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ON THE MATHEMATICAL ASPECTS OF AN URBAN RETAIL MODEL

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1. Introduction

This paper will be devoted to the subject of mathematical modelling. In fact we will give a survey of the mathematical aspects of a model which is well known in economic geography and was first developed by Huff (1964) and Lakshmanen and Hansen (1965). In this model, it is assumed that we are considering an economic region, e.g. a city, with m living zones L_i $(i = l, \ldots, m)$ and n shopping centers S_j $(j = l, \ldots, n)$.

It is assumed that the total expenditure on goods by inhabitants of zone L_i in a certain period is equal to a fixed amount O_i. How the inhabitants divide their expenditure over the different shopping centers will of course depend on the distance c_{ij} between the living zone and the different shopping centers but also on the attractiveness of the centers. We will assume that W_j is a measure of the attractiveness of shopping center S_j. The attractiveness will of course depend on the size of the center, the diversity of products available, the prices in the center, parking facilities and so on. From this it is immediately clear that it will be very hard to measure W_j in reality.

Now the model assumes that the inhabitants of zone L_i spend

$$O_i W_j^{\alpha} C_{ij} / \sum_{l=1}^{n} W_l^{\alpha} C_{il} \tag{1}$$

in shopping center j, where α is a positive constant and

$$C_{ij} = \exp(-\beta c_{ij}) \tag{2}$$

where also β is a positive constant. We see immediately that if shopping center S_j is further away or less attractive the inhabitants will spend less money in this center. The parameter β is a measure reflecting ease of travel. If for example the fuel prices rise, also β must be increased, since people are less willing to travel for their shopping.

Now let D_j be the total amount which will be spent in shopping center S_j. We see from (1) that

$$D_j = \sum_{i=1}^{m} O_i(W_j^{\alpha} C_{ij} / \sum_{l=1}^{n} W_l^{\alpha} C_{il}), \quad j = 1, \ldots, n, \tag{3}$$

Equation (3) can be seen as the demand side of the model. It will be clear that the costs of shopping center S_j will depend on the attractiveness W_j. It will be assumed that the costs B_j linearly depend on W_j. Hence

$$B_j = kW_j, \quad j = 1, \ldots, n, \tag{4}$$

with k a positive constant.

An equilibrium for the model will be defined as a $W = (W_1, \ldots, W_n) \geqslant 0$ such that

$$D_j = B_j, \quad j = 1, \ldots, n, \tag{5}$$

i.e. the expenditures are equal to the costs for each shopping center S_j.

In this paper we will be mainly interested in the mathematical aspects of the equations (1)–(5). These were already studied by Wilson (1981) but we will use a more profound mathematical approach. We will be interested in the existence and uniqueness of positive equilibria. Furthermore we will consider dynamic and stochastic extensions of the model.

Since we will only concentrate on the mathematical aspects of the model, we will first give the interested reader some references about the empirical applications. The parameters α and β of the model were estimated by several people. In the first place we should mention Wilson and his collaborators. They estimated the parameters for the Leeds region in England and performed a lot of simulations for this region. See e.g. Clarke and Wilson (1986) and the references mentioned in this paper. Lombardo and Rabino (1983) did the same for Rome, while in van Est and Rijk (1981) the model has been estimated for the city of Eindhoven in the Netherlands. The model has not only been used to describe the location of shopping facilities. Clarke and Wilson used the model to advise the authorities of the Piemonte region in Italy to plan the health care facilities in their region. An interesting application of the model is given by Rihll and Wilson (1985). They use the model to study the settlement structure in ancient Greece. Furthermore Wilson (1986) gives a good recent survey of the state of the art on the empirical sides of the model. This paper is organised as follows. In the next section we will devote our attention to the question of existence and uniqueness of a positive equilibrium and to a dynamic extension of the model. In Section 3 we will study the dependence of the equilibria on the parameters of the model, while in Section 4 we will study a stochastic extension of the model. Finally in the last section we will give some conclusions. Since this paper is only a survey of results which are already known, we will not give proofs but we will refer the interested reader to some papers in which the proofs can be found.

2. Existence and uniqueness

As we said in the introduction, we are interested in solutions of equations (1) to (5), which can be resumed as follows

$$\sum_{i=1}^{m} O_i(W_j^\alpha C_{ij}/\sum_{l=1}^{n} W_l^\alpha C_{il})-kW_j = 0, \quad j = 1, \ldots, n, \tag{6}$$

where the O_i's and C_{ij}'s are positive constants.

Hence we have n equations in the n unknows W_j. If we put for example $W_n = 0$, the n-th equation is fulfilled and we can forget W_n in all other equations. In this way we are reduced to $n-1$ equations in $n-1$ unknowns. Hence we immediately see that $W=(0, \ldots, 0)$ is a solution and that lower dimensional solutions give rise to higher dimensional solutions if we add some zeros. This is the reason that we will only consider positive solutions $W=(W_1, \ldots, W_n) > 0$ (i.e. $W_j > 0$ for all $j = 1, \ldots, n$) of equations (6).

THEOREM 1. *The equations* (6) *always have a positive solution.*

Proofs of this result can be found in Rijk and Vorst (1983 a, b) and Chudzynska and Skodkowski (1984). In Rijk and Vorst (1983 a) the theorem is proved by applying Brouwer's fixed point theorem to a system of equations which is equivalent to (6). One cannot apply Brouwer's fixed point theorem immediately to the equations (6) since we are interested in solutions contained in the open domain

$$\boldsymbol{R}_+^n = \{W_1, \ldots, W_n | \ W_j > 0 \text{ for all } j\},$$

while Brouwer's fixed point theorem typically works for closed domains. And as we have seen before, the closure of $\boldsymbol{R}_+^n$ always contains the trivial solution in which we are not interested. So if we apply Brouwer's theorem to $\boldsymbol{R}_+^n$ we only learn that a solution exists but this might be the trivial one.

Now we come to the uniqueness of the positive solutions. We have the following result

THEOREM 2. *If* $\alpha < 1$ *there exists a unique positive solution of the equations* (6).

Proofs of this result can be found in Rijk and Vorst (1983 b), Chudzynska and Slodkowski (1984) and Vorst (1985). In Rijk and Vorst (1983 b) this result is proved by using the Poincaré–Hopf index theorem (see e.g. Milnor (1965) or Guillemin and Pollack (1974)). To apply the Poincaré–Hopf index theorem one needs a dynamical system. Such a dynamical model was already proposed earlier for the urban retail model. See for example Wilson (1981) and Beaumont, Clarke and Wilson (1981). It will be assumed that the W_j's follow the following differential equations

$$\dot{W}_j = f_j(W_j)(D_j-kW_j) = F_j(W) \tag{7}$$

where $f_j(W_j) > 0$ for all $W_j > 0$.

The specifications for $f_j(W_j)$ which are most used in the literature about the urban retail model are

(8) $$f_j(W_j) = \varepsilon$$

or

(9) $$f_j(W_j) = \varepsilon W_j$$

where ε is a positive constant.

The system of differential equations (7) states that if the demand is higher than the costs for shopping center S_j, this center will increase its attractivity, while if the demand is lower it will decrease its attractivity. If one uses (9) instead of (8) it simply means that one assumes that more attractive shopping centers react faster to their demand than the less attractive shopping centers. Since more attractive centers most of the time will be very large, one might also argue that they will react slower. Hence one might also be interested in

(10) $$f_j(W_j) = \varepsilon / W_j.$$

Now we can return to the sketch of the proof of Theorem 2 as given in Rijk and Vorst (1983 b). It is clear that positive solutions of (6) correspond exactly with constant solutions or equilibria of the system of differential equations (7). One can show that there exists a bounded closed convex domain T in $\boldsymbol{R}^n_+$ such that the vectorfield F with specification (8) points inward on the boundary of T and such that there cannot be any solution of (6) lying in $\boldsymbol{R}^n_+ \setminus T$. Furthermore one can prove that in an equilibrium W^* of (7) we have that

(11) $$\det\left(-\frac{\partial F_j}{\partial W_i}(W^*) \right) > 0$$

and hence all equilibria have index 1. Since by the Poincaré–Hopf index theorem the sum of the indexes over all equilibria is 1, there must be exactly one equilibrium. This finishes our proof. From the proof of inequality (11) one can also deduce that the unique equilibrium is locally stable, not only if one uses specification (8) for f_j but also if one uses (9) or (10). If one wants to apply the above method in the case that $\alpha > 1$ the problem is that inequality (11) no longer holds and one can only deduce that there must be at least one equilibrium which gives an alternative proof of Theorem 1. Since we know that there exists a unique positive equilibrium which is locally stable if $\alpha < 1$ one might be interested in the global stability of the unique positive equilibrium. The global stability of the unique positive equilibrium has been proved in Vorst (1985) for (7) with specification (8), (9) or (10). The proof heavily relies on the fact that (7) with specification (10) is a Lagrangian or gradient system. Using this fact it is also possible to give an alternative proof of Theorem 2 (see appendix 1 of Vorst (1985)).

In the case $\alpha > 1$ it has been shown in Rijk and Vorst (1983 b) that it is possible that there exists more than one positive solution of the equations (6). We will come back to this in the next section, but we can remark here already that the model behaves quite differently if $\alpha < 1$ then if $\alpha > 1$. This is of course very important in practical applications of the model.

3. Dependence of the equilibria on the model parameters

Let

$$\begin{aligned}
\boldsymbol{A} &= \{\alpha \in \boldsymbol{R} | \ 0 < \alpha < 1\}; \\
\boldsymbol{B} &= \{O = (O_i) \in \boldsymbol{R}^m | \ O_i \geq 0 \text{ for all } i, \ O \neq 0\}; \\
\boldsymbol{C} &= \{C = (C_{ij}) \in \boldsymbol{R}^{mn} | \ C_{ij} > 0\}; \\
\boldsymbol{D} &= \{k \in \boldsymbol{R} | \ k > 0\}; \\
\boldsymbol{W} &= \{W = (W_j) \in \boldsymbol{R}^n | \ W_j > 0 \text{ for all } j\}.
\end{aligned}$$

We have the following result

THEOREM 3. *There exists a continuous function* G: $\boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C} \times \boldsymbol{D} \to \boldsymbol{W}$ *such that for all* $(\alpha, O, C, k) \in \boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C} \times \boldsymbol{D}$, $G(\alpha, O, C, k)$ *is the unique positive solution of equations* (6) *where the parameters are equal to* (α, O, C, k).

The proof of this theorem can be found in Kaashoek and Vorst (1984) and is just an application of the implicit function theorem. Hence we see that the model has a unique solution if $\alpha < 1$, which continuously depends on the parameters of the model and which is also globally stable according to the preceeding section. One might say that the model behaves smoothly if $\alpha < 1$.

We will now concentrate our attention to the case where $\alpha > 1$.

The first result is a negative one.

THEOREM 4. *If* $\alpha > 1$ *and there are more shopping centers than living zones (i.e.* $n > m$*) there cannot be any positive asymptotically stable equilibrium for the model given by* (7).

The proof of this result can be found in Kaashoek and Vorst (1984). From this we learn that it is only possible to have a stable equilibrium if some shopping centers disappear until we have less shopping centers then living zones. This is completely different for the situation with $\alpha < 1$. The next result tells us that the situation is not too bad in some sense. Let

$$\boldsymbol{A}^* = \{\alpha \in \boldsymbol{R} | \ \alpha > 1\}.$$

THEOREM 5. *There exists a set* M *of measure zero in the parameter space* $\boldsymbol{A}^* \times \boldsymbol{B} \times \boldsymbol{C} \times \boldsymbol{D}$ *such that for every combination of parameters which is not lying in* M, *the equations* (6) *have only a finite number of positive solutions.*

The proof of this result can also be found in Kaashoek and Vorst (1984). In the same paper the following special case has been studied. Assume that we have two zones which are both shopping center and living zone. Hence $n = m = 2$. Furthermore assume $\alpha = 2$, $C_{11} = C_{22}$; $C_{22} > C_{12}$; $C_{12} = C_{21}$; $C_{21} > 0$. Let $C = C_{12}/C_{22} < 1$. By scaling we might assume that $(O_1 + O_2)/k = 1$. Define $Q = O_1/k - \frac{1}{2}$ hence $-\frac{1}{2} \leqslant Q \leqslant \frac{1}{2}$. The equations (6) become for this special case

$$\frac{(Q+\frac{1}{2})W_1^2}{W_1^2 + W_2^2 C} + \frac{(\frac{1}{2}-Q)W_1^2 C}{W_1^2 C + W_2^2} - W_1 = 0, \tag{12}$$

$$\frac{(Q+\frac{1}{2})W_2^2 C}{W_1^2 + W_2^2 C} + \frac{(\frac{1}{2}-Q)W_2^2}{W_1^2 C + W_2^2} - W_2 = 0. \tag{13}$$

It immediately follows from (12) and (13) that $W_2 = 1 - W_1$. Using this in equation (12) we can reduce the equations to just one polynomial equation of degree 5 in W_1. $W_1 = 0$ and $W_1 = 1$ (i.e. $W_2 = 0$) are of course trivial solutions of this equation. The parameters in the equation are Q and C. In Figure 1 a) we have depicted the dependence of the W_1-value of the equilibria on the parameters Q and C. It is clear that for some combinations of the parameter values the equilibrium will not be unique and in fact figure b) shows that we have a cusp catastrophe for a certain combination of the parameter values. By

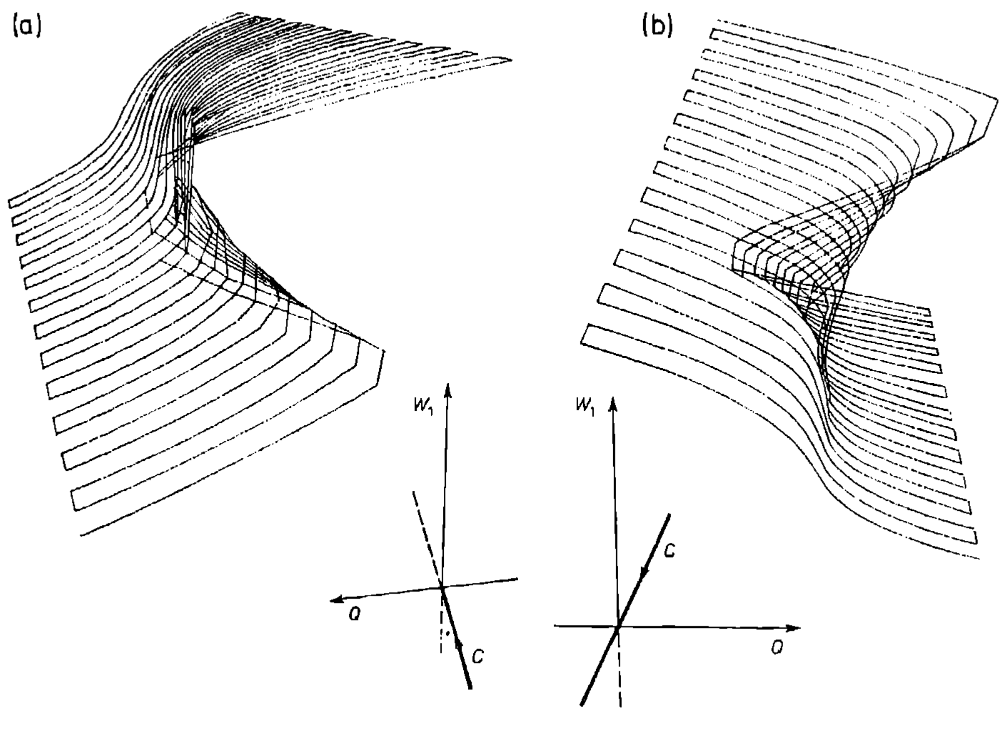

Fig. 1

studying also the dynamic extension of the model in this special case one can show that at most one of the equilibria is stable and that if there is only one equilibrium this one is unstable. Assume that the parameters are such that we have three positive equilibria. Only one of this three will be stable. Let the system be in this stable situation, hence we have two shopping centers. Now, if the fuel prices decrease, the following might happen. A decrease in β (parameter of the fuel prices) gives rise to an increase in C and we eventually come in a situation with only one positive equilibrium and besides, this equilibrium will be unstable. The system will now suddenly move to a boundary equilibrium, i.e. an equilibrium with $W_1 = 0$ or $W_2 = 0$, depending on the value of Q. This means that one of the centers disappears. In reality this kind of phenomena has been often encountered. One might think of the disappearance of a lot of small shops due to the improved transportation possibilities for individuals. Such an improvement has the same effect as a decrease in the fuel prices since in both cases large distances are easier to overcome and form no longer a dominant factor in the decision where to do the shopping. In Kaashoek and Vorst (1984) other economic implications of the cusp catastrophe for this model have been given.

4. A stochastic version of the model

Since the urban retail model essentially only uses distances, expenditures of inhabitants and attractiveness of shopping centers as variables, it is clear that it can only be a very rough approximation of reality. There will be a lot of other uncertain factors which will influence the shopping decisions of the individuals. A more realistic model will therefore have a stochastic component. In Vorst (1985) the following extension of the model has been proposed:

$$dW_j = \varepsilon W_j(D_j - kW_j)dt + \sum_{i=1}^{n} g_{ji}(W)dZ_i(t) \tag{14}$$

where the $Z_i(t)$ are independent Wiener processes. Hence we have a system of stochastic differential equations. The behaviour of the time paths of solutions of the equations (14) will heavily depend on the specification for g_{ji} one uses. In Vorst (1985) we considered two specifications. The first one is the following:

$$dW_j = \varepsilon W_j(D_j - kW_j)dt + \delta(D_j - k_j W_j)dZ_j(t), \tag{15}$$

where δ is a positive constant.

The equations (15) state that if one is closer to an equilibrium the stochastic influences will be less than if one is further away form an equilibrium.

In Vorst (1985) the following result has been proved using a well-known stability result which can for example be found in Schuss (1980), p. 128.

THEOREM 6. *If $\alpha < 1$ and δ is small enough then the unique positive equilibrium $\bar{W}$ of the equations* (6), *is a globally stochastically asymptotically stable equilibrium of the system of stochastic differential equations* (15).

Instead of assuming that the stochastic influences reduce if one approaches an equilibrium one might assume that the stochastic influences remain constant in size. This happens if one uses the following system of stochastic differential equations:

$$dW_j = \varepsilon(D_j - kW_j)dt + \sum_{i=1}^{n} g_{ji}\, dZ_i(t) \tag{16}$$

where the g_{ji}'s are constants. In this case one can show that eventually always at least one of the shopping centers must disappear or one of them must become immensely large. For a precise statement of this result the reader is referred to Vorst (1985). A lot of research about the stochastic version remains to be done, since most research has been devoted to the deterministic versions of the model.

Conclusions. In this paper we describe the mathematics behind a model which has been developped by economic geographers to describe the location of shopping facilities. The model tries to describe the location of shopping centers using the expenditures of the inhabitants of a city and the distances between shopping centers and living zones as exogeneous variables. The location of the shopping centers is the solution of a set of n non-linear equations in n unknowns. We gave a survey on the mathematical results which might be useful in the application of the model in real world situations. In the first place we focussed our attention on existence and uniqueness of solutions. We have seen that if the parameter $\alpha < 1$ we have a unique solution, while if $\alpha > 1$ we don't have unique solutions in general. In this case the dependence of the solutions on the parameters is no longer continuous. We also considered dynamic and stochastic extensions of the model.

In the economic geographic literature there are a lot of models which are not very well investigated from a mathematical point of view. Also for users of these models it is very important to know whether solutions exist and/or are unique. Also the dependence on the parameters is very important. Birkin and Wilson (1986) describe some of these models for industrial location. Sikdar and Karmeska (1982) give a stochastic model for immigration and population growth. These are by far not the only examples in the economic geographic literature but might be a good start for an interested mathematician.

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