ANNALES
POLONICI MATHEMATICI
XXVIII (1973)

Entire functions of bounded index over non-Archimedean field

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Abstract. The aim of this paper is to develope a non-Archimedean theory for entire functions of bounded index. The theory proceeds on entirely different lines although many of the results obtained in this case agree with those in the classical case. Some interesting departures can also be seen. In this paper we have established a sufficient condition for the functions of bounded index. This result is used to construct a non-polynomial entire function of bounded index with any given index greater than or equal to 1. It is shown that entire functions of zero index other than constants do not exist. Some more results treating the class of entire functions of bounded index as a subset of the space of entire functions are obtained.

1. Introduction. The idea of an entire function of bounded index in the complex plane was introduced by B. Lepson [8] in connection with his study of the differential equations of infinite order. Following his preliminary investigations Fred Gross [3], S. M. Shah [10] and others established some further properties of these functions. The purpose of this paper and the papers to follow is to extend the theory of functions of bounded index to the non-Archimedean case. We make use of the idea of the maximum term of an entire function. The theory bears some analogy to what is obtained in the classical case although entirely different techniques are needed to prove the results. Interesting departures also occur.

2. Notation and statement of results. Let K be a non-Archimedean non-trivial valued complete field which is algebraically closed and has characteristic zero, where the valuation is of rank 1. Throughout we assume that the non-Archimedean valuation $|\ |$ is from K to R, the field of real numbers. Then, an entire function with coefficients from K and z taking its values from K can be written as a power series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z_1)(z-z_1)^n,$$

where

$$a_n(z_1) = \frac{f^n(z_1)}{n!}.$$

DEFINITION. An entire function $f(z)$ is said to be *of bounded index* iff there exists an integer k such that for all $z \in K$

$$\text{(A)} \quad \operatorname{Max}\left(|f(z)|, \left|\frac{f^1(z)}{1!}\right|, \ldots, \left|\frac{f^k(z)}{k!}\right|\right) \geqslant \left|\frac{f^j(z)}{j!}\right|, \quad j = 0, 1, 2, \ldots$$

($f^0(z)$ denotes $f(z)$). We shall say that $f(z)$ is of index k if k is the smallest integer for which the above inequality holds. An entire function which is not of bounded index is said to be of unbounded index.

For an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and for every r there exists a term $|a_n| r^n$ ($|z| = r$) which is greater than or equal to the remaining terms of the sequence $(|a_n| r^n)_{n=0}^{\infty}$ and is denoted by $\mu(r, f)$. The index of the maximum term is called the *rank of the maximum term* for $|z| = r$ and is denoted by $\nu(r, f)$. If for a given r there is but one term equal to the maximum term, we call r an *ordinary point*. But, if for a given r the terms equal to the maximum term are not less than two, we call r a *critical point*. It is useful to note that $\mu(r, f) = |f(z)|$ for z, $|z| = r$ being an ordinary point. Let us say that an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is of *bounded* (0)-*index* k iff k is the least positive integer for which (A) is satisfied for all $z \in K$, where $|z| = r$ are ordinary points of $f(z)$.

In the following theorem we obtain a sufficient condition for an entire function to be of bounded index.

THEOREM 1. *Let $C(f)$ and $C(f^1)$ denote the critical points of $f(z)$ and $f^1(z)$ respectively. If for an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$,*

(i) $|k|\,|a_k| \geqslant |a_j|$, $j = 0, 1, \ldots$, *where $k \geqslant 1$ is the least positive integer for which the inequality holds,*

(ii) $C(f) \cap C(f^1) = \emptyset$ *(an empty set),*

(iii) *for every $z \in C(f)$, $|z| \geqslant 1$,*

then $f(z)$ is of bounded index k.

Every polynomial is of bounded index. In Theorem 2, we give a method of constructing a non-polynomial entire function of any given index $k \geqslant 1$. It may be noted that such functions can also be constructed by using the Weierstrass product theorem. But we prefer to follow the methods laid down in Chapter 3, 9 and use Theorem 1 to prove Theorem 2.

THEOREM 2. *Given a positive integer $k \geqslant 1$, there exists an entire function of bounded index with index k.*

THEOREM 3. (i) *An entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of bounded* (0) *index $k \geqslant 1$ can be expressed as a sum of an entire function of bounded* (0) *index* 0 *and a polynomial of degree k.*

(ii) *An entire function* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, *of bounded* (0) *index* $k \geqslant 1$ *can be expressed as a product of an entire function of bounded* (0)*-index* 0 *and a polynomial of degree* k.

It is known that an entire function having no zeros is a constant. See [2], Proposition 3, p. 114. An entire function which has zeros of arbitrarily high order is obviously of unbounded index. So, if it is of bounded index, then it has zeros of finite order. Hence, it is clear that there exists no entire function which is of bounded index with index 0. For, if $z = a$ is a zero of order ν, then at the same point $z = a$ we have $|f(z)| = 0$, whereas $|f^{\nu+1}(z)/(\nu+1)!| > 0$. In fact, we prove much more in the following theorem. For convenience, we write

$$C_f(z) = \operatorname*{Max}_{n\geqslant 0}\left(\left|\frac{f^n(z)}{n!}\right|\right)$$

for the point z.

THEOREM 4. *There exists no non-constant entire function* $f(z)$ *for which*

$$C_f(z) = \left|\frac{f^N(z)}{N!}\right| \tag{B}$$

for all $z \in K$ *and for any positive integer* N.

Let Γ denote the space of all entire functions topologized into a metric space with the metric defined by the functional

$$\|f-g\| = \sup(|a_0-b_0|, |a_n-b_n|^{1/n}, n \geqslant 1)$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ belonging to Γ. In addition to the usual triangular inequality, $\|\ldots\|$ satisfies the ultra metric inequality by virtue of the non-Archimedean valuation. Also, let A denote the set of all entire functions which are of bounded index. We now study A as a subset of Γ. The space Γ in the complex case has been studied exaustively by V. G. Iyer [4], [5], [6] and [7]. Most of these results are also true in the non-Archimedean case, and they follow by essentially the same arguments. We refer to such results without giving any proof.

We prove the following two theorems for the set A.

THEOREM 5. A_k *is a closed set*; $A_k = \{f \in A \mid f$ *is of index* $\leqslant k\}$.

It is natural to ask the following question: Given any two entire functions which are of bounded index, is it true that their sum is also of bounded index? We answer this question in the negative. In fact, we prove the following

THEOREM 6. A *is non-linear.*

By an analogue of Theorem 3 (see [4]) and Theorem 4 we immediately get

COROLLARY 1. *If a sequence of entire functions $(f_n(z))$ of bounded index $\leqslant k$ converges uniformly to $f(z)$ in $|z| \leqslant r$, where r is finite, then $f(z)$ is of bounded index $\leqslant k$.*

3. Proofs of the theorems. Proof of Theorem 1 requires the following

LEMMA 1. *An entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, is of bounded (0) index k, iff k is the least positive integer for which*

$$|a_k| \geqslant |a_j|, \quad j = 0, 1, \ldots \tag{1}$$

Proof. Let (1) hold. For all $z \in K$, $|z| = r \geqslant 1$, where r is an ordinary point of $f(z)$, we have

$$\begin{aligned} |f(z)| &= \mu(r, f) = |a_{\nu(r,f)}| r^{\nu(r,f)} \\ &\geqslant |a_{\nu(r,f^j)}| \, |c_j^{\nu(r,f^j)}| \; r^{\nu(r,f^j)-j} \\ &= \frac{\mu(r, f^j)}{|j!|} \geqslant \left| \frac{f^j(z)}{j!} \right|, \end{aligned} \tag{2}$$

$j = 1, 2, \ldots$ So, if we assume that $f(z)$ is of bounded (0) index $t \neq k$, then there is at least a $z \in K$, $|z| = r < 1$, where r is an ordinary point of $f(z)$ such that

$$\left| \frac{f^t(z)}{t!} \right| > \left| \frac{f^k(z)}{k!} \right|,$$

if $t > 0$. From (1) and the above inequality, observing that any $r < 1$ is an ordinary point of $f^k(z)$, we get

$$|a_k| \geqslant \frac{\mu(r, f^t)}{|t!|} > \frac{\mu(r, f^k)}{|k!|} = |a_k|,$$

which is an impossibility. If $t = 0$, then, as $z = 0$ is an ordinary point of $f(z)$, we have $|f(0)| = |a_0| > |f^k(0)/k!| = |a_k|$, which refutes our assumption. So $f(z)$ is of bounded (0) index k.

Let $f(z)$ be of bounded (0) index k. We proceed to establish (1). And this completes the proof of the lemma. Suppose (1) holds with t instead of k, $t \neq k$. Then, for all $z \in K$, $|z| = r < 1$ we have

$$\left| \frac{f^t(z)}{t!} \right| = \frac{\mu(r, f^t)}{|t!|} = |a_t| \geqslant |a_{\nu(r,f^j)}| \, |c_j^{\nu(r,f^j)}| \; r^{\nu(r,f^j)-j} = \frac{\mu(r, f^j)}{|j!|} \geqslant \left| \frac{f^j(z)}{j!} \right|,$$

$j = 0, 1, \ldots$ So, from (2) and the above inequality it follows that $f(z)$ is of bounded (0) index t. Thus, we have a contradiction and hence (1) holds for $f(z)$.

Proof of Theorem 1. From Lemma 1 and (i) it follows that $f(z)$ is of bounded (0) index k. So, in view of (ii) and (iii), to prove the result we show that

$$|f^1(z)| \geqslant \left|\frac{f^j(z)}{j!}\right|, \tag{3}$$

$j = 1, 2, \ldots$ for all z, $|z| = r \geqslant 1$, which are ordinary points of $f^1(z)$. We consider the following two cases: When $\mu(r, f^j) = |a_t| |t(t-1) \ldots (t-j+1)| r^{t-j}$, $t \geqslant j+1$, for all $r \geqslant 1$ which are ordinary points of $f^1(z)$ and when $\mu(r, f^j) = |a_j| |j!|$ for $j > 1$. In the first case we have $\nu(r, f^j) - 1 \geqslant j$, so that

$$\begin{aligned} |f^1(z)| &= \mu(r, f^1) = |a_{\nu(r,f^1)}| |\nu(r, f^1)| r^{\nu(r,f^1)-1} \\ &\geqslant |a_{\nu(r,f^j)}| |\nu(r, f^j)| \, |c_j^{\nu(r,f^j)-1}| r^{\nu(r,f^j)-1} \\ &= \frac{\mu(r, f^j)}{|j!|} r^{j-1} \\ &\geqslant \left|\frac{f^j(z)}{j!}\right|, \quad j \geqslant 1. \end{aligned}$$

In the second case

$$|f^1(z)| = \mu(r, f^1) \geqslant |a_k| |k| r^{k-1} \geqslant |a_j| = \frac{\mu(r, f^j)}{|j!|} \geqslant \left|\frac{f^j(z)}{j!}\right|, \quad j > 1.$$

So, (3) holds for all $z \in K$, $|z| = r \geqslant 1$ which are ordinary points of $f^1(z)$.

Proof of Theorem 2. As the valuation is non-trivial and any valuation which is non-Archimedean on the rationals is p-adic ([1], p. 13), we can choose positive integers

$$1 \leqslant k = k_1 < k_2 < \ldots$$

such that

$$\left|\frac{k_t}{k_{t+1}}\right| > 1, \quad t = 1, 2, \ldots$$

Let

$$\begin{aligned} R_{k_1} &= 2, \\ R_{k_{t+1}} &= \operatorname{Max}\left(\left|\frac{k_{t-1}}{k_{t+1}}\right|^3 R_{k_t}^{k_{t+1}-k_t}, (k_{t+1}^{k_{t+1}})\right), \quad t = 2, 3, \ldots, \quad R_{k_2} > \frac{2}{|k_1|}. \end{aligned}$$

Also let

$$1 < a_1(k_1) < \left|\frac{k_{t-1}}{k_t}\right|, \quad t = 1, 2, \ldots$$

and

$$a_1(k_t) R_{k_t} = R'_{k_t}, \quad t = 1, 2, \ldots$$

Now define

$$f(z) = \sum_{t=1}^{\infty} a_{k_t} z^{k_t},$$

where

$$\frac{1}{2} = \frac{1}{R_{k_1}} > |a_{k_1}| > \frac{1}{R_{k_1}},$$

(4)
$$\frac{|k_1||a_{k_1}|}{R_{k_2}} > |a_{k_2}| > \frac{|a_{k_1}|}{R'_{k_2}}$$

and

$$\frac{|a_{k_{t-1}}|}{R_{k_t}} > |a_{k_t}| > \frac{|a_{k_{t-1}}|}{R'_{k_t}}, \quad t = 3, 4, \ldots$$

As

$$|a_{k_t}|^{1/k_t} < \left(\frac{2}{|k_1|} R_{k_t} \ldots R_{k_{t-1}} R_{k_t}\right)^{-1/k_t}$$

$$< R_{k_t}^{-1/k_t} < \frac{1}{k_t} \to 0, \quad \text{as } t \to \infty,$$

$f(z)$ is an entire function. Obviously,

$$S_1 = \left(\left|\frac{a_{k_t}}{a_{k_{t+1}}}\right|^{1/(k_{t+1}-k_t)}\right)_{t=1}^{\infty} \quad \text{and} \quad S_2 = \left(\left|\frac{a_{k_t} k_t}{a_{k_{t+1}} k_{t+1}}\right|^{1/(k_{t+1}-k_t)}\right)_{t=1}^{\infty}$$

are the sets of critical points of $f(z)$ and $f^1(z)$ respectively. We now show that these are the only critical points of $f(z)$ and $f^1(z)$, respectively. Suppose there exists a $r \notin S_1$ which is a critical point of $f(z)$. Then, for this r there should exist at least two terms equal to $\mu(r, f)$. So, let

$$\mu(r, f) = |a_{k_{t_1}}| r^{k_{t_1}} = |a_{k_t}| r^{k_t}, \quad t_1 < t-1.$$

Then we have

(5)
$$r^{k_t - k_{t_1}} = \left|\frac{a_{k_{t_1}}}{a_{k_t}}\right| = \left|\frac{a_{k_{t_1}}}{a_{k_{t_1+1}}}\right| \ldots \left|\frac{a_{k_{t-1}}}{a_{k_t}}\right|.$$

Now

(6)
$$\mu(r, f) = |a_{k_t}| r^{k_t}.$$

As $r \notin S_1$, there exists a $A > 1$ such that $r = A \left|\frac{a_{k_{t-1}}}{a_{k_t}}\right|^{\frac{1}{k_t - k_{t-1}}}$. From (4), (5) and (6), we get

$$A^{k_t - k_{t_1}} R_{k_t}^{\frac{k_t - k_{t_1}}{k_t - k_{t-1}}} < A^{k_t - k_{t_1}} \left|\frac{a_{k_{t-1}}}{a_{k_t}}\right|^{\frac{k_t - k_{t_1}}{k_t - k_{t-1}}} = \left|\frac{a_{k_{t_1}}}{a_{k_{t_1}+1}}\right| \ldots \left|\frac{a_{k_{t-1}}}{a_{k_t}}\right|$$

$$< R'_{k_t} R'_{k_{t-1}} \ldots R'_{k_{t_1}}.$$

This yields $$A^{k_t-k_{t_1}}\, a_1(k_t)a_1(k_{t-1}) < a_1(k_t)a_1(k_{t-1}),$$

which is an impossibility. So, S_1 consists of all the critical points of $f(z)$. Let

$$a_1(k_t) < a_2(k_t) < \left|\frac{k_{t-1}}{k_t}\right|, \quad t = 2, 3, \ldots,$$

and

$$R_{k_t}a_2(k_t) = r_{k_t}, \quad \left|\frac{k_{t-1}}{k_t}\right| a_1(k_t) r_{k_t} = r'_{k_t}, \quad t = 1, 2, \ldots$$

Then, clearly

(7) $$R'_{k_t} < r_{k_t} < \left|\frac{k_{t-1}}{k_t}\right|\left|\frac{a_{k_{t-1}}}{a_{k_t}}\right| < r'_{k_t} < R_{k_{t+1}}.$$

Proceeding as above and making use of the above inequality instead of (4), we can show that the set of all critical points of $f^1(z)$ is S_2. From (4) and (7) we see that $f(z)$ and $f^1(z)$ have no common critical points. Clearly, (i) and (iii) of Theorem 1 are satisfied. So, the result follows from Theorem 1.

Proof of Theorem 3. (i) Let

$$f_0(z) = a_k + a_1 z + \ldots + a_0 z^k + a_{k+1} z^{k+1} + \ldots,$$
$$p(z) = z^k(a_k - a_0) + (a_0 - a_k).$$

From Lemma 1, we have $a_k - a_0 \neq 0$ and $f_0(z)$ is of bounded (0) index 0. So, $p(z)$ is of degree k. Clearly, $f(z) = f_0(z) + p(z)$. Hence the result is proved.

(ii) Let $b_0, b_1, \ldots, b_n, \ldots$ all belong to K and $|b_0| \geqslant |b_j|$, $j = 1, 2, \ldots, k$, $|b_0| \geqslant |a_k|$. Now define

$$c_0 = \frac{a_0}{b_0}, \quad c_1 = \frac{a_1}{b_0} - \frac{c_0 b_1}{b_0}, \quad \ldots, \quad c_k = \big(a_k - (b_1 c_{k-1} + \ldots + c_k b_0)\big)/b_0$$

and

$$b_{k+1} = \big(a_{k+1} - (b_1 c_k + b_2 c_{k-1} + \ldots + b_k c_1)\big)/c_0, \ldots$$

Clearly, $|b_{k+j}| \leqslant |a_k| \leqslant |b_0|$, $j = 1, 2, \ldots$ So, if $f_0(z) = \sum_{n=0}^{\infty} b_n z^n$ and $p(z) = \sum_{n=0}^{\infty} c_k z^k$, then from Lemma 1 it follows that $f_0(z)$ is of bounded (0) index 0 and $p(z)$ is a polynomial of degree k. Clearly, $f(z) = f_0(z)p(z)$. So, the result is proved.

Proof of Theorem 4. The case $N = 0$ is discussed earlier in Section 2. So, we assume $N \geqslant 1$. If possible, let there be a non-constant entire function for which (B) holds for all $z \in K$. Then

$$\left|\frac{f^j(z)}{j!}\right| \leqslant \left|\frac{f^N(z)}{N!}\right|$$

for all $z \in K$, and $j = 0, 1, \ldots$ $f^N(z)$ cannot vanish anywhere in K. For, $f^N(z) = 0$ for some point z, implies by the above inequality that $f^j(z) = 0$ for all $j = 0, 1, \ldots$ Hence, $f(z)$ vanishes identically in K, which refutes our supposition. So, from a known result [2] refered to earlier, $f^N(z)$ is a constant, which means that $f(z)$ is a polynomial or a constant (when $f^N(z) \equiv 0$, $N = 1$). Now, if $f(z)$ is a polynomial, we have, as (B) holds for all $z \in K$ and $j = 0, 1, 2, \ldots$

$$\mu(r, f) \leqslant \left| \frac{f^N(z)}{N!} \right| = B = \text{a constant.}$$

Since $\mu(r, f)$ is a monotonic increasing function of r which can be taken to be greater than B for an arbitrarily large r, we have an impossibility. So, the result is proved.

Proof of Theorem 5. To prove that A_k is closed, we prove that CA_k is open. Let $f(z) \in CA_k$. For any $z \in K$, let B denote the set of all $f^j(z)$'s for which $f^j(z) = 0$, if such $f^j(z)$'s exist and

$$m = \operatorname{Min}\big(|f^j(z)|, f^j(z) \notin B\big).$$

Let

$$0 < \varepsilon \leqslant \operatorname{Min}\left(\frac{1}{2}, \frac{m}{1 + r(1+m)}\right),$$

where r is fixed. If $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \Gamma$ be such that $\|g(z)\| < \varepsilon$, then

$$|g(z)| \leqslant \sum_{n=0}^{\infty} |b_n| r^n < \frac{(\varepsilon - 1)(1 - \varepsilon r) + 1}{(1 - \varepsilon r)}$$

and

$$|g^j(z)| \leqslant \sum_{\nu=j}^{\infty} |b_\nu| |c_j^\nu| r^{\nu-j} \leqslant \sum_{\nu=j}^{\infty} |b_\nu| r^{\nu-j} < \frac{\varepsilon^j}{(1 - \varepsilon r)},$$

$j = 1, 2, \ldots$ As $\varepsilon < 1$, we have

(8) $$|g(z)| < m, \quad |g^j(z)| < m, \quad j = 1, 2, \ldots$$

Hence,

(9) $$|f^j(z) + g^j(z)| = \begin{cases} |f^j(z)| & \text{if} \quad f^j(z) \notin B, \\ |g^j(z)| & \text{if} \quad f^j(z) \in B, j = 0, 1, \ldots \end{cases}$$

If possible, let

$$h(z) = f(z) + g(z)$$

be of bounded index. Then, for all $z \in K$, there is a positive integer k such that

(10) $$\operatorname{Max}\left(|h(z)|, \left|\frac{h^1(z)}{1!}\right|, \ldots, \left|\frac{h^k(z)}{k!}\right|\right) \geqslant \left|\frac{h^j(z)}{j!}\right|, \quad j = 0, 1, \ldots$$

As $f(z)$ is of index greater than k, we have a $z_1 \in K$ and a positive integer $l > k$ such that

$$\left|\frac{f^l(z_1)}{l!}\right| > \operatorname{Max}\left(\left|f(z_1)\right|, \left|\frac{f^1(z_1)}{1!}\right|, \ldots, \left|\frac{f^k(z_1)}{k!}\right|\right).$$

Clearly, $f^l(z_1) \neq 0$. So, from (9) we have $|f^l(z_1)| = |h^l(z_1)|$. Also, from (8) we have

$$|g^j(z_1)| < m \leqslant |f^l(z_1)|, \quad j = 0, 1, \ldots, k.$$

So

$$\left|\frac{g^j(z_1)}{j!}\right| < \left|\frac{f^l(z_1)}{l!}\right| = \left|\frac{h^l(z_1)}{l!}\right|,$$

since $|l!| \leqslant |j!|$, $j = 0, 1, \ldots, k$. Hence

$$\left|\frac{h^l(z_1)}{l!}\right| > \operatorname{Max}\left(|h(z_1)|, \left|\frac{h^1(z_1)}{1!}\right|, \ldots, \left|\frac{h^k(z_1)}{k!}\right|\right).$$

This contradicts (10). So, $h(z) \in CA_k$. Hence, CA_k is open.

Proof of Theorem 6. Let

$$f(z) = a_0 + a_1 z + \sum_{j=1}^{\infty} a_{n_j} z^{n_j},$$

where

$$2 \leqslant n_1 < n_2 < \ldots, \quad \left|\frac{n_j}{n_{j+1}}\right| > 1, \quad n_0 = 1,$$

$$R_{n_1} = 2,$$

$$R_{n_{j+1}} = \max\left(\left|\frac{n_{j-1}}{n_{j+1}}\right| R_{n_j}^{n_{j+1}-n_j}, (n_{j+1})^{n_{j+1}}\right), \quad j = 1, 2, \ldots,$$

$$1 < a_1(n_j) < \left|\frac{n_{j-1}}{n_j}\right|, \quad j = 1, 2, \ldots,$$

$$a_1(n_j) R_{n_j} = R'_{n_j}, \quad j = 1, 2, \ldots,$$

$$|a_0| = |a_1| = a, \quad 1 < a < 2, \quad \frac{1}{2} = \frac{1}{R_{n_1}} > |a_{n_1}| > \frac{1}{R'_{n_1}},$$

and

$$\frac{|a_{n_{j-1}}|}{R_{n_j}} > |a_{n_j}| > \frac{|a_{n_{j-1}}|}{R'_{n_j}}, \quad j = 2, 3, \ldots$$

Proceeding as in Theorem 2, we can show that $f(z)$ is of bounded index 1. Now let

$$g(z) = \prod_{i=1}^{\infty} \left(1 - \frac{z}{\lambda_i}\right)^i = 1 + \sum_{k=1}^{\infty} b_k z^k,$$

$$b_k = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k} (-1)^k (\varrho_{i_1} \cdots \varrho_{i_k})^{-1},$$

where

$$\lambda_1 = \varrho_1 = 1,$$

$$1+\mathrm{Max}\big(n^{n^2}, C(n_j, n)\big) > |\lambda_n| = |\varrho_{n(n+1)/2}| = \ldots = |\varrho_{[n(n+1)/2]-n}|$$

$$> \mathrm{Max}\big(n^{n^2}, C(n_j, n)\big), \quad n_j \geqslant \frac{n(n+1)}{2} > n_{j-1}, \; j = 2, 3, \ldots$$

and

$$A(n_j, n) = \left|a_{n_j}\varrho_1 \cdots \varrho_{\frac{n(n+1)}{2}-n}\right|^{-1/\left(n_j+n-\frac{n(n+1)}{2}\right)},$$

$$B(n_j, n) = \mathop{\mathrm{Max}}_{0\leqslant\nu\leqslant n-1, 0\leqslant t\leqslant n_j}\left(\left|\left(\frac{n(n+1)}{2}-\nu-t-1\right)\Big/\left(\frac{n(n+1)}{2}-\nu\right)\right| \times\right.$$

$$\left.\times\left|\Big(n_j\ldots(n_j-t+1)\Big)\Big/\left(\frac{n(n-1)}{2}-\nu\right)\ldots\left(\frac{n(n+1)}{2}-\nu-t\right)\right|^{-1/\left(n_j+n-\frac{n(n+1)}{2}\right)}\right)$$

$$C(n_j, n) = \mathrm{Max}\big(A(n_j, n), A(n_j, n)B(n_j, n)\big).$$

As

$$|b_k|^{1/k} = \Big|\sum_{1\leqslant i_1<i_2<\ldots<i_k}(-1)^k(\varrho_{i_1}\cdots\varrho_{i_k})^{-1}\Big| \quad \text{for} \quad \frac{n(n+1)}{2}-n \leqslant k$$

$$\leqslant \frac{n(n+1)}{2} = |\varrho_1\cdots\varrho_k|^{-1/k} < \left(\frac{1}{n^{n^2}}\right)^{\frac{2}{(n^2+n)}} \to 0, \quad \text{as} \quad n\to\infty,$$

$g(z)$ is an entire function. Obviously, it is of unbounded index. In what follows we show that $f(z)+g(z)$ is of bounded index with index 1. Since $-f(z)$ is of bounded index 1, we have $f(z)+g(z)-f(z) = g(z)$ is of unbounded index. So, the result is proved. We now proceed to show that $f(z)+g(z)$ is of bounded index 1. First, we prove for all $|z| = r$, $z\in K$,

$$\mu(r, f^t) > \mu(r, g^t), \quad t = 0, 1, \ldots \tag{11}$$

Let G denote all r for which $\mu(r, g) = 1$. Then, for all $r\in G$ we have

$$\frac{\mu(r, f)}{\mu(r, g)} \geqslant |a_0| > 1,$$

so that (11) holds for all $r\in G$, when $t = 0$. For $r\in CG$, if $\mu(r, g) = |b_N|r^N$, then there is a positive integer n such that

$$\mu(r, g) = \left|b_{\frac{n(n+1)}{2}-\nu}\right| r^{\frac{n(n+1)}{2}-\nu}, \quad 0 \leqslant \nu \leqslant n-1,$$

and r satisfies the inequality

$$r \geqslant \left|\varrho_{\frac{n(n+1)}{2}-\nu}\right|.$$

Now

$$\frac{\mu(r,f)}{\mu(r,g)} \geqslant \left|a_{n_j}\varrho_1\cdots\varrho_{\frac{n(n+1)}{2}-\nu}\right| r^{n_j+\nu-\frac{n(n+1)}{2}}$$

$$\geqslant \left|a_{n_j}\varrho_1\cdots\varrho_{\frac{n(n+1)}{2}-n}\right| \left|\varrho_{\frac{n(n+1)}{2}-n+1}\right|^{n_j-n-\frac{n(n+1)}{2}}$$

$$> A_j(n_j,n)^{n_j+n-\frac{n(n+1)}{2}} \left|a_{n_j}\varrho_1\cdots\varrho_{\frac{n(n+1)}{2}-n}\right| = 1,$$

so that (11) holds with $t = 0$ for $r\in CG$ also. Similarly, we can prove (11) when $t = 1, 2, \ldots$ For the function $f(z)$, for all z for which $|z| = r$ are ordinary points of $f(z)$, we have

$$\mu(r,f) > \frac{\mu(r,f^j)}{|j!|}, \quad j = 1, 2, \ldots \tag{12}$$

and for all z, $|z| = r$ being critical points of $f(z)$ (which are ordinary points of $f^1(z)$), we have

$$\mu(r,f^1) > \frac{\mu(r,f^j)}{|j!|}, \quad j = 2, 3, \ldots \tag{13}$$

From (11)–(13) we have clearly

$$\mathrm{Max}\big(|f(z)+g(z)|, |f^1(z)+g^1(z)|\big) = \left|\frac{f^j(z)+g^j(z)}{j!}\right|, \quad j = 0, 1, \ldots,$$

for all $z\in K$. So, $f(z)+g(z)$ is of bounded index 1.

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Reçu par la Rédaction le 30. 10. 1971