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CHARACTERISTIC CLASSES OF MULTIFOLIATIONS

BY

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In this paper we present results concerning product structures, almost flag and flag structures and multifoliations. Characteristic classes of these structures are considered from several points of view. In Section 1 a vanishing theorem for the characteristic classes of product structures is proved. In Section 2 flag manifolds are discussed. We define new characteristic classes, prove a vanishing theorem for them and point out obstructions to the existence of a product structure defining a given flag structure. In Section 3 we deal with similar problems for almost flag structures and in Section 4 for multifoliate structures. The exotic characteristic classes are discussed in Section 5. In the last section we present results on residues of flag manifolds which are generalizations of Heitsch's results from [7].

The geometrical objects considered in this paper are smooth, i.e. of class C^∞. We assume the knowledge of the basic definitions from [2], [9].

Notation. Throughout the paper, if V is a vector space and $V_1, \ldots, V_k$ is a set of subspaces of V ordered by inclusion, then $\mathrm{GL}(V; V_1, \ldots, V_k)$ (resp. $\mathrm{gl}(V; V_1, \ldots, V_k)$) denotes the Lie group of linear isomorphisms of V preserving each of the spaces V_i, $i = 1, \ldots, k$ (resp. the Lie algebra of linear mappings preserving each of the subspaces V_i, $i = 1, \ldots, k$). For $V = R^q$ and $V_i = R^{q_i}$ the Lie group $\mathrm{GL}(R^q; R^{q_1}, \ldots, R^{q_k})$ is denoted by $\mathrm{GL}(q; q_1, \ldots, q_k)$ and $\mathrm{gl}(R^q; R^{q_1}, \ldots, R^{q_k})$ by $\mathrm{gl}(q; q_1, \ldots, q_k)$. Then, of course, $q_i < q_{i+1}$. If we consider the vector space as the direct sum of subspaces $V_1, \ldots, V_k$ then the Lie group of linear isomorphisms of V preserving $V_1, \ldots, V_k$ is denoted by $\mathrm{GL}(V_1, \ldots, V_k)$ and its Lie algebra of linear mappings preserving $V_1, \ldots, V_k$ by $\mathrm{gl}(V_1, \ldots, V_k)$. In case of $V_i = R^{q_i}$ we denote $\mathrm{GL}(R^{q_1}, \ldots, R^{q_k})$ by $\mathrm{GL}(q_1, \ldots, q_k)$ and $\mathrm{gl}(R^{q_1}, \ldots, R^{q_k})$ by $\mathrm{gl}(q_1, \ldots, q_k)$. The algebra of invariant polynomials of a Lie algebra of a Lie group G is denoted by $I(G)$.

1. Characteristic classes of product structures. First of all we are going to compute $I(\mathrm{GL}(V_1, \ldots, V_k))$, where $V = V_1 \oplus \ldots \oplus V_k$. Since

$$g = \mathrm{gl}(V_1, \ldots, V_k) = \mathrm{gl}(V_1) \otimes \ldots \otimes \mathrm{gl}(V_k),$$

$$I(G) = I(\mathrm{GL}(V_1, \ldots, V_k)) = I(\mathrm{GL}(V_1)) \otimes \ldots \otimes I(\mathrm{GL}(V_k)).$$

Now we proceed to prove a vanishing theorem for product structures. By a *product structure* we understand an integrable $\mathrm{GL}(V_1, \ldots, V_k)$-structure. Let $F = (F_1, \ldots, F_k)$ be such a structure, $q_i = \dim F_i$, $g_i = \sum_{j \leqslant i} q_j$, $g_k = n$. There exists an adapted atlas to this structure. A product structure gives a reduction of the structure group of the tangent bundle TM to the group $\mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_k)$. The integrability of the $\mathrm{GL}(V_1, \ldots, V_k)$-structure assures that infinitesimal automorphisms of this structure are locally of the form

$$\sum f_i \, \partial_i$$

where $\partial_j(f_i) = 0$ for $g_{r-1} < i \leqslant g_r$, $j \leqslant g_{r-1}$ or $j > g_r$.

On the tangent bundle we introduce a connection ∇ as follows. Let ∇^i be any connection on F_i. Let $s \in F_i$, $F^i = \bigoplus_{i \neq j} F_j$, $\mathscr{U}_F$ the sheaf of germs of infinitesimal automorphisms of F. Then we put

$$\nabla_X s = [X, s] \quad \text{for} \quad X \in F^i \cap \mathscr{U}_F,$$

$$\nabla_X s = \nabla^i_X s \quad \text{for} \quad X \in F_i.$$

To prove that ∇ is well defined we have to show that it is independent of the choice of an infinitesimal automorphism X. Indeed, if

$$X = \sum f_j \, \partial_j, \quad j \leqslant g_i \text{ or } j > g_{i+1},$$

then

$$[X, s] = [\sum f_j \, \partial_j, s] = \sum f_j [\partial_j, s] - \sum s(f_j) \, \partial_j = \sum f_j [\partial_j, s]$$

which proves independence of the choice of X.

Note. The connection ∇ is a Bott connection for each foliation F_i.

Now we will prove several lemmas which will allow us to formulate and prove a vanishing theorem.

LEMMA 1.1. *$R(X, Y)s = 0$ for $X, Y \in F^i$, $s \in F_i$, where R is the curvature tensor of the connection ∇.*

Proof. We have

$$\begin{aligned} R(X, Y)s &= (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})s \\ &= [X, [Y, s]] - [Y, [X, s]] - [[X, Y], s] = 0 \end{aligned}$$

as $[X, s]$ and $[Y, s]$ belong to F_i.

LEMMA 1.2. *The Chern–Weil homomorphism annihilates $I^r(\mathrm{GL}(V_i))$ for $r > q_i$.*

Proof. Let $\Omega = (\Omega^r_s)$ be the curvature form pulled back to the base manifold (locally). Since $R(X, Y)s = 0$ for $s \in F_i$, $X, Y \in F^i$, locally we have

$$\Omega^t_s = \sum a^{ts}_r b_r,$$

where b_r are forms vanishing on F^i; $g_{i-1} < s, t \leqslant g_i$. The desired result now follows directly from the definition of the Weil homomorphism – in computation we take into account components Ω^t_s of Ω for $g_{i-1} < t, s \leqslant g_i$, which we denote by Ω^i – and the fact that $(\Omega^i)^r = 0$ for $r > q_i$.

LEMMA 1.3. *Let Γ^v_{ts} be Christoffel symbols for the connection ∇^i and let $g_{i-1} < v, t, s \leqslant g_i$. If $\partial_r \Gamma^v_{ts} = 0$ for $r \leqslant g_{i-1}$, $r > g_i$, then $\Omega^i(X, Y) = 0$ for any Y, and X belonging to F^i.*

Proof. Since we have proved that $R^i(X, Y) = 0$ for $X, Y \in F^i$, it is only necessary to prove that $R^i(X, Y) = 0$ for $Y \in F_i$ and $X \in F^i$. We have

$$R^i(X, Y)s = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})s.$$

Let $s = \sum_{g_{i-1}+1}^{g_i} f_t \partial_t$ and $X = \partial_r$, $Y = \partial_m$, $r \leqslant g_{i-1}$ or $r > g_i$, $g_{i-1} < m \leqslant g_i$. Then

$$\begin{aligned} R^i(X, Y)s &= [\partial_r, \nabla^i_{\partial_m} \sum f_t \partial_t] - \nabla^i_{\partial_m}[\partial_r, s] \\ &= [\partial_r, \sum (\partial_m(f_t)\partial_t + f_t \Gamma^v_{mt} \partial_v)] \\ &= \sum (\partial_r \partial_m(f_t)\partial_t + \partial_r(f_t)\Gamma^v_{mt}\partial_v + f_t \partial_r \Gamma^v_{mt} \partial_v) \\ &= \sum f_t \partial_r(\Gamma^v_{mt})\partial_v = 0. \end{aligned}$$

Now we are in a position to state a vanishing theorem.

THEOREM 1.4. *Let $F = (F_1, \ldots, F_k)$ be a product structure, $q_i = \dim F_i$. Then the Chern–Weil homomorphism of the foliated bundle $L(TM/F_i)$ factorizes through that of the reduction to the group $\mathrm{GL}(V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_k)$. The Chern–Weil homomorphism of the reduced bundle annihilates $I^r(\mathrm{GL}(V_j))$ for $r > q_j$. If the space of leaves of the foliation $F^j - M/F^j$ is a paracompact manifold, then the Chern–Weil homomorphism annihilates $I^r(\mathrm{GL}(q_i))$ for $r > [q_i/2]$.*

Proof. We have only to prove the last statement. Since M/F^j is a paracompact manifold, there is a connection on M/F^j. Since the bundle F_j is an inverse image by the projection p_j: $M \to M/F^j$ of the tangent bundle, the connection on M/F^j induces a connection ∇^j on F_j having the properties required in Lemma 1.3.

COROLLARY 1.5. *If F^j is a compact foliation (i.e. F^j has only compact leaves) and M/F^j is a Hausdorff manifold, then the Chern–Weil homomorphism annihilates $I^r(\mathrm{GL}(V_j))$ for $r > [q_j/2]$.*

Proof. The Theorem 4.1 of [3] asserts that the projection p_j: $M \to M/F^j$ is closed. Since the image of a paracompact manifold by a closed map is paracompact, we get the result from the second part of Theorem 1.4.

2. Characteristic classes of flag manifolds.

Definition. A *flag structure on a manifold* is a system of foliations $F = (F_1, \ldots, F_k)$ which is ordered by inclusion, i.e. $F_i \subset F_{i+1}$. Let $q_i = \operatorname{codim} F_i$ for $i \geqslant 2$ and $q_1 = \dim F_1$. A flag structure on M induces a reduction of the structure group of the bundle TM/F_1 to the group $\mathrm{GL}(n-q_1;\, q_2, \ldots, q_k)$, where n is the dimension of the manifold M.

Now we will compute the algebra of invariant polynomials on the Lie algebra $\mathrm{gl}(n-q_1;\, q_2, \ldots, q_k)$.

Proposition 2.1. *The algebra of invariant polynomials on the Lie algebra* $\mathrm{gl}(n-q_1;\, q_2, \ldots, q_k)$ *is isomorphic to the algebra of invariant polynomials on the Lie algebra* $\mathrm{gl}(q_2, \ldots, q_k - q_{k-1}, n - q_1 - q_k)$, *i.e.*

$$I(\mathrm{GL}(n-q_1;\, q_2, \ldots, q_k)) = I(\mathrm{GL}(q_2, \ldots, q_k - q_{k-1}, n - q_1 - q_k)).$$

Since the proof of the proposition is not essential for the understanding of the paper we defer it to the appendix.

Remark. Unfortunately, the pair $(\mathrm{gl}(n-q), \mathrm{gl}(q_2, \ldots, q_k))$ is not a reductive pair, so we cannot apply Kamber–Tondeur's theory.

Our next step is to prove a vanishing theorem for flag structures. Via an adapted atlas to the flag structure and a partition of the unity it is possible to construct a Riemannian metric on the bundle TM/F_1 which is nondegenerate on each F_i/F_1.

Let $S_i =: F_{i+1} \cap F_i^{\perp}$. Then $S_i \oplus F_i = F_{i+1}$ and S_i is isomorphic to F_{i+1}/F_i. It is possible to choose a Bott connection in the bundle TM/F_i preserving F_{i+1}/F_i. Let us denote the connection induced in the bundle S_i by ∇^i. The bundle TM/F_1 is isomorphic to the bundle $\bigoplus S_i$. Then define ∇ as

$$\nabla_X s = \nabla^i_X s \quad \text{for} \quad s \in S_i.$$

In view of Proposition 2.1 we can consider the algebras $I(\mathrm{GL}(q_2))$, $I(\mathrm{GL}(q_i - q_{i+1}))$, $I(\mathrm{GL}(n - q_k - q_1))$ as subalgebras of the algebra $I(\mathrm{GL}(n - q_1;\, q_2, \ldots, q_k))$. The image (by the Chern–Weil homomorphism) of an element of one of these subalgebras depends only on the corresponding component of the curvature form. But $(\Omega^i)^r = 0$ for $r > n - q_i - q_1$ for $i \leqslant k$. Therefore we have the following proposition.

Proposition 2.2. *Let* $F = (F_1, \ldots, F_k)$ *be a flag structure. Then the Chern–Weil homomorphism of the reduction of the normal bundle of the foliation* F_1 *to the structure group* $\mathrm{GL}(n-q_1;\, q_2, \ldots, q_k)$ *annihilates*

$$I^r(\mathrm{GL}(q_2)) \quad \textit{for} \quad r > n - q_2 - q_1,$$

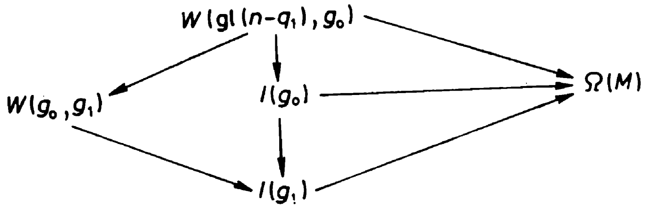
$$I^r(\mathrm{GL}(q_{i+1} - q_i)) \quad \textit{for} \quad r > n - q_i - q_1 \quad \textit{for} \quad 2 < i < k$$

and

$$I^r(\mathrm{GL}(n - q_k - q_1)) \quad \textit{for} \quad r > n - q_k - q_1.$$

Combining Proposition 2.2 with Theorem 1.4 we obtain obstructions to the existence of a product structure inducing the given flag structure.

Let $g_0 = \mathrm{gl}(n-q_1;\ q_2, \ldots, q_k)$ and $g_1 = \mathrm{gl}(q_2, \ldots, q_k - q_{k-1}, n-q_k-q_1)$. Then we have the following factorization of the Weil homomorphism



THEOREM 2.3. *Let $F = (F_1, \ldots, F_k)$ be a flag structure. The necessary condition for the existence of a product structure $F' = (F'_1, \ldots, F'_{k+1})$ such that $F_1 = F'_1$, $F_i = \bigoplus_{j\leq i} F'_j$ is the vanishing of the following characteristic classes*

$$\Delta_* f \quad \text{for} \quad f \in I^r(\mathrm{GL}(q_2)),\ n-q_1 \geq r > q_2,$$
$$f \in I^r(\mathrm{GL}(q_{i+1}-q_i)),\ n-q_i-q_1 \geq r > q_{i+1}-q_i.$$

Both Proposition 2.2 and Theorem 1.4 do not imply vanishing for other characteristic classes than those given by polynomials $f \in I^r(\mathrm{GL}(q))$ for $r > q = n-q_1$.

Now we introduce a notion of the flag bundle.

Definition. A principal G-bundle $P \to M$ is called a *flag bundle* if it is equipped with a G-equivariant involutive subbundle H such that

(i) $H_u \cap Gu = 0$ for any point u of P,

(ii) there exist involutive G-equivariant subbundles H_i $(i = 1, \ldots, k)$ of H ordered by inclusion.

Since H_i are G-equivariant and involutive, the projections F_i of H_i onto the base manifold M define a flag structure F. Unfortunately, the theory of flag bundles is rather trivial as Proposition 2.4 shows.

For the sake of simplicity, from now on we use terms F-flag bundle and F_i-foliated bundle for the corresponding flag (resp. foliated) bundles with the flag structure F (resp. foliation F_i) on M.

PROPOSITION 2.4. *Let $F = (F_1, \ldots, F_k)$ be a flag structure on a manifold M. A principal G-bundle $P \xrightarrow{p} M$ is an F-flag bundle if $P \to M$ is an F_k-foliated bundle.*

Proof. The result follows from the following lemma.

LEMMA 2.5. *Let P be an F-foliated bundle and F_1 be a subbundle of the bundle F. Then F_1 is a foliation iff $H \cap H^{-1} F_1$ is a flat partial connection.*

Proof. Let F_1 be a foliation, $X_i \in F_1$. Then

$$H \ni [\tilde{X}_1, \tilde{X}_2] = [X_1, X_2]^{\sim} \in H^{-1} F_1.$$

If $H \cap H^{-1} F_1$ is a flat partial connection, then

$$H^{-1} F_1 \ni [\tilde{X}_1, \tilde{X}_2] \quad \text{and} \quad p_* [\tilde{X}_1, \tilde{X}_2] = [X_1, X_2] \in F_1.$$

Proposition 2.4 indicates a rather close relation between characteristic classes of F_i-foliated bundles P when P is an F-flag bundle.

COROLLARY 2.6. *The characteristic classes of an F_i-foliated bundle P are characteristic classes of the F-flag bundle P and of the F_{i+1}-foliated bundle P.*

Proof. An adapted connection to the F-flag bundle (i.e. the F_k-foliated bundle) is also an adapted connection to the F_i-foliated bundle.

3. Characteristic classes of almost flag structures.

Definition. An *almost flag structure* F is a system of distributions $(F_1, \ldots, F_k)$ of constant dimension ordered by inclusion. Let $\dim F_i = q_i$.

We say that an almost flag structure F is *of type* $(r_1, \ldots, r_k)$ if $r_i = \sup \operatorname{codim} C_i$; C_i is the distribution generated by the sheaf $F_i \cap \mathcal{U}_F$, where $\mathcal{U}_F$ is the sheaf of infinitesimal automorphisms of the flag structure F.

We define a connection ∇ in the bundle TM/F_1 as follows. Let S_i be a subbundle of TM such that $F_i \oplus S_i = F_{i+1}$, $F_k \oplus S_k = TM$. Then the bundle S_i is isomorphic to the bundle F_{i+1}/F_i and TM/F_1 is isomorphic to $S_1 \oplus \ldots \oplus S_k$. By S^i we denote the bundle $\bigoplus_{j \geq i} S_j$. Let $X \in TM$, then by X^i we denote the S^i-component of X and by X_i the S_i-component.

Let ∇^i be a connection in S_i defined in the following way. For $Y \in S_i$ and $X \in F_i$ we put

$$\nabla^i_X Y = [X, Y]^i - [X, Y_i]^{i+1};$$

for X in S^i we take any connection in S_i.

LEMMA 3.1. *If R_i is the curvature tensor of the connection ∇^i, then $R_i(X, Y) = 0$ for X, $Y \in C_i$.*

Proof.

$$\begin{aligned}
R_i(X, Y)s &= (\nabla^i_X \nabla^i_Y - \nabla^i_Y \nabla^i_X - \nabla^i_{[X,Y]})s \\
&= \nabla^i_X([Y, s]^i - [Y, s_i]^{i+1}) - \nabla^i_Y([X, s]^i - [X, s_i]^{i+1}) \\
&\quad -[[X, Y], s]^i + [[X, Y], s_i]^{i+1} \\
&= [X, [Y, s]^i]^i - [X, [Y, s]_i]^{i+1} - [X, [Y, s_i]^{i+1}]^i \\
&\quad -([Y, [X, s]^i]^i - [Y, [X, s]_i]^{i+1} - [Y, [X, s_i]^{i+1}]^i) \\
&\quad -[[X, Y], s]^i + [[X, Y], s_i]^{i+1}
\end{aligned}$$

$$
\begin{aligned}
&= [X, [Y, s]^i]^i - [Y, [X, s]^i]^i - [[X, Y], s]^i \\
&\quad -[X, [Y, s]_i]^{i+1} - [Y, [X, s]_i]^{i+1} - [[X, Y], s_i]^{i+1} \\
&\quad -[X, [Y, s_i]^{i+1}]^i + [Y, [X, s_i]^{i+1}]^i \\
&= 0.
\end{aligned}
$$

A connection ∇ on $S = S_1 \oplus \ldots \oplus S_k$ is given by

$$\nabla_X s = \nabla^i_X s \quad \text{for} \quad s \in S_i.$$

The Chern–Weil homomorphism on $I(\mathrm{GL}(q_{i+1} - q_i))$ depends only on the S_i-component of the curvature tensor. Therefore by Lemma 3.1 we have the following proposition.

PROPOSITION 3.2. *Let $F = (F_1, \ldots, F_k)$ be an almost flag structure of type $(r_1, \ldots, r_k)$. Then the Chern–Weil homomorphism annihilates $I^r(\mathrm{GL}(q_{i+1} - q_i))$ for $r > r_i$.*

4. Characteristic classes of multifoliate structures. The simplest non-trivial multifoliate structure is the following: two foliations F_1, F_2 having a foliation as the intersection. We are going to formulate a vanishing theorem for characteristic classes of such a structure.

It follows from [11] and [14] that the distribution $F_1 \cap F_2$ has the following properties:

(i) $F_1 \cap F_2$ is involutive,

(ii) $\dim(F_1 \cap F_2) = \text{const}$ iff $\dim(F_1 + F_2) = \text{const}$,

(iii) if $\dim(F_1 + F_2) = \text{const}$, then $F_1 \cap F_2$ is a foliation.

PROPOSITION 4.1. *The Chern–Weil homomorphism of the flag structure $(F_1, F_1 \cap F_2)$ annihilates $I^r(\mathrm{GL}(q_1 - q_0))$ for $r > n - q_2$, where $q_i = \dim F_i$, $q_0 = \dim(F_1 \cap F_2)$.*

Proof. Since the vector bundle $(F_1 + F_2)/F_2$ is isomorphic to the vector bundle $F_1/(F_1 \cap F_2)$, the result follows from Theorem 2.2.

Note. A similar, but slightly weaker result was obtained by Andrzejczak in [1]. He considered only foliations F_1, F_2 such that $F_1 + F_2 = TM$.

5. Exotic characteristic classes of almost flag structures. For all details on exotic characteristic classes and convexity of sets of connections see [12]. We will prove convexity of sets of connections related to a given almost flag structure and draw some conclusions from this fact for characteristic classes.

PROPOSITION 5.1. *Let F be an almost flag structure. The set of F-connections, i.e. connections adapted to F* (see §3), *is convex.*

Proof. One of the basic facts used in the construction of F-connections is the existence of subbundles S_i of the tangent bundle TM such that $S_i \oplus F_i = F_{i+1}$. Such a sequence of subbundles is given by a sequence of projections s_i

$$0 \longrightarrow F_i \overset{s_i}{\rightleftarrows} F_{i+1} \longrightarrow F_{i+1}/F_i \longrightarrow 0$$

Let ∇, ∇' be two connections in whose construction we have used such two sequences of subbundles S and S' given by two sequences of projections s_i and s_i', respectively. Since $T(M \times I) = p^{-1}\, TM \oplus 1$, where p: $T(M\times I) \to TM$ is the natural projection and 1 is the vector bundle tangent to its fibre, put $\bar{F}_i = p^* F_i \oplus 1$. Let $s_i^t = ts_i' + (1-t)s_i$.

Define $\bar{S}_i$ as

$$\bar{S}_i | M \times \{t\} = S_i^t, \quad S_i^t = \ker s_i^t.$$

Then

$$\bar{F}_{i+1} = \bar{S}_i \oplus \bar{F}_i.$$

If F is of type $(r_1, \ldots, r_k)$, then $\bar{F}$ is of type $(r_1, \ldots, r_k)$.

Let $\bar{\nabla}$ be an F-connection in whose construction the bundles $\bar{S}_i$ have been used. Then the F-connections $\bar{\nabla}^0 = \bar{\nabla} | M \times \{0\}$ and $\bar{\nabla}^1 = \bar{\nabla} | M \times \{1\}$ are homotopic. We must show that $\bar{\nabla}^0$ is homotopic to the connection ∇ and $\bar{\nabla}^1$ to the connection ∇'. Let $S_i^0 = p^* S_i$. The connection $[\bar{\nabla}^0, \nabla]$ is an F-connection in whose construction the subbundles S_i^0 are used and defines a homotopy between the connections $\bar{\nabla}^0$ and ∇. The same can be done for the other pair, which ends the proof.

PROPOSITION 5.2. *The set of metric connections adapted to the almost flag structure is convex.*

Proof. A Riemannian metric on TM/F_1 is said to be *adapted* to F if, for any i, $F_i \oplus F_i^{\perp} \cap F_{i+1} = F_{i+1}$. Let g_0, g_1 be two Riemannian metrics on TM/F_1 adapted to F, and ∇^0 be a g_0-connection and ∇^1 a g_1-connection.

Choose ε such that $g^t = tg_1 + (1-t)g_0$ $(-\varepsilon < t < 1+\varepsilon)$ is a Riemannian metric on TM/F_1. The metric g^t defines a Riemannian metric $\tilde{g}$ on $TM/F_1 \times (-\varepsilon, 1+\varepsilon) \to M \times (-\varepsilon, 1+\varepsilon)$ by

$$\tilde{g} | M \times \{t\} = g^t.$$

Let $\tilde{\nabla}$ be a $\tilde{g}$-connection, and $\tilde{\nabla}^t = \tilde{\nabla} | TM/F_1 \times \{t\}$. $\tilde{\nabla}$ defines a homotopy between $\tilde{\nabla}^0$ and $\tilde{\nabla}^1$. We have to show that there are homotopies between ∇^0, $\tilde{\nabla}^0$ and ∇^1, $\tilde{\nabla}^1$. Let $\tilde{g}_0$ be a Riemannian metric on $TM/F_1 \times (-\varepsilon, 1+\varepsilon)$ defined as the inverse image of g_0 by the natural projection. The connection $[\nabla^0, \tilde{\nabla}^0]$ is a $\tilde{g}_0$-connection defining a homotopy between ∇^0 and $\tilde{\nabla}^0$. The same can be done for the other pair of connections.

Let $G = \mathrm{GL}(n-q_1;\, q_2-q_1, \ldots, q_k-q_1)$, then

$$I(G) = \bigotimes I(\mathrm{GL}(q_{i+1}-q_i)) \otimes I(\mathrm{GL}(n-q_k)),$$

$J = \bigotimes J_i\ (> r_i)$, $J' = \bigotimes J_i'$ where $J_i' = \mathrm{Id}\,\{c_{2j+1}^i\}$. $J_i\ (> r_i)$ denotes the ideal generated by $c_{j_1}^i \ldots c_{j_a}^i$ such that $\sum j_v > r_i$. Since the curvature of an

F-connection has the property $(\Omega^i)^{r_i+1} = 0$, we have a well defined homomorphism from $W(J, J') = I(G)/J \otimes I(G)/J' \otimes \bigwedge I^+(G)$ to ΩM denoted by $\Delta(\nabla^0, \nabla^1)$, where ∇^0 is an F-connection, ∇^1 a metric connection adapted to F used in the construction of Δ. $\Delta(\nabla^0, \nabla^1)$ defines in cohomology

$$\Delta_*(\nabla^0, \nabla^1)\colon H(W(J, J')) \to H(M).$$

Elements $\Delta_*(\nabla^0, \nabla^1)(c^i_{j_1} \dots c^i_{j_a} \otimes h^i_{k_1} \wedge \dots \wedge h^i_{k_b}) \in H(M)$ are called *exotic characteristic classes* of the almost flag structure F.

PROPOSITION 5.3. *The inclusion of a subalgebra* $\otimes_i R[c^i_j] \otimes \bigwedge \{h^i_a\}$ *(a odd) into* $W(J, J')$ *induces isomorphism in cohomology.*

Proof. It is essentially Theorem 6.1 of [12].

The Vey theorem gives the basis of $H(W(J, J'))$. See Theorem 6.3 of [12] or [4].

THEOREM 5.4. *Cohomology classes of cocycles of the form*

$$c^i_{j_1} \dots c^i_{j_a} \otimes h^i_{k_1} \wedge \dots \wedge h^i_{k_b}$$

such that

$$j_1 + \dots + j_a + k^i_0 > r_i, \qquad k^i_0 \leqslant j^i_0,$$

form a basis of $H(W(J, J'))$.

Because of the above theorem we can define the algebra WO_F whose cohomology is equal to that of the algebra $W(J, J')$, i.e.

$$WO_F = WO_{q_2-q_1} \otimes \dots \otimes WO_{n-q_k}$$

with the product differential.

Now we can formulate the following rigidity theorem.

THEOREM 5.5. *If* $j^i_1 + \dots + j^i_a + k^i_0 > r_i + 1$, *then* $\Delta_*(\nabla^0, \nabla^1)(c^i_J h^i_K)$ *depends only on the arc-component of the connection* ∇^0 *in the space of connections having the property* $(\Omega^i)^{r_i+1} = 0$.

Proof. It is essentially Theorem 9.1 of [12].

PROPOSITION 5.6. *The cohomology class* $\Delta_*(\nabla^0, \nabla^1)f$ *is independent of the choice of* ∇^0 *— the* F*-connection — and* ∇^1 *— the metric connection adapted to* F.

Proof. Proposition follows from Theorem 7.1 of [12] and Propositions 5.1 and 5.2.

Note. The only non-rigid characteristic classes are given by (cf. [6])

$$c^i_{j_1} \dots c^i_{j_a} \otimes h^i_{k_1} \wedge \dots \wedge h^i_{k_b} \quad \text{for} \quad j_1 + \dots + j_a + k^i_0 = r_i + 1.$$

6. Characteristic classes and residues. The results of this section are based on results of J. L. Heitsch announced in [7].

Let $F = (F_1, \dots, F_k)$ be a flag structure, X an infinitesimal automorphism of F such that if X is tangent to F_k at a point m, then it is also tangent to F_1 (property S). Such a point m is called a *singular point* of X. Then the set A_X of all singular points of X is a union of leaves of the foliation F_k.

We assume that $A_X = \bigcup F(m_j)$, where $F(m_j)$ are closed and separated leaves of the foliation F_k, and that at no other point of the manifold the vector field X is tangent to F_k. On an open subset $M - A_X$ of M there is a new flag structure $F' = (F_1 + X, \ldots, F_k + X)$. It is possible to embed open normal disc bundles $D_j \supset F(m_j)$ in such a way that their closures $\bar{D}_j$ are disjoint embedded normal disc bundles.

Let U be an open neighbourhood of the set $M - \bigcup D_i$. A U-X-F-connection is an F-connection constructed by the use of a set of subbundles $S = (S_1, \ldots, S_k)$ with the following properties:

(i) $F_i \oplus S_i = F_{i+1}$ on M,
$F'_i \oplus S_i = F'_{i+1}$ on $M - A_X$,

(ii) $F_k \oplus S_k = TM$ on M,
there exists a subbundle S'_k on $M - A_X$ such that
$S'_k \oplus X = S_k$ on $M - A_X$.

Let ∇_1 be an F-connection constructed with the help of the bundles $S_1, \ldots, S_k$. Let $\hat{\nabla}$ be a connection in the bundle TM/F_1 whose covariant derivative is defined as follows:

$$\hat{\nabla}_Y s = [Y, s]_i,$$

for Y a section of the bundle F'_i, s a section of the bundle S_i, $i < k$;

$$\hat{\nabla}_Y s = [Y, s]_{S'_k},$$

for Y a section of the bundle F'_k, s a section of the bundle S'_k;

$$\hat{\nabla}_Y X = 0,$$

for any vector $Y \in F'_k$.

We do not impose any conditions on the covariant derivative along other vectors but we take ∇^k such that $\nabla^k S'_k \subset S'_k$ and $\nabla^k X = 0$. Therefore, choosing a family of subbundles $S_1, \ldots, S_k, S'_k$ with the properties (i), (ii) and a family of connections ∇^i in the bundles $S_1, \ldots, S_k$, respectively, we can construct a connection $\hat{\nabla}$ in the bundle $S_1 \oplus \ldots \oplus S_k$ isomorphic to the bundle TM/F_1. The connection induced in the bundle TM/F_1 by the connection $\hat{\nabla}$ will be denoted by $\nabla(S, \nabla^i)$. When possible, we shall omit S and ∇^i and write ∇_0.

Take an open neighbourhood U of the set $M - \bigcup D_i$. Let V_0, V_1 be two open sets such that $U \subset \bar{U} \subset V_1 \subset \bar{V}_1 \subset V_0$. Put $V_2 = M - \bar{V}_1$. Take a partition of the unity $\{f_0, f_2\}$ relative to $\{V_0, V_2\}$. Then $f_0 | U \equiv 1$. Put

$$\nabla = f_0 \nabla_0 + f_1 \nabla_1.$$

We call the connection ∇ a *U-X-F-connection.* U indicates that on the set

U the connection ∇ is equal to the connection $\nabla(S, \nabla^i)$. When we do not need to specify the open set U we say that a connection is an *X-F-connection.*

LEMMA 6.1. *Let ∇ be a U-X-F-connection, Ω its curvature form. Then*

$$(\Omega^i)^r = 0 \quad \text{on } U \quad \text{for} \quad r > n - q_i - 1,\ i \geqslant 2.$$

Proof. It is a consequence of the fact that on the open set U the connection ∇ is equal to $\nabla(S, \nabla^i)$.

Let I_i be an ideal of $WO_{q_{i+1}-q_i}$, considered as a subalgebra of WO_F, generated by the elements of the form $c^i_{j_1} \ldots c^i_{j_a}$, $\sum j_v = n - q_i$.

LEMMA 6.2. *If $f \in I_i$, then $df = 0$ and $\Delta(\nabla, \nabla')f = 0$ on U for any U-X-F-connection ∇.*

The proof is straightforward.

THEOREM 6.3. *Let $F = (F_1, \ldots, F_k)$ be an oriented flag structure on an oriented n-dimensional manifold M. Let X be an infinitesimal automorphism of F with the property S, whose set of singular points A_X is a finite union of closed and separated leaves, $A_X = \bigcup N_j$. Let $f \in I_i$ be an element of degree r. Then f, F, X determine a cohomology class*

$$\mathrm{Res}_f(F, X, N_i) \in H^{r-n+q_k}(N_i; R),$$

the residue of f, X and F at N_i, such that

(i) $\mathrm{Res}_f(F, X, N_i)$ *depends only on the germs of F and X at N_i,*

(ii) $\sum_i j^* \mathrm{Res}_f(F, X, N_i) = \Delta_*(\nabla, \nabla')(f)$, *where j^* is the composition*

$$H^{r-n+q_k}(N_i; R) \xrightarrow{t^i} H^r_c(D_i; R) \xrightarrow{e_i} H^r(M; R)$$

of the Thom isomorphism and the map given by trivially extending a form with fibre compact support; $H^r_c(D_i; R)$ denotes the cohomology of the complex of forms with compact fibre support.

Proof. For each N_i choose an embedded open normal disc bundle D_i. Let ∇ be a U-X-F-connection and ∇' be a metric connection adapted to F. Lemma 6.1 implies that the form $\Delta(\nabla, \nabla')(f)$ vanishes on U. Therefore it defines a differential form on D_j with compact fibre support. $\Delta(\nabla, \nabla')(f) \mid D_j$ is a closed form and defines a cohomology class $[\Delta(\nabla, \nabla')(f) \mid D_j] \in H^r_c(D_j; R)$. Integration over the fibre defines the map

$$t_{D_j}\colon H^r_c(D_j; R) \to H^{r-n+q_k}(N_j; R).$$

We set

$$\mathrm{Res}_f(F, X, N_j) = t_{D_j}[\Delta(\nabla, \nabla')(f) \mid D_j].$$

LEMMA 6.4. $\mathrm{Res}_f(F, X, N_j)$ *does not depend on the choice of an X-F-connection.*

Proof. Using the same technique as in Proposition 5.1 we can prove that the set of X-F-connections is convex. It is sufficient for proving the lemma.

LEMMA 6.5. $\mathrm{Res}_f(F, X, N_j)$ *does not depend on the choice of the disc bundle* D_j.

Proof. See Lemma 3.18 of [7].

Combining Lemmas 6.4 and 6.5 we get the first part of the theorem. Since t^j is the inverse of t_{D_j}(cf. [5]), the second part of the theorem is true.

Now we would like to express relations between exotic characteristic classes and residues of F and X.

Let N be a singular manifold of X, i.e. $N \subset A_X$. Then F is a flag structure on an oriented $n-q_k$ disc bundle M over the oriented manifold N. The singular set of X is precisely N. On $M-N$ we have a flag structure $F' = (F_1+X, \ldots, F_k+X)$ defining a characteristic mapping

$$\Delta_*: H(WO_{F'}) \to H(M-N; R).$$

By a we denote the composition of the injection of $WO_{F'}$ into WO_F with the differential of WO_F:

$$a(c_{j_1} \ldots c_{j_s} \otimes h_{k_1} \wedge \ldots \wedge h_{k_v}) = c_{k_1} c_{j_1} \ldots c_{j_s} \otimes h_{k_2} \wedge \ldots \wedge h_{k_v}$$

for each element $c_{j_1} \ldots c_{j_s} \otimes h_{k_1} \wedge \ldots \wedge h_{k_v}$ of the Vey basis of $WO_{F'}$. Observe that $a(f) = 0$ iff f is a rigid element of $H(WO_{F'})$.

Let us choose an embedded open normal disc bundle D of N in M such that its closure is contained in M. The inclusion $i: S \to M-N$ of the boundary S of $\bar{D}$ is a homotopy equivalence. Denote by $t_S: H^s(S; R) \to H^{s-n+q_k+1}(N; R)$ integration over the fibre of the oriented $n-q_k-1$ sphere bundle S.

For the objects defined above we have the following theorem.

THEOREM 6.6. *Let* $f \in WO_{F'}$ *be an element of the Vey basis. Then*

$$t_S i^* \Delta_{F'}(f) = \mathrm{Res}_{af}(F, X, N).$$

Note. The theorem links characteristic classes of the flag structure F' with residues of the pair (F, X). The proof, using methods similar to those used by J. Heitsch, will appear in the forthcoming paper.

Appendix

The proof of Proposition 2.1. First we shall show that a mapping f of $I(\mathrm{GL}(n-q_1; q_2, \ldots, q_k))$ is determined by its values on the set Δ of diagonal matrices and that the image of the correspondence is the set $P(q_2, q_3-q_2, \ldots, n-q_k-q_1)$ of polynomials symmetric in

$$\{x_i\}_{i=1}^{q_2}, \ldots, \{x_i\}_{i=n-q_k-q_1}^{n-q_1}.$$

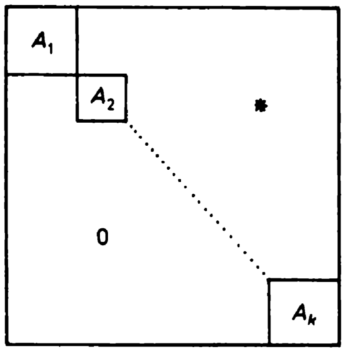
The second part of the above is obvious as we can just interchange the elements of the basis by matrices from the considered group within the range described above. Now we shall show that the mapping

$$a\colon I(\mathrm{GL}(n-q_1;q_2,\ldots,q_k))\to P(q_2,q_3-q_2,\ldots,n-q_k-q_1)$$

defined by $af=f\,|\,\Delta$ is surjective.

The algebra $P(q_2,q_3-q_2,\ldots,n-q_k-q_1)$ is isomorphic to the tensor product of $P(q_2),\ldots,P(n-q_k-q_1)$, where $P(s)$ denotes the algebra of symmetric polynomials in s variables. Then $P(s)$ is isomorphic to the algebra of polynomials $R[g_1,\ldots,g_s]$ where g_i is the ith elementary symmetric function.

Since the matrices of the Lie algebra $\mathrm{gl}(n-q_1;q_2,\ldots,n-q_k-q_1)$ are of the form



we can define the following Ad-invariant mappings p_i:

$$p_i\colon \mathrm{gl}(n-q_1;q_2,\ldots,n-q_k-q_1)\to \mathrm{gl}(g_{i+1}-q_i),$$
$$p_i A = A_i.$$

Then the mapping k_i given by

$$k_i A = \det(\mathrm{Id}_{q_{i+1}-q_i}+tp_i A)$$

is Ad-invariant.

Denote $c_j(p_i A)$ by c_j^i, then $ac_j^i=g_j^i$ where g_j^i is the jth elementary symmetric function of $q_{i+1}-q_i$ variables placed in the ith place by the isomorphism of polynomial algebras.

To prove that the mapping a is injective we have to change the base field to C. The algebra $I(\mathrm{GL}(n-q_1;q_2,\ldots,n-q_k-q_1))$ can be injectively mapped into $I_C(\mathrm{GL}(n-q_1;q_2,\ldots,n-q_k-q_1)\otimes C)$ and $P(q_2,\ldots,n-q_k-q_1)$ into $P_C(q_2,\ldots,n-q_k-q_1)$. Since we have the following commutative diagram

$$\begin{array}{ccc}
I(GL(n-q_1;q_2,\ldots,n-q_k-q_1)) & \longrightarrow & I_C(GL(n-q_1;q_2,\ldots,n-q_k-q_1)\otimes C) \\
\Big\downarrow a & & \Big\downarrow a_C \\
P(q_2,\ldots,n-q_k-q_1) & \longrightarrow & P_C(q_2,\ldots,n-q_k-q_1)
\end{array}$$

it is sufficient to show that a_C is injective.

Let T denote the space of upper-triangular matrices. Such a matrix is diagonalisable if it has different entries on the diagonal. Therefore the diagonalisable upper-triangular matrices form a dense subset of the set T. They are also semi-simple, hence they can be diagonalised in $GL(n-q_1; q_2, \ldots, n-q_k-q_1)$. Since any matrix of $GL(n-q_1; q_2, \ldots, n-q_k-q_1)$ is adjoint in the same group to an upper-triangular matrix, the set of $GL(n-q_1; q_2, \ldots, n-q_k-q_1)$-diagonalisable matrices forms a dense subset of the group $GL(n-q_1; q_2, \ldots, n-q_k-q_1)$. Therefore, if $a_C f = 0$, then $f \equiv 0$ (f is a continuous mapping equal to zero on a dense subset).

The above allows us to define the desired isomorphism as follows: Let $c_{ij} \in I(GL(q_2)) \otimes I(GL(q_3-q_2)) \otimes \ldots \otimes I(GL(n-q_k-q_1))$ be the mapping $1 \otimes c_j \otimes 1$ where c_j is on the ith place, and define the isomorphism

$$d\colon I(GL(n-q_1; q_2, \ldots, n-q_k-q_1)) \to I(GL(q_2)) \otimes \ldots \otimes I(GL(n-q_k-q_1))$$

by putting $dc_j^i = c_{ij}$.

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