FUNCTIONAL CONTINUITY
OF COMMUTATIVE \( m \)-CONVEX \( B_0 \)-ALGEBRAS
WITH COUNTABLE MAXIMAL IDEAL SPACES

BY

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All algebras in this paper are commutative algebras over the field of complex numbers. A \( B_0 \)-algebra is a locally convex completely metrizable topological algebra. The topology of such an algebra \( A \) can be given by means of an increasing sequence of seminorms

\[
\|x\|_1 \leq \|x\|_2 \leq \ldots
\]

for all elements \( x \) in \( A \). The requirement of joint continuity of multiplication in a topological algebra means that the seminorms (1) can be chosen so that

\[
\|xy\|_i \leq \|x\|_{i+1} \|y\|_{i+1}
\]

for \( i = 1, 2, \ldots \), and all elements \( x \) and \( y \) in \( A \).

A \( B_0 \)-algebra is said to be \textit{locally multiplicatively-convex algebra} (shortly an \textit{\( m \)-convex algebra}) if instead of (2) we have stronger relations

\[
\|xy\|_i \leq \|x\|_i \|y\|_i
\]

for all elements \( x, y \in A \), \( i = 1, 2, \ldots \). In case when the algebra in question possesses the unit element \( e \), the seminorms satisfying (1) and (3) can be so chosen that they also satisfy

\[
\|e\|_i = 1
\]

for all natural \( i \). The \( m \)-convex \( B_0 \)-algebras are also called \textit{Fréchet algebras}. These algebras were introduced and studied in [1] and [5]. One of basic questions posed in [5] (Question 1, p. 50), considered also by S. Mazur, asks whether in a commutative \( m \)-convex \( B_0 \)-algebra \( A \) all its multiplicative-linear functionals are automatically continuous. This problem, still unsolved, called attention of many authors. For an up to day bibliography the reader is referred to the book [3] devoted to this subject. The topological algebras in which all multiplicative-linear functionals must be automatically continuous are called by Michael \textit{functionally continuous}. Since there is a one-to-one
correspondence between multiplicative-linear functionals and maximal modular ideals, of codimension one, given by \( f \mapsto \ker f \). the problem of functional continuity of \( m \)-convex \( B_0 \)-algebras is equivalent to the question whether in such algebras all such ideals are closed. We denote by \( \mathfrak{M}(A) \) the maximal ideal space of \( A \), i.e. the set of all its closed maximal modular ideals, or all its continuous multiplicative-linear functionals provided with the Gelfand topology. We also denote by \( \mathfrak{M}^*(A) \) the space of all maximal modular ideals of codimension one (multiplicative-linear functionals) in \( A \). If \( A \) is a \( Q \)-algebra, i.e. the algebra with open set of invertible elements in case when \( A \) possesses the unit element, or open set of quasi-invertible elements in general, then all its maximal modular ideals are closed and of codimension one, provided the algebra in question is a \( B_0 \)-algebra. Otherwise, as we showed in [6] there are always maximal ideals which are of infinite codimension and dense in \( A \). The strongest result on the problem of functional continuity of \( m \)-convex \( B_0 \)-algebras seems to be the result of Arens [2] stating that if such an algebra is finitely generated (i.e. there is a finite number of elements in \( A \) (the system of generators) such that the algebra of all polynomials in these elements is dense) then it is functionally continuous. We shall need this result in the following form:

**Theorem A.** Let \( A \) be a commutative complex \( m \)-convex \( B_0 \)-algebra and let \( F \in \mathfrak{M}^*(A) \). Then for each finite number of elements \( x_1, x_2, \ldots, x_n \in A \) there is a functional \( f \) in \( \mathfrak{M}(A) \) such that

\[
F(x_i) = f(x_i)
\]

for \( i = 1, 2, \ldots, n \).

Using this result we prove in this paper that if the maximal ideal space \( \mathfrak{M}(A) \) of a commutative \( m \)-convex \( B_0 \)-algebra \( A \) is at most countable, then \( A \) is a functionally continuous algebra. As a corollary we obtain a result of Husain and Liang [4] stating that commutative \( m \)-convex \( B_0 \)-algebras with orthogonal Schauder bases are functionally continuous. Our result reads as follows.

**Theorem.** Let \( A \) be a commutative \( m \)-convex \( B_0 \)-algebra with at most countable maximal ideal space. Then all multiplicative linear functionals in \( A \) are continuous.

**Proof.** Without loss of generality we can assume that the algebra \( A \) possesses the unit element \( e \). Otherwise we could consider the algebra \( A_1 \), obtained from \( A \) by adjoining the unit \( e \). It is the direct sum \( A_1 = A \oplus C e \) provided with seminorms given by the formula \( \| x + \lambda e \| = \| x \|_n + |\lambda | \) for \( x \in A \) and \( \lambda \in C \). All elements in \( \mathfrak{M}^*(A) \) extend to elements of \( \mathfrak{M}^*(A_1) \) by setting \( f(x + \lambda e) = f(x) + \lambda \). Also the cardinality of \( \mathfrak{M}(A_1) \) is the same as that of \( \mathfrak{M}(A) \). We can also assume that the space \( \mathfrak{M}(A) \) is infinite, otherwise, by Theorem 13.6 in [5], \( A \) is a \( Q \)-algebra and the conclusion follows.
Let then $\mathcal{M}(A) = \{f_1, f_2, \ldots\}$. We shall construct an element $z$ in $A$ such that

\begin{equation}
 f_i(z) \neq f_j(z)
\end{equation}

for all natural $i \neq j$. To this end observe that for each natural $k$ there is an element $y_k \in A$ such that

\begin{equation}
 f_i(y_k) = 0
\end{equation}

for $i < k$, and

\begin{equation}
 f_k(y_k) = 1
\end{equation}

for all indices $k$. In fact, for a given natural $m$ and $n$, $m \neq n$, we can find an element $x$ in $A$ such that

\[ x = f_m(x) \neq f_n(x) = \beta. \]

Setting

\[ y_{m,n} = \frac{x - \alpha e}{\beta - \alpha} \]

we have $f_m(y_{m,n}) = 0$ and $f_n(y_{m,n}) = 1$. Finally setting $y_k = \prod_{i < k} y_{i,k}$ we obtain an element satisfying relations (7) and (8).

We shall construct inductively a sequence $\{z_n\}$ of elements of $A$ satisfying

\begin{equation}
 f_k(z_m) = f_k(z_n)
\end{equation}

for all indices $k, m, n$ satisfying $k \leq m \leq n$;

\begin{equation}
 f_k(z_n) \neq f_l(z_n)
\end{equation}

for all indices $k, l, n$ satisfying $k, l \leq n$;

\begin{equation}
 \|z_{n+1} - z_n\| \leq 2^{-n}
\end{equation}

for all natural $n$. To this end we put $z_1 = 0$ and assuming that we have already constructed elements $z_1, z_2, \ldots, z_n$ satisfying relations (9), (10), (11) we construct the element $z_{n+1}$ in the following way. If $f_{n+1}(z_n) \neq f_k(z_n)$ for all $k \leq n$ we simply put $z_{n+1} = z_n$. If $f_{n+1}(z_n) = f_k(z_n)$ for some $k \leq n$ we put $z_{n+1} = z_n + \alpha y_{n+1}$, where the complex scalar $\alpha$ is chosen so that $f_{n+1}(z_{n+1}) = f_{n+1}(z_n) + \alpha \neq f_l(z_n)$ for all $i \leq n$, and $|\alpha| \cdot \|y_{n+1}\| \leq 2^{-n}$. The relations (4), (7) and (8) now show that we have (9), (10), and (11) for all involved indices not greater than $n+1$. The induction follows. The relation (11) and the completeness of $A$ imply that the sequence $\{z_n\}$ converges in $A$. Define $z = \lim z_n$. By (9) we have $f_k(z_m) = f_k(z)$ for $k \leq m$ and thus the relation (10) implies the relation (6). Assume now in (5) $n = 2$, $x_1 = z$ and $x_2 = x$. 

arbitrary element in $A$. Relation (5) implies that there exists an index $i_0$ such that
\begin{equation}
F(z) = f_{i_0}(z)
\end{equation}
and
\begin{equation}
F(x) = f_{i_0}(x).
\end{equation}
But relation (12) determines the index $i_0$ uniquely, since the element $z$ separates between the points of $\mathfrak{M}(A)$. By (13) we have $F = f_{i_0}$, since $x$ was an arbitrary point in $A$. Conclusion follows.

An $m$-convex $B_0$-algebra is said to possess an orthogonal basis if there is a Schauder basis $\{e_i\}$ in $A$ such that $e_i e_j = 0$ for $i \neq j$. Thus each element $x$ in $A$ can be written uniquely as $x = \sum_{i=1}^{\infty} \alpha_i(x) e_i$, the series being convergent in $A$ and the coefficients $\alpha_i(x)$ being continuous linear functionals on $A$. Clearly any such algebra is commutative. Husain and Liang [4], cf. also [3], Theorem 3.47, have shown that every such algebra is functionally continuous. We shall obtain this result as a corollary to our theorem. In fact, let $f$ be a non-zero continuous multiplicative-linear functional on $A$. There exists an index $i_0$ such that $f(e_{i_0}) \neq 0$. Otherwise for any element $x$ in $A$ we have
\begin{equation}
f(x) = f\left(\sum_{i=1}^{\infty} \alpha_i(x) e_i\right) = \sum_{i=1}^{\infty} \alpha_i(x) f(e_i) = 0
\end{equation}
and $f = 0$. Also for $i \neq i_0$ we have $0 = f(e_{i_0} e_i) = f(e_{i_0}) f(e_i)$ and so $f(e_i) = 0$. This implies $f(x) = \sum_{i=1}^{\infty} \alpha_i(x) f(e_i) = \alpha_{i_0}(x) f(e_{i_0})$. Thus $f$ is a scalar multiple of $\alpha_{i_0}$, which implies that the space $\mathfrak{M}(A)$ is at most countable. We obtain

**Corollary** (Husain and Liang). *Let $A$ be an $m$-convex $B_0$-algebra with an orthogonal basis. Then $A$ is functionally continuous.*

**Remark.** The above corollary is formulated in a slightly more general way than the result in [4], since it is assumed there that the basis is unconditional, which is irrelevant in our proof.

**REFERENCES**


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