma 18, we shall, for each j, select four distinct points p_j, y_j, q_j, and x_j of $J_{\alpha ij}$. In this section, we introduce some terminology for the cases that arise, and establish some lemmas for later use.

Suppose that $j = 1, 2, \ldots,$ or n_m. We shall denote $J_{\alpha ij}$ by J^j. Let p_j, y_j, q_j and x_j denote four distinct points of J_j.

J_j is of *type* 1 (relative to the choices made) if and only if x_j and y_j belong to different ones of the two arcs of J_j from p_j to q_j. Let a_j and c_j denote the arcs $p_j \hat{q}_j y_j$ and $q_j \hat{p}_j x_j$, respectively, of J_j. Note that a_j and c_j are disjoint. Then if both a_j and c_j have oscillation 2, (a) they have opposite patterns and (b) if a_j has pattern AB, $y_j \hat{p}_j q_j$ has pattern B or $\emptyset$, and $x_j \hat{q}_j p_j$ has pattern A or $\emptyset$. Analogous results hold for pattern BA. Suppose one of a_j and c_j has oscillation 3, a_j, for example. Then c_j has oscillation at most one, and the arc $p_j q_j x_j$ also has oscillation at most one. Recall that, in $h[W_{ai}]$, a_j and $p_j q_j x_j$ are homotopic with fixed endpoints. Analogous results hold in other cases.

Suppose J_j is of type 1. Let b_j and d_j denote the arcs $y_j \hat{p}_j q_j$ and $x_j \hat{q}_j p_j$ on J. Note that b_j and d_j are disjoint. Then analogous results hold for b_j, d_j, $q_j \hat{p}_j x_j$, and $p_j \hat{q}_j y_j$.

J_j is of *type* 2 if and only if x_j and y_j belong to the same arc of J_j from p_j to q_j, and are in the order $p_j x_j y_j q_j$ on that arc. Let b_j and d_j denote the arcs $y_j \hat{p}_j q_j$ and $x_j \hat{q}_j p_j$, respectively, of J_j. Then b_j and d_j are disjoint. If both b_j and d_j have oscillation 2, then (a) they have the same pattern and (b) if b_j has pattern AB, $y_j \hat{p}_j x_j$ has pattern A or $\emptyset$, and $p_j \hat{x}_j q_j$ has pattern B or $\emptyset$. Analogous results hold for pattern BA. If b_j has oscillation 3, then d_j has oscillation at most one, and the arc $y_j x_j q_j$ of J_j has oscillation at most one and is homotopic, with fixed endpoints, in $h[W_{ai}]$, to b_j.

If J_j is of type 2, then any two arcs, one from p_j to y_j and the other from q_j to x_j, intersect. However, by the hypothesis of Section 8, there exist arcs from p_j to y_j and from q_j to x_j on J_j and of oscillation at most 2. We have, however, the following result, which may be established by an analysis of cases.

PROPOSITION 3. *Suppose that J_j is of type 2 and for each choice of arcs $p_j y_j$ and $q_j x_j$ on J_j, both have oscillation at least 2. Then there exist arcs $p_j y_j$ and $q_j x_j$ on J_j such that the following condition holds:*

(*) *Both have oscillation 2 and either* (a) *$p_j y_j$ and $q_j x_j$ have the same pattern, and if this pattern is AB, then the arc $p_j \hat{x}_j q_j$ has pattern A or $\emptyset$, and the arc $x_j \hat{p}_j y_j$ has pattern B or $\emptyset$, and similarly for BA, or* (b) *$p_j y_j$ and $q_j x_j$ have opposite patterns, and if $p_j y_j$ has pattern AB, then the arc $p_j \hat{q}_j x_j$ has pattern A or $\emptyset$, and the arc $q_j \hat{p}_j y_j$ has pattern B or $\emptyset$.*

J_j is of *type* 3 if and only if x_j and y_j belong to the same arc of J_j from p_j to q_j, and are in the order $p_j y_j x_j q_j$ on that arc. In this case, conclusions analogous to those of type 2 hold, but with x_j and y_j interchanged.

Suppose that $j = 1, 2, \ldots,$ or n_m, and that p_j, y_j, q_j, and x_j are four distinct points of J_j. Two arcs on J_j, one from p_j to y_j and the other from q_j to x_j, are an *admissible choice* of such arcs if and only if the following hold: (i) Neither arc has oscillation greater than 2. (ii) If J_j is of type 1 or 3 and both arcs have oscillation 2, they are disjoint. (iii) If J_j is of type 2 and there exist two arcs on J_j, one from p_j to y_j and one from q_j to x_j such that one has oscillation at most one, then $p_j y_j$ and $q_j x_j$ are two such arcs. Note that if J_j is of type 2 and both arcs $p_j y_j$ and $q_j x_j$ of an admissible choice have oscillation 2, then Proposition 3 necessarily applies, and hence condition (*) holds.

An *admissible choice* of arcs on J_j, one from p_j to x_j and the other from q_j to y_j, is defined similarly.

LEMMA 17. *Suppose* $j = 1, 2, \ldots,$ *or* n_m *and some arc of* J_j *from* p_j *to* q_j (*oriented positively from* p_j *to* q_j) *has pattern* AB, A, *or* $\emptyset$. *Suppose* p_jy_j *and* q_jx_j *are admissible choices. If both* p_jy_j *and* q_jx_j *have oscillation* 2, *then either* (i) p_jy_j *has pattern* AB *or* (ii) *both* p_jy_j *and* q_jx_j *have pattern* BA *and some arc of* J_j *from* p_j *to* q_j *has pattern* $\emptyset$ *or* B.

Analogous results hold for the arcs y_jq_j and x_jp_j, and, relative to both pairs of arcs, for the patterns BA, B, and $\emptyset$.

Now we describe a construction to be used several times in the sequel. (We assume the hypothesis of Section 8.) Suppose $r \in R$, $s \in R$, $s \neq r$, and $s \neq \hat{r}$. For each j, let J_j denote J_{aij}. For each j, let p_j^* be a point of $J_{r_j}^* \cap V_{ai}^*(r)$, let q_j^* be a point of $J_{\hat{r}_j}^* \cap V_{ai}^*(\hat{r})$, and let p_j and q_j denote $h\Phi_{ai}(p_j^*)$ and $h\Phi_{ai}(q_j^*)$, respectively. For each j, let x_j^* be a point of $J_{s_j}^* \cap V_{ai}^*(s)$, let y_j^* be a point of $J_{\hat{s}_j}^* \cap V_{ai}^*(\hat{s})$, and let x_j and y_j denote $h\Phi_{ai}(x_j^*)$ and $h\Phi_{ai}(y_j^*)$, respectively. We assume that p_j, y_j, q_j, and x_j are distinct.

Since $s \neq r$, there is an integer k_0 such that $s_{k_0} \neq r_{k_0}$ so that $s_{k_0} \neq \hat{r}_{k_0}$. Since $s \neq \hat{r}$, there is an integer l_0 such that $s_{l_0} \neq \hat{r}_{l_0}$ and so $s_{l_0} = r_{l_0}$. Thus $k_0 \neq l_0$. Note that $\hat{s} \neq r$ and $\hat{s} \neq \hat{r}$.

Choose integers k and l such that $s_k = \hat{r}_k$ and $s_l = r_l$. By the preceding paragraphs, such integers exist.

Now let a_k and c_k be an admissible choice of arcs on J_k from p_k to y_k, and from q_k to x_k, respectively. Let a_k^* and c_k^* be the arcs on $J_{kr_k}^*$ and $J_{k\hat{r}_k}^*$, respectively, covering a_k and c_k. Let b_l and d_l be an admissible choice of arcs on J_l from y_l to q_l, and from x_l to p_l, respectively. Let b_l^* and d_l^* be the arcs on $J_{ls_l}^*$ and $J_{l\hat{s}_l}^*$, respectively, that cover b_l and d_l. Choose points p^*, y^*, q^*, and x^* of $V_{ai}^*(r)$, $V_{ai}^*(s)$, $V_{ai}^*(\hat{r})$, and $V_{ai}^*(\hat{s})$, respectively. Join p^* to p_k^* by an arc in $V_{ai}^*(r)$ and y^* to y_k^* by an arc in $V_{ai}^*(\hat{s})$; let a^* be the union of these and a_k^*. Join y^* to y_l^* by an arc in $V_{ai}(\hat{s})$ and q^* to q_l^* by an arc in $V_{ai}(\hat{r})$; let b^* be the union of these and b_l^*. Join q^* to q_k^* by an arc in $V_{ai}^*(\hat{r})$ and x^* to x_k^* by an arc in $V_{ai}^*(s)$; let c^* be the union of these and c_k^*. Join x^* to x_l^* by an arc in $V_{ai}^*(s)$ and p^* to p_l^* by an arc in $V_{ai}^*(r)$; let d^* be the union of these and d_l^*. If necessary, adjust a^*, b^*, c^*, and d^* slightly so that $a^* \cup b^* \cup c^* \cup d^*$ is a simple closed curve. There is a function f from the square disc Δ_0 into W_{ai}^* sending p_0, y_0, q_0, and x_0 to p^*, y^*, q^*, and x^*, respectively, and sending a_0, b_0, c_0, and d_0 onto a^*, b^*, c^*, and d^*, respectively. We suppose that $h\Phi_{ai}f|\mathrm{Bd}\,\Delta_0$ is a homeomorphism. Let Δ^* denote $f[\Delta_0]$.

Let p, y, q, x, a, b, c, d, and Δ denote the images under $h\Phi_{ai}$ of p^*, y^*, q^*, x^*, a^*, b^*, c^*, d^*, and Δ^*, respectively. Orient a, b, c, and d so that the positive directions are from p to y, y to q, q to x, and x to p, respectively. See Figure 16.

The singular disc Δ that results will be the Δ of the *Δ-construction*

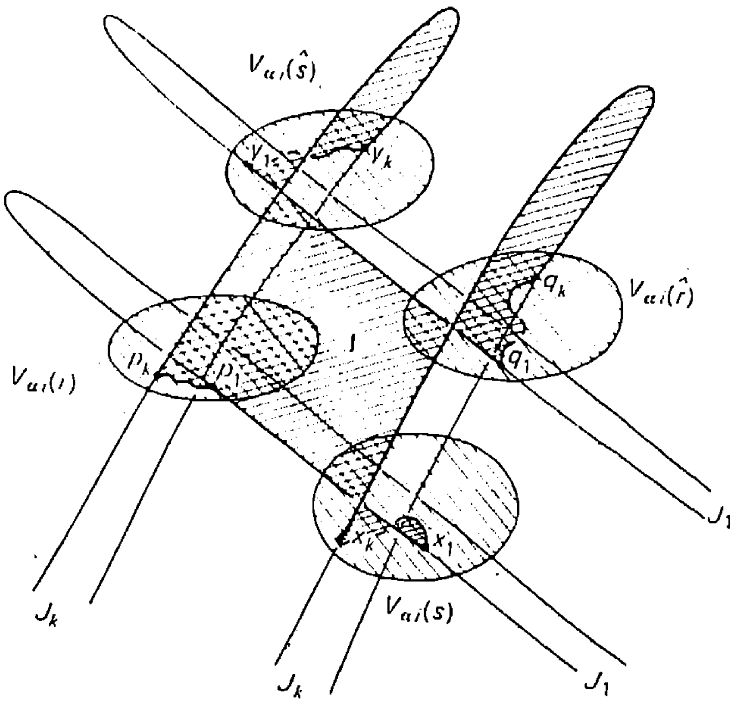

Fig. 16

relative to k and l. Note that the hypothesis of Section 9 will be satisfied. In the instances where we apply the Δ-construction, we will keep the notation used above.

12. Similar patterns

In this and the next section, we establish the main lemma for the second step of the proof of Lemma 3. Property II is defined in Section 13.

LEMMA 18. *Suppose the hypothesis of Section* 8. *Then* $h[W_{ai}]$ *has Property* I *or Property* II.

Proof. Suppose $r \in R$. If $j = 1, 2, \ldots,$ or n_m, let J_j denote J_{aij}, let p_j^* be a point of $J_j^* \cap V_{ai}^*(r)$, and let q_j be a point of $J_j^* \cap V_{ai}^*(\hat{r})$. Let p_j and q_j denote $h\Phi_{ai}(p_j^*)$ and $h\Phi_{ai}(q_j^*)$, respectively.

Case 1a. For each j, at least one of the arcs on J_j from p_j to q_j (oriented positively from p_j to q_j) has pattern AB, A, B, or $\emptyset$.

Suppose first that there exists an arc ζ in $h[W_{ai}]$ admissible from $V_{ai}(r)$ to $V_{ai}(\hat{r})$ and having pattern AB, A, B, or $\emptyset$, where ζ is oriented positively from its end in $V_{ai}(r)$ to its end in $V_{ai}(\hat{r})$.

For each j, there is, by the hypothesis of this case, an arc β_j on J_j from p_j to q_j and of pattern AB, A, B, or $\emptyset$. For each j, let L_j denote the union of β_j, ζ, and suitable connecting arcs in $V_{ai}(r)$ and $V_{ai}(\hat{r})$.

If ζ has pattern A or $\emptyset$, then by Lemma 14, $h[W_{ai}]$ has Property I.

We consider some subcases.

Subcase 1a (i). There exist two distinct integers k and l, $1 \leqslant k \leqslant n_m$, $1 \leqslant l \leqslant n_m$, such that if $j = k$ or l, at least one of the arcs on J_j from p_j to q_j (oriented positively from p_j to q_j) has pattern AB, A, or $\emptyset$.

Now we shall define s. Let $s_k = \hat{r}_k$ and if $j \neq k$, let $s_j = r_k$; in particular, $s_l = r_l$. Then we may carry out the Δ-construction relative to k and l. We use the notation of Section 11; the hypothesis of Section 9 is satisfied.

First, we may assume that Δ satisfies the AB-hypothesis. For, by definition of admissible choices, each of a, b, c, and d has oscillation at most 2, and if Δ does not satisfy the AB-hypothesis, then by Lemma 15, $h[W_{ai}]$ has Property I.

Hence, Δ satisfies the AB-hypothesis. By Lemma 16, either $h[W_{ai}]$ has Property I, or Δ satisfies the AB-condition. We suppose the latter.

Suppose first that Δ satisfies the AB-condition relative to a and c. Then there exist points u and v of a and c, respectively, and an arc uv in Δ such that (1) pu, uy, qv, and vx each intersects at most one of A and B, (2) the arc uv misses $A \cup B$, and (3) the arc $pu \cup uv \cup vq$ is admissible relative to $V_{ai}(r)$ and $V_{ai}(\hat{r})$.

By Lemma 17, either a has pattern AB, or both a and c have pattern BA. Let ζ denote $pu \cup uv \cup vq$; we suppose ζ is an arc, and clearly ζ is admissible from $V_{ai}(r)$ to $V_{ai}(\hat{r})$. Orient ζ positively from p to q. If a has pattern AB, ζ has pattern A or AB; if both a and c have pattern BA, then ζ has pattern B.

Hence by previous remarks, $h[W_{ai}]$ has Property I.

Suppose Δ satisfies the Δ-condition relative to b and d. Then an analogous construction can be made, applying Lemma 17 to J_l.

Subcase 1a (ii). Subcase 1a (i) does not hold.

There exists an integer k such that if $j \neq k$, neither arc of J_j from p_j to q_j (oriented positively from p_j to q_j) has pattern AB, A, or $\emptyset$. Since one such arc of J_j has pattern AB, A, B, or $\emptyset$, then one such arc has pattern B. The other arc of J_j from p_j to q_j has pattern B, BA, or BAB.

Let l be an integer different from $k, 1 \leqslant l \leqslant n_m$. Define s so that $s_k = \hat{r}_k$ and if $j \neq k$, $s_j = r_j$; in particular, $s_l = r_l$. Then we may carry out the Δ-construction relative to k and l.

As in Subcase 1a (i), we may assume that Δ satisfies the AB-condition.

First suppose that Δ satisfies the AB-condition relative to a and c. Let u and v be points of a and c, respectively, and let uv be an arc in Δ such that (1) pu, uy, qv, and vx each intersects at most one of A and B, (2) uv misses $A \cup B$, and (3) $pu \cup uv \cup vq$ is admissible from $V_{ai}(r)$ to $V_{ai}(\hat{r})$. Let ζ denote $pu \cup uv \cup vq$. Note that ζ has oscillation at most 2.

If ζ has a pattern different from BA, then by a result above, $h[W_{ai}]$ has Property I. It follows that a has pattern BA and c has pattern AB.

Suppose J_k is of type 1. Then each arc of J_k from u to v has oscillation 1. Let L_k denote the union of uv and one of the arcs of J_k from u to v. For any point z of L_k, the z-based oscillation of L_k is one.

Suppose J_k is of type 3. Then a contradiction results with the hypothesis of Section 8.

Suppose J_k is of type 2. Now if one of the arcs of J_k from p_k to q_k has pattern AB, A, or $\emptyset$, then by Lemma 17, either a has pattern AB, or a and c both have pattern BA. Since a has pattern BA and c has pattern AB, neither arc of J_k from p_k to q_k has pattern AB, A, or $\emptyset$. Hence one such arc has pattern B. Let β_k be an arc of J_k from p_k to q_k of pattern B.

Let L_k denote the union of ζ, β_k, and connecting arcs in $V_{ai}(r)$ and $V_{ai}(\hat{r})$. If z is a point of arc uv, L_k has z-based oscillation at most 2.

If $j \neq k$, let β_j be an arc of J_j from p_j to q_j and having pattern B. Let L_j denote the union of ζ, β_j, and connecting arcs in $V_{ai}(r)$ and $V_{ai}(\hat{r})$. If z is a point of arc uv, L_j has z-based oscillation at most 2.

Let L denote $\bigcup_{j=1}^{n_m} L_j$; we assume each L_j is a simple closed curve and if $i \neq j$, $L_i \cap L_j = \{z\}$. It may be proved that L is a homotopy centerline of $h[W_{ai}]$. Hence $h[W_{ai}]$ has Property I.

Now suppose Δ satisfies the AB-condition relative to b and d. Again, the only case to be considered is that in which b has pattern BA and d has pattern AB.

Note that if $j \neq k$, then the arc yy_jq_jq is homotopic in $h[W_{ai}]$, with fixed endpoints, to b. Now we shall prove the following: If $j \neq k$, then each arc y_jq_j of J_j (oriented positively from y_j to q_j) has pattern BA.

First, each arc of J_j from y_j to q_j has oscillation at least 2. Suppose some such arc β has oscillation 0 or 1. Now replace b by β, and construct a square Δ' having sides a, β, c, and d. If Δ' does not satisfy the AB-condition, then $h[W_{ai}]$ has Property I. Hence Δ' satisfies the AB-condition, clearly relative to a and c. By an argument similar to that used above, $h[W_{ai}]$ has Property I.

Second, each arc of J_j from y_j to q_j has oscillation 2, for if some such arc has oscillation 3, some such arc has oscillation 0 or 1.

Third, each such arc has pattern BA. If for some $j \neq k$, some arc μ of J_j from p_j to q_j has pattern AB, then by Theorem 7 of [5], there is an arc β in $h[W_{ai}]$, homotopic in $h[W_{ai}]$ to b with fixed endpoints, and of oscillation 0 or 1. Then the argument above applies.

By a similar argument, if $j \neq k$, each arc of J_j from x_j to p_j has pattern AB.

Suppose $j \neq k$. If J_j is of type 1, then since b and d are disjoint and of oscillation 2, and p_j and q_j separate, on J_j, x_j and y_j, it follows that neither arc of J_j from p_j to q_j has pattern B. Thus J_j is not of type 1.

If J_j is of type 2, then since b and d are disjoint and we have $p_j x_j y_j q_j$, a contradiction with the hypothesis of Section 8 results.

Suppose J_j is of type 3. Suppose the arc $p_j \hat{y}_j q_j$ has pattern B. Now the arc $y_j x_j q_j$ has pattern BA. Hence $p_j \hat{q}_j y_j$ has pattern B or $\emptyset$. Then $y_j p_j q_j$ has pattern B, a contradiction. Suppose the arc $p_j y_j q_j$ has pattern B. In that case $y_j x_j q_j$ has pattern B or $\emptyset$, a contradiction.

Hence it is impossible for Δ to satisfy the AB-condition relative to b and d. This concludes Case 1a.

Case 1b. For each j, at least one of the arcs $p_j q_j$ of J_j (oriented positively from p_j to q_j) has pattern BA, B, A, or $\emptyset$.

PROPOSITION 4. *Suppose that Case 1 does not apply. Then there exist distinct integers k and l such that (1) each arc of J_k from p_k to q_k (oriented positively from p_k to q_k) has pattern AB and (2) each arc of J_l from p_l to q_l (oriented positively from p_l to q_l) has pattern BA.*

Proof. Suppose that there is a j_0 such that if $j \neq j_0$, some arc of J_j from p_j to q_j has oscillation at most one. Now some arc of J_{j_0} from p_{j_0} to q_{j_0} has oscillation at most 2. Then Case 1 applies.

Consider those integers t such that each arc of J_t from p_t to q_t has oscillation 2 (the two arcs of such J_t have the same pattern). By the paragraph above, there are at least two such t's. For each such t, select an arc on J_t from p_t to q_t and of oscillation 2. If each such arc has pattern AB, Case 1 applies, and similarly if each such arc has pattern BA. Thus Proposition 4 holds.

Case 2. There exist distinct integers k and l such that (1) each arc of J_k from p_k to q_k (oriented positively from p_k to q_k) has pattern AB, (2) each arc of J_l from p_l to q_l (oriented positively from p_l to q_l) has pattern BA, and (3) if j is neither k nor l, some arc of J_j from p_j to q_j has oscillation at most one.

We select s as follows: Recall that $n_m \geqslant 4$. Let t be some integer such that $t \leqslant n_m$, $t \neq k$, and $t \neq l$. Let $s_k = \hat{r}_k$, $s_t = \hat{r}_t$, but if j is neither k nor t, $s_j = r_j$. Note that there is an integer w such that $w \leqslant n_m$ and w is distinct from k, l, and t; thus $s_w = r_w$.

We may carry out the Δ-construction relative to k and l. As in the argument for Case 1a, we may assume that Δ satisfies the AB-condition.

The hypothesis of Lemma 17 is satisfied, and thus by Lemma 17, if both a and c have oscillation 2, then a has pattern AB, and if both b and d have oscillation 2, then d has pattern AB (recall that on d, the positive orientation is form x_l to p_l). Here we use the fact that neither arc of J_l from p_l to q_l has oscillation 0 or 1.

Suppose Δ satisfies the AB-conclusion relative to a and c. First suppose that b and d both have oscillation 2. Then by the results of Section 12, either (1) the arc $p_l y_l$ of J_l not containing q_l has pattern B or $\emptyset$, or (ii) some arc $p_l q_l$ of J_l has pattern B or $\emptyset$. Hence (i) holds. There exist a point u of a, a point v of c, and an arc uv in Δ such that the subarc pu of a has pattern A, uv has pattern $\emptyset$, and the subarc vq of c has pattern A or B. If vq has pattern A, Lemma 14 applies. So we assume vq has pattern B.

Let β_l denote the arc of J_l from p_l to y_l not containing p_l; β_l has pattern B or $\emptyset$. Let L_l be $up \cup \beta_l \cup yu$, plus suitable connecting arcs. By the hypothesis of Section 8, one of the arcs of J_k from u to v has oscillation at most 2; let β_k denote such an arc. Let L_k be $uv \cup \beta_k$. For all other j, let β_j be an arc of J_j from p_j to q_j with oscillation at most one, and let L_j be $up \cup \beta_j \cup qv \cup uv$ together with suitable connecting arcs. Then $\bigcup_{j=1}^{n_m} L_j$ yields a homotopy centerline for $h[W_{ai}]$ with centerpoint u. For each j, the u-based oscillation of L_j is at most 2. This fact depends on the fact that vq has pattern B.

Now suppose that at most one of b and d has oscillation 2, and hence the other has oscillation at most one. Now consider $p_t y_t$. We may assume that each arc of J_t from p_t to y_t has pattern AB, for if not, then by Theorem 7 of [5], there exists an arc ζ from p to y, homotopic in $h[W_{ai}]$ to a, and of oscillation at most one. Then we may use $\zeta \cup b \cup c \cup d$ in place of $a \cup b \cup c \cup d$ to carry out the Δ-construction, and then Lemma 15 applies. By a similar argument, each arc of J_t from q_t to x_t has the same pattern as c. Now c has oscillation 2. Thus J_t is of type 2 or 3, for if not, then both arcs of J_t from p_t to q_t would be of oscillation 2 (recall that some arc of J_t from p_t to q_t has oscillation at most one). Since both $p_t y_t$ and $q_t x_t$ have oscillation 2 and $p_t y_t$ has pattern AB, then $q_t x_t$ has pattern AB. If J_t is of type 2, this may be established as follows: First, since each arc on J_t from p_t to y_t has pattern AB, then $p_t x_t y_t$ does. Thus $p_t x_t q_t$ cannot have oscillation at most one (recall that since J_t is of type 2, $y_t \epsilon p_t x_t q_t$). Since some arc of J_t from p_t to q_t has oscillation at most one, $p_t \hat{x}_t q_t$ does. Since $p_t q_t y_t$ has pattern AB, $p_t \hat{x}_t q_t$ has pattern A or $\emptyset$. Since $q_t p_t x_t$ has oscillation 2, its pattern is thus AB.

Thus c also has pattern AB. Since one of b and d has oscillation at most one, Δ satisfies the AB-condition relative to a and c. Therefore, there exist points u of a, v of c, and an arc uv in Δ such that the subarcs pu of a and vq of c both have pattern A, and uv has pattern $\emptyset$. Then the arc $pu \cup uv \cup vq$ has oscillation at most one, and thus Lemma 14 applies.

This concludes Case 2 in case Δ satisfies the AB-conclusion relative to a and c. Since $n_m \geqslant 4$, $w \neq l$, and $s_w = r_w$, a similar argument holds if Δ satisfies the AB-conclusion relative to b and d. This concludes Case 2.

13. Property II

Suppose that L_0 is a homotopy centerline of W_{ai}, and $j = 1, 2, \ldots,$ or n_m. Then there is precisely one simple closed curve (one loop) of L_0 that links the simple closed curve $\mathrm{Bd}\,D_{aij}$ (see Figure 12 for D_{aij}). This loop is the *j-th loop of* L_0. If L is a homotopy centerline of $h[W_{ai}]$, then $h^{-1}[L]$ is a homotopy centerline of W_{ai}, and by the *j-th loop of* L is meant $h[L_j]$ where L_j is the jth loop of $h^{-1}[L]$.

The statement that $h[W_{ai}]$ has *Property* II means that if M and N are distinct positive integers such that $M \leqslant n_m$ and $N \leqslant n_m$, there exist a homotopy centerline L and distinct positive integer k and t such that (1) $k \leqslant m_n$ and $t \leqslant m_n$, (2) $\{M, N\} \cap \{k, t\}$ has at most one element, and (3) if L has centerpoint p, then the kth loop of L has p-based oscillation at most 4, the tth loop of L has p-based oscillation at most 2, and every other loop of L has oscillation 0.

Case 3. Neither Case 1 nor Case 2 holds.

Then there exist three distinct integers σ, τ, and ω such that (1) each arc of J_σ from p_σ to q_σ has pattern AB, (2) each arc of J_τ from p_τ to q_τ has pattern BA, and (3) each arc of J_ω from p_ω to q_ω has oscillation 2.

Suppose M and N are distinct integers such that $M \leqslant n_m$ and $N \leqslant n_m$. We shall now select k and t. Let k be an element of $\{\sigma, \tau, \omega\} - \{M, N\}$. Consider the pattern of each arc of J_k from p_k to q_k. At least one element t of $\{\sigma, \tau, \omega\} - \{k\}$ has the property that the pattern of each arc of J_t from p_t to q_t is distinct from the pattern of each arc of J_k from p_k to q_k. Hence the patterns associated with J_k and J_t are distinct. Note that $\{M, N\} \cap \{k, t\}$ has at most one element.

Now we shall define s. Let s_k be r_k, let s_t be r_t, and for every other j, let s_j be $\hat{r}_j$. Let l be some integer such that $l \leqslant n_m$, $l \neq k$, and $l \neq t$. Then we may carry out the Δ-construction relative to k and l.

Suppose that the following holds:

(+) Each arc of J_k from p_k to q_k has pattern AB, and each arc of J_t from p_t to q_t has pattern BA.

We shall show that there is an arc λ in $h[W_{ai}]$ such that (1) λ has oscillation at most one, and (2) either (i) λ is from p_k to x_k and homotopic, in $h[W_{ai}]$, to d with fixed endpoints, or (ii) λ is from q_k to y_k and homotopic, in $h[W_{ai}]$ with fixed endpoints, to b. To prove this, suppose first that each of $p_k x_k$, $p_t x_t$, $q_k y_k$, and $q_t y_t$ has oscillation 2. Then by Lemma 17, $p_k x_k$ has pattern AB and $p_t x_t$ has pattern BA; we can apply Lemma 17 since $(+)$ holds. By Theorem 7 of [5], there is an arc λ in $h[W_{ai}]$ of oscillation at most one, homotopic in $h[W_{ai}]$ to d with fixed endpoints. On the other hand, if one of $p_k x_k$, $p_t x_t$, $q_k y_k$, and $q_t y_t$ fails to have oscillation 2, then by definition of admissible choices, it has oscillation at most one. Then there is an arc λ as described above. Analogous results hold for the other possible patterns associated with J_k and J_t.

If λ is from p_k to x_k, let μ denote $a \cup b \cup c \cup \lambda$, and if λ is from q_k to y_k, let μ denote $a \cup \lambda \cup c \cup d$; we may assume μ is a simple closed curve such that μ bounds a singular disc $\hat{\Delta}$ in $h[W_{ai}]$. We use the previous notation Δ for $\hat{\Delta}$; in effect, we may assume that b or d has oscillation at most one.

Now Δ satisfies the AB-hypothesis. If not, then Lemma 15 applies and $h[W_{ai}]$ has Property I. Further, since λ has oscillation at most one, Δ satisfies the AB-conclusion relative to a and c. Thus a and c both have oscillation 2.

Suppose a has pattern AB. Then we shall show that for each j distinct from k and t, any arc of J_j from p_j to y_j and of oscillation at most 2 has pattern AB. Choose any such arc of J_j. If it does not have pattern AB, there is an arc $p_j v_j$ of J_j from p_j to q_j, of oscillation at most 2, and not having pattern AB. Then with the aid of Theorem 7 of [5], it follows that there is an arc δ_j from p_j to y_j in $h[W_{ai}]$, homotopic in $h[W_{ai}]$ to $p_j y_j$ with fixed endpoints, and of oscillation at most one. Then by using δ_j, c, λ, and the proper one of b and d, we may apply Lemma 15. Thus we may assume that for each j different from k and t, each arc of J_j from p_j to y_j has pattern AB.

By a similar argument, for each j different from k and t, each arc of J_j from q_j to x_j has the same pattern as c.

There exist points u and v of a and c, respectively, and an arc uv in Δ such that (1) each of pu, uy, qv, and ux intersects only one of A and B, (2) the arc uv misses $A \cup B$, and (3) the arc uv lifts to an arc in W^*_{ai}.

If c has pattern AB, then $pu \cup uv \cup vq$ is an arc in $h[W_{ai}]$ admissible from $V_{ai}(r)$ to $V_{ai}(\hat{r})$, and of oscillation one. Thus Lemma 14 applies.

Suppose c has pattern BA. Then for each j different from k and t, J_j is of type 1 or 2, since type 3 would contradict the hypothesis of Section 8. Hence for each such j, the arc β_j of J_j from y_j to q_j not containing p_j has pattern B or $\emptyset$, and the arc ω_j from p_j to x_j not containing q_j has pattern A or $\emptyset$. If J_j is of type 1, this is easily verified. If J_j is of type 2, this may be established using results of Section 11. Note that each arc of J_j from p_j to y_j and from q_j to x_j has oscillation at least 2. Note that $uy \cup \beta_j \cup qv$ and $up \cup \omega_j \cup xv$ are homotopic in $h[W_{ai}]$ with fixed endpoints. By Theorem 6 of [5], for each such j, there is an arc ϱ_j in $h[W_{ai}]$, from u to v, homotopic in $h[W_{ai}]$ to β_j with fixed endpoints, and of pattern $\emptyset$. For each such j, let L_j denote $uv \cup \varrho_j$. Then L_j has oscillation 0.

Let β_k denote an arc of J_k from p_k to q_k. Let L_k denote $\beta_k \cup pu \cup \cup uv \cup vq$ plus suitable connecting arcs. Since β_k (oriented positively from p_k to q_k) has pattern AB, L_k has u-based oscillation 2.

Let β_t denote an arc of J_t from p_t to q_t. Let L_t denote $\beta_t \cup pu \cup \cup uv \cup vq$ plus suitable connecting arcs. Since β_t has oscillation at most 2, L_t has u-based oscillation at most 4.

Let L denote $\bigcup_{j=1}^{n_m} L_j$. Then L is a homotopy centerline of $h[W_{ai}]$ with centerpoint u, the kth loop of L has u-based oscillation at most 2, the tth loop of L has u-based oscillation at most 4, and each other loop of L has oscillation 0. By the choice of k and t, $\{k, t\} \cap \{M, N\}$ has at most one element. Thus $h[W_{ai}]$ has Property II.

A similar argument holds if a has pattern BA. Also, if β_k and β_t have patterns BA and AB, respectively, a similar argument applies. This concludes the proof of Case 3. Lemma 18 is therefore established.

14. Construction of homotopy-$h[\Gamma_a]$'S

In this section, we shall prove the lemma which provides the third main step in the proof of Lemma 3.

LEMMA 19. *Suppose that h is a homeomorphism from E^3 onto E^3, m is a non-negative integer, and a is a stage m index. Suppose that each of $h[W_{a1}]$ and $h[W_{a2}]$ has Property* I *or Property* II. *Then there is a homotopy-$h[\Gamma_a]$ Γ in $h[T_a]$ such that Γ does not have Property* X.

Proof. Let $C_1, C_2, \ldots,$ and C_{n_m} be 3-cells in $\operatorname{Int} T_a$ as shown in Figure 17. Suppose L_1 and L_2 are homotopy centerlines of $h[W_{ai}]$. Then if $i = 1$ or 2, $h^{-1}[L_i]$ is a homotopy centerline of W_{ai}.

For each j such that $j = 1, 2, \ldots,$ or n_m, we shall construct an arc e_j in $W_{a1} \cup W_{a2} \cup C_j$ with one endpoint on the jth loop of $h^{-1}[L_1]$ and the other on $h^{-1}[L_2]$ and missing $h^{-1}[A \cup B]$. We shall use $e_1, e_2, \ldots,$ and e_{n_m} to construct Γ.

Suppose then that $j = 1, 2, \ldots,$ or n_m, and let L_{1j} and L_{2j} denote the jth loops of L_1 and L_2, respectively. L_{1j} and L_{2j} are linked, and each is homotopic to 0 in $W_{a1} \cup W_{a2} \cup C_j$. It follows from Theorem 3 of [5] that there is an arc e_j from a point of $h^{-1}[L_{1j}]$ to a point of $h^{-1}[L_{2j}]$ in $W_{a1} \cup W_{a2} \cup C_j$ such that e_j misses $h^{-1}[A \cup B]$. For each j, let ε_j denote $h[e_j]$. Then ε_j joins the jth loops of L_1 and L_2 and lies in $h[W_{a1} \cup W_{a2} \cup C_j]$.

We show now how to construct a homotopy-$h[\Gamma_a]$ in $h[T_a]$. Suppose p_1 and p_2 are the centerpoints of L_1 and L_2, respectively. For each j, let x_j and y_j denote the endpoints of ε_j on L_{1j} and L_{2j}, respectively; we suppose that $x_j \neq p_1$ and $y_j \neq p_2$. Now suppose that for each j, β_j is any arc on L_{1j} from x_j to p_1, and δ_j is any arc on L_{2j} from y_j to p_2. Let σ_j denote $\beta_j \cup \varepsilon_j \cup \delta_j$; we assume σ_j is an arc. Let Γ denote $\bigcup_{j=1}^{n_m} \sigma_j$. We can easily establish the following proposition.

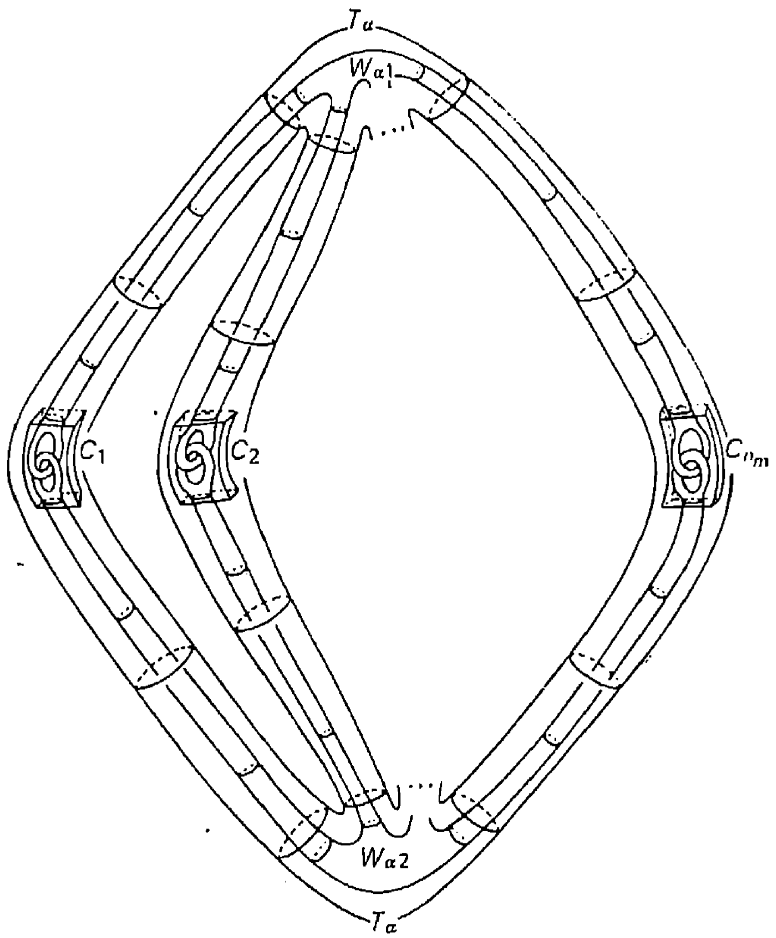

Fig. 17

PROPOSITION 5. *Γ is a homotopy-$h[\Gamma_\alpha]$ in $h[T_\alpha]$, and for each j, σ is the j-th strand of Γ.*

We shall apply this scheme of construction to obtain a Γ as required in Lemma 19.

Case 1. Each of $h[W_{\alpha 1}]$ and $h[W_{\alpha 2}]$ has Property I. We assume L_1 and L_2 selected so that if $i = 1$ or 2 and $j = 1, 2, \ldots,$ or n_m, each loop of L_i has p_i-based oscillation at most 2. Suppose $j = 1, 2, \ldots,$ or n_m. By Lemma 10, there is an arc β_j on L_{1j} from x_j to p_1 and of oscillation at most one, and there is an arc δ_j on L_{2j} from y_j to p_2 and of oscillation at most one. Let σ_j denote $\beta_j \cup \varepsilon_j \cup \delta_j$; clearly σ_j has oscillation at most 2.

Let Γ denote $\bigcup_{j=1}^{n_m} \sigma_j$; by Proposition 5, Γ is a homotopy-$h[\Gamma_a]$ in $h[T_a]$. Since each strand of Γ has oscillation at most 2, it is easy to verify that Γ does not have Property X.

Case 2. $h[W_{a1}]$ has Property I and $h[W_{a2}]$ has Property II.

We assume L_1 selected so that for each j, each loop of L_1 has p_1-based oscillation at most 2, and L_2 so that for some k and l, L_{2k} has p_2-based oscillation at most 4, L_{2l} has p_2-based oscillation at most 2, and for each other j, L_{2j} has oscillation 0. For each j, let β_j be an arc on L_{1j} from x_j to p_1 and of oscillation at most one. Let δ_k be an arc on L_{2k} from y_k to p_2 and of oscillation at most 2, let δ_l be an arc on L_{2l} from y_l to p_2 and of oscillation at most one, and for each other j, let δ_j be any arc on L_{2j} from y_j to p_2. For each j, let σ_j denote $\beta_j \cup \varepsilon_j \cup \delta_j$. σ_k has oscillation at most 3, σ_l has oscillation at most 2, and for every other j, σ_j has oscillation at most one. Let Γ denote $\bigcup_{i=1}^{n_m} \sigma_j$. Then Γ is a homotopy-$h[\Gamma_a]$ in $h[T_a]$. It is easy to see that Γ does not have Property X.

Case 3. $h[W_{a1}]$ has Property I and $h[W_{a2}]$ has Property II. This is similar to Case 2.

Case 4. Each of $h[W_{a1}]$ and $h[W_{a2}]$ has Property II.

We shall select L_1 and L_2 as follows: Since $h[W_{a1}]$ has Property II, there exists a homotopy centerline L_1 of $h[W_{a1}]$, and integers M and N such that (1) the Mth loop L_{1M} of L_1 has p_1-based oscillation at most 4, (2) the Nth loop L_{1N} of L_1 has p_1-based oscillation at most 2, and (3) if j is neither M nor N, the jth loop of L_1 has oscillation 0. Now we use the full strength of Property II. There exist a homotopy centerline L_2 of $h[W_{a2}]$ and distinct integers k and t such that (1) $\{k, t\} \cap \{M, N\}$ has at most one element, (2) the kth loop of L_2 has p_2-based oscillation at most 4, (3) the tth loop of L_2 has p_2-based oscillation at most 2, and (4) every other loop of L_2 has oscillation 0. Let β_M be an arc on L_{1M} from x_M to p_1 and of oscillation at most 2. Let β_N be an arc on L_{1N} from x_N to p_1 and of oscillation at most one. For each other j, let β_j be any arc on L_{1j} from x_j to p_1. Let δ_k be an arc on L_{2k} from y_k to p_2 and of oscillation at most 2. Let δ_t be an arc on L_{2t} from y_t to p_2 and of oscillation at most one. For every other j, let δ_j be any arc on L_{2j} from y_j to p_2. For each j, let σ_j denote $\beta_j \cup \varepsilon_j \cup \delta_j$. Let Γ denote $\bigcup_{j=1}^{n_m} \sigma_j$. Γ is a homotopy-$h[\Gamma_a]$ of $h[T_a]$.

If $M = k$, then $t \neq N$, and in this case, σ_M has oscillation at most 4, each of σ_t and σ_N has oscillation at most one, and for every other j, σ_j has oscillation 0. It is easily seen that in this instance, Γ does not have Property X.

If $M = t$, then $k \neq N$, and in this case, σ_M has oscillation at most 3,

σ_N has oscillation at most one, σ_k has oscillation at most 2, and for every other j, σ_j has oscillation 0. In this case, Γ does not have Property X.

If $N = k$, the situation is analogous to that for $M = t$. Suppose $N = t$; then $M \neq k$. In this case, each of σ_N, σ_M, and σ_j has oscillation at most 2, and for every other j, σ_j has oscillation 0. In this case, Γ does not have Property X.

In every other case, there exist at most two strands with oscillation at most 2, and every other strand has oscillation at most one. In these cases, Γ does not have Property X.

This concludes the proof of Lemma 19.

15. Proof of Lemma 3

Suppose Lemma 3 is false. Then there exist a homeomorphism h from E^3 onto E^3, a non-negative integer m, and a stage m index a such that (1) each homotopy-$h[\Gamma_a]$ in $h[T_a]$ has Property X, but (2) if $i = 1$ or 2 and $j = 1, 2, \ldots,$ or n_m, there is a homotopy-$h[\Gamma_{aij}]$ γ_{aij} in $h[T_{aij}]$ such that γ_{aij} does not have Property X. Now for each i and j, orient γ_{aij}.

Suppose now that $i = 1$ or 2. By Lemma 9, there exist integers k_{ai} and l_{ai} such that $1 \leqslant k_{ai} < l_{ai} \leqslant n_m$ and for each j, each of the k_{ai}th and l_{ai}th strands of γ_{aij} has oscillation at most 2, and if both have oscillation 2, they have the same pattern. Let J_{aij} denote the union of the k_{ai}th and l_{ai}th strands of γ_{aij}.

Suppose that if $i = 1$ or 2, and $j = 1, 2, \ldots,$ or n_m, K_{aij} denotes the union of the k_{ai}th strands of Γ_{aij}. Then J_{aij} is homotopic, in $h[T_{aij}]$, to $h[K_{aij}]$. It follows from properties of the k_{ai}th and l_{ai}th strands of γ_{aij} and Lemma 8, that for each i and j, and each two points x and y of J_{aij}, there is an arc on J_{aij} from x to y and of oscillation at most 2.

Thus the hypothesis of Section 8 is satisfied. By Lemma 18, if $i = 1$ or 2, $h[W_{ai}]$ has Property I or Property II. By Lemma 19, there is a homotopy-$h[\Gamma_a]$ Γ in $h[T_a]$ such that Γ does not have property X. This is a contradiction, and thus Lemma 3 is established.

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