ANNALES
POLONICI MATHEMATICI
XXVII (1973)

Derivative of generalized quadratic mean function of entire functions defined by Dirichlet series

by J. S. GUPTA and MANJIT SINGH (Jammu, India)

1. Let E be the set of mappings $f\colon C \to C$ (C is the complex field such that the image under f of an element $s \in C$ is

$$f(s) = \sum_{n\in N} a_n e^{s\lambda_n}$$

with

$$\limsup_{n\to+\infty} \frac{\log n}{\lambda_n} = D < +\infty,$$

and $\sigma_c^f = +\infty$ (σ_c^f is the abscissa of convergence of the Dirichlet series defining f); N is the set of natural numbers $0, 1, 2, \ldots$, $\langle\lambda_n\colon n\in N\rangle$ is a strictly increasing unbounded sequence of non-negative reals, $s = \sigma + it$, $\sigma, t \in R$ (R is the field of reals), and $\langle a_n\colon n \in N\rangle$ is a sequence in C. Since the Dirichlet series defining f converges for all values of s, f is an entire function. Also, since $D < +\infty$, we have ([4], p. 168) $\sigma_a^f = +\infty$ (σ_a^f is the abscissa of absolute convergence of the Dirichlet series defining f), and that f is bounded on each vertical line $\sigma = \sigma_0$.

For any $r \in R$, the generalized quadratic mean functions W_r and $W_{r,m}$ of an entire function $f\in E$ and that of its m-th derivative $f^{(m)}$, $m \in Z_+$ (Z_+ is the set of positive integers), are defined, respectively, as

$$W_r(\sigma, f) = \lim_{T\to+\infty} \frac{1}{2Te^{r\sigma}} \int_0^\sigma \int_{-T}^{T} |f(x+it)|^2 e^{rx}\,dx\,dt, \qquad \forall \sigma < \sigma_c^f, \tag{1.1}$$

and

$$W_{r,m}(\sigma, f^{(m)}) = \lim_{T\to+\infty} \frac{1}{2Te^{r\sigma}} \int_0^\sigma \int_{-T}^{T} |f^{(m)}(x+it)|^2 e^{rx}\,dx\,dt, \qquad \forall \sigma < \sigma_c^f. \tag{1.2}$$

The object of this paper is to study a few properties of the derivatives of the functions W_r and $W_{r,m}$ with respect to σ. At the outset we give an alternative and shortest possible proof of a result already proved in ([2], Theorem 5).

THEOREM 1. *If $f \in E$ is an entire function of Ritt order ϱ and lower order λ, and W'_r is the derivative of W_r with respect to σ, then*

$$\lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\big(W'_r(\sigma,f)/W_r(\sigma,f)\big)}{\sigma} = \begin{matrix}\varrho\\ \lambda\end{matrix}. \tag{1.3}$$

Proof. By definition

$$W_r(\sigma,f) = \lim_{T\to+\infty} \frac{1}{2Te^{r\sigma}} \int_0^\sigma \int_{-T}^{T} |f(x+it)|^2 e^{rx}\,dx\,dt$$
$$= \frac{1}{e^{r\sigma}} \int_0^\sigma I_2(x,f)e^{rx}\,dx,$$

where I_2 is the quadratic mean function of f ([1], p. 520). Therefore

$$W'_r(\sigma,f) = I_2(\sigma,f) - rW_r(\sigma,f).$$

Hence

$$\frac{W'_r(\sigma,f)}{W_r(\sigma,f)} = \frac{I_2(\sigma,f)}{W_r(\sigma,f)}\left(1 - \frac{r}{I_2(\sigma,f)/W_r(\sigma,f)}\right)$$

or

$$\log\big(W'_r(\sigma,f)/W_r(\sigma,f)\big) = \log\left(\frac{I_2(\sigma,f)}{W_r(\sigma,f)}\right) + \log\left(1 - \frac{r}{I_2(\sigma,f)/W_r(\sigma,f)}\right). \tag{1.4}$$

But ([3], Theorem 2),

$$\lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\big(I_2(\sigma,f)/W_r(\sigma,f)\big)}{\sigma} = \begin{matrix}\varrho\\ \lambda\end{matrix},$$

and ([3], Lemma 3), $I_2(\sigma,f)/W_r(\sigma,f)$ increases with σ. Hence, from (1.4), we get

$$\lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\big(W'_r(\sigma,f)/W_r(\sigma,f)\big)}{\sigma} = \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\big(I_2(\sigma,f)/W_r(\sigma,f)\big)}{\sigma} = \begin{matrix}\varrho\\ \lambda\end{matrix},$$

which proves the theorem.

THEOREM 2. *If $f \in E$ is an entire function of lower order λ such that $\lambda \geqslant \delta > 0$, and $W'_{r,m}$ is the derivative of $W_{r,m}$ with respect to σ, then, for all $\sigma > \sigma_0$,*

$$W_r(\sigma,f) < W'_r(\sigma,f) < 4W'_{r,1}(\sigma,f^{(1)}) < \ldots < 4^m W'_{r,m}(\sigma,f^{(m)}) < \ldots \tag{1.5}$$

In order to prove this theorem we need the following:

LEMMA. *If $f \in E$ is an entire function of Ritt order ϱ and lower order λ, and $\lambda \geqslant \delta > 0$, then, for all $\sigma > \sigma_0$,*

$$W_r(\sigma,f) < W'_r(\sigma,f). \tag{1.6}$$

Proof. From (1.3), we have

$$\lim_{\sigma\to+\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\log\big(W_r'(\sigma,f)/W_r(\sigma,f)\big)}{\sigma}=\frac{\varrho}{\lambda}.$$

Hence, for any $\varepsilon\in R_+$ (R_+ is the set of positive reals) and for all $\sigma>\sigma_0(\varepsilon,f)$,

$$W_r(\sigma,f)e^{\sigma(\lambda-\varepsilon)}<W_r'(\sigma,f)<W_r(\sigma,f)e^{\sigma(\varrho-\varepsilon)}. \tag{1.7}$$

Since $\lambda\geqslant\delta>0$, (1.6) follows from (1.7).

Proof of Theorem 2. We have, from (1.2),

$$\begin{aligned}W_{r,1}(\sigma,f^{(1)})&=\lim_{T\to+\infty}\frac{1}{2Te^{r\sigma}}\int_0^\sigma\int_{-T}^{T}|f^{(1)}(x+it)|^2e^{rx}\,dx\,dt\\&=\lim_{T\to+\infty}\frac{1}{2Te^{r\sigma}}\int_0^\sigma\int_{-T}^{T}\left(\lim_{\varepsilon\to0}\left|\frac{f(x+it)-f\big(x(1-\varepsilon)+it\big)}{\varepsilon x}\right|^2\right)e^{rx}\,dx\,dt\\&\geqslant\lim_{T\to+\infty}\frac{1}{2Te^{r\sigma}}\int_0^\sigma\int_{-T}^{T}\left(\lim_{\varepsilon\to0}\frac{|f(x+it)|-\big|f\big(x(1-\varepsilon)+it\big)\big|}{\varepsilon x}\right)^2e^{rx}\,dx\,dt.\end{aligned}$$

Since, by Minkowski's inequality ([6], p. 384),

$$\begin{aligned}&\Big(\int_{-T}^{T}\big(|f(x+it)|-\big|f\big(x(1-\varepsilon)+it\big)\big|\big)^2dt\Big)^{1/2}\\&\qquad\geqslant\bigg(\Big(\int_{-T}^{T}|f(x+it)|^2dt\Big)^{1/2}-\Big(\int_{-T}^{T}\big|f\big(x(1-\varepsilon)+it\big)\big|^2dt\Big)^{1/2}\bigg),\end{aligned}$$

it follows that

$$\begin{aligned}W_{r,1}(\sigma,f^{(1)})&\geqslant\lim_{T\to+\infty}\lim_{\varepsilon\to0}\frac{1}{2Te^{r\sigma}\varepsilon^2}\Bigg(\int_0^\sigma\bigg(\Big(\int_{-T}^{T}|f(x+it)|^2dt\Big)^{1/2}-\\&\qquad-\Big(\int_{-T}^{T}\big|f\big(x(1-\varepsilon)+it\big)\big|^2dt\Big)^{1/2}\bigg)^2\frac{e^{rx}}{x^2}\,dx\Bigg)\\&\geqslant\lim_{T\to+\infty}\lim_{\varepsilon\to0}\frac{1}{2Te^{r\sigma}\varepsilon^2\sigma^2}\Bigg(\int_0^\sigma\bigg(\Big(\int_{-T}^{T}|f(x+it)|^2e^{rx}dt\Big)^{1/2}-\\&\qquad-\Big(\int_{-T}^{T}\big|f\big(x(1-\varepsilon)+it\big)\big|^2e^{rx}dt\Big)^{1/2}\bigg)^2dx\Bigg).\end{aligned}$$

Again, using Minkowski's inequality, we get

$$W_{r,1}(\sigma,f^{(1)}) \geqslant \lim_{\varepsilon\to 0}\lim_{T\to+\infty}\frac{1}{2Te^{r\sigma}\varepsilon^2\sigma^2}\Bigg(\Big(\int_0^{\sigma}\int_{-T}^{T}|f(x+it)|^2e^{rx}dxdt\Big)^{1/2} - $$
$$-\Big(\int_0^{\sigma}\int_{-T}^{T}|f(x(1-\varepsilon)+it)|^2e^{rx}dxdt\Big)^{1/2}\Bigg)^2$$
$$\geqslant \lim_{\varepsilon\to 0}\left(\frac{\big(W_r(\sigma,f)\big)^{1/2}-\big(W_r(\sigma(1-\varepsilon),f)\big)^{1/2}}{\varepsilon\sigma}\right)^2$$
$$=\left(\frac{d}{d\sigma}\big(W_r(\sigma,f)\big)^{1/2}\right)^2$$
$$=\frac{1}{4}\,\frac{\big(W_r'(\sigma,f)\big)^2}{W_r(\sigma,f)}.$$

Using the lemma, we get

$$W_r'(\sigma,f) < 4W_{r,1}(\sigma,f^{(1)}) \quad \text{for all}\,\sigma > \sigma'.$$

Again using the lemma, we get

$$W_r'(\sigma,f) < 4W_{r,1}'(\sigma,f^{(1)}) \quad \text{for all } \sigma > \sigma_1. \tag{1.8}$$

Inequalities similar to (1.8) can be deduced for the higher derivatives of f; after combining them all and (1.6) the theorem follows for all $\sigma > \sigma_0$, where

$$\sigma_0 = \sup\{\sigma_1, \sigma_2, \ldots, \sigma_m, \ldots\}.$$

THEOREM 3. *If $f\in E$ is an entire function, and $\sigma_2 > \sigma_1$, then*

$$\frac{W_r'(\sigma_1,f)}{W_r(\sigma_1,f)} \leqslant \frac{\log W_r(\sigma_2,f)-\log W_r(\sigma_1,f)}{\sigma_2-\sigma_1} \leqslant \frac{W_r'(\sigma_2,f)}{W_r(\sigma_2,f)}. \tag{1.9}$$

Proof. We know from ([2], Theorem 1), that log W_r is an increasing convex function of σ. Hence log W_r is differentiable almost everywhere with an increasing derivative; the set of points where the right derivative is greater than the left derivative is of Lebesgue measure zero. This fact enables us to write log W_r in the following form:

$$\log W_r(\sigma,f) = \log W_r(\sigma_0,f) + \int_{\sigma_0}^{\sigma}\frac{W_r'(x,f)}{W_r(x,f)}\,dx \tag{1.10}$$

for an arbitrary σ_0. From (1.10) we get

$$\log W_r(\sigma_2,f) = \log W_r(\sigma_1,f) + \int_{\sigma_1}^{\sigma_2}\frac{W_r'(x,f)}{W_r(x,f)}\,dx.$$

Hence

$$\log W_r(\sigma_2,f) \leqslant \log W_r(\sigma_1,f) + \frac{W_r'(\sigma_2,f)}{W_r(\sigma_2,f)}(\sigma_2-\sigma_1) \tag{1.11}$$

and

$$\log W_r(\sigma_2,f) \geqslant \log W_r(\sigma_1,f) + \frac{W_r'(\sigma_1,f)}{W_r(\sigma_1,f)}(\sigma_2-\sigma_1). \tag{1.12}$$

From (1.11) and (1.12) follows (1.9).

THEOREM 4. *If $f \in E$ is an entire function of Ritt order ϱ and lower order λ, and $\lambda_{N(\sigma,f)}$ is the exponent corresponding to the maximum term $\mu(\sigma,f)$, for* Re$(s) = \sigma$, *in the Dirichlet series defining f, then for any $\varepsilon \in R_+$ such that $\varepsilon = \varepsilon(\sigma,f)$ tends to zero as σ tends to plus infinity, and for almost all $\sigma > \sigma_0(\varepsilon,f)$,*

$$W_r'(\sigma,f) > \frac{W_r(\sigma,f)\log W_r(\sigma,f)}{(1+\varepsilon)\sigma}, \tag{1.13}$$

$$W_r'(\sigma,f) > \frac{W_r(\sigma,f)\log W_r(\sigma,f)}{\left(1-\dfrac{\lambda}{\varrho}+\varepsilon\right)\sigma}, \tag{1.14}$$

and

$$W_r'(\sigma,f) > \frac{W_r(\sigma,f)\log W_r(\sigma,f)}{\left(\dfrac{1}{\lambda}-\dfrac{1}{\varrho}+\varepsilon\right)\log \lambda_{N(\sigma,f)}}. \tag{1.15}$$

Proof. We have, for the left derivative of $\log W_r(\sigma,f)$,

$$\frac{W_r'(\sigma,f)}{W_r(\sigma,f)} = \frac{d}{d\sigma}\big(\log W_r(\sigma,f)\big) \geqslant \frac{\log W_r(\sigma,f)-\log W_r(\sigma_1,f)}{\sigma-\sigma_1}$$
$$> \frac{\log W_r(\sigma,f)}{(1+\varepsilon)\sigma},$$

where $\sigma_1 < \sigma$. $\varepsilon = \varepsilon(\sigma,f)$ is positive and tends to zero as σ tends to plus infinity, proving (1.13).

In order to establish (1.14) we shall apply the following result of ([2], p. 308):

$$\limsup_{\sigma\to+\infty} \frac{\log W_r(\sigma,f)}{\sigma\lambda_{N(\sigma,f)}} \leqslant 2\left(1-\frac{\lambda}{\varrho}\right). \tag{1.16}$$

From (1.16) we get, for any $\varepsilon \in R_+$ and $\sigma > \sigma_0(\varepsilon,f)$,

$$\log W_r(\sigma,f) < 2\left(1-\frac{\lambda}{\varrho}+\varepsilon\right)\sigma\lambda_{N(\sigma,f)} \tag{1.17}$$
$$= 2\left(1-\frac{\lambda}{\varrho}+\varepsilon\right)\sigma\frac{\mu'(\sigma,f)}{\mu(\sigma,f)},$$

for almost all $\sigma > \sigma_0(\varepsilon, f)$, in view of ([5], Lemma 1). Also, since ([1], p. 521)

$$I_2(\sigma, f) = \sum_{n \in N} |a_n|^2 e^{2\sigma\lambda_n},$$

we have

$$\begin{aligned} W_r(\sigma, f) &= \frac{1}{e^{r\sigma}} \int_0^\sigma I_2(x, f) e^{rx} dx \\ &= \frac{1}{e^{r\sigma}} \int_0^\sigma \Big(\sum_{n \in N} |a_n|^2 e^{2x\lambda_n}\Big) e^{rx} dx \\ &\geqslant \big(\mu(\sigma, f)\big)^2 \frac{1}{r} (1 - e^{-r\sigma}). \end{aligned} \tag{1.18}$$

Taking logarithm of both sides in (1.18) and differentiating with respect to σ, we get

$$\frac{W_r'(\sigma, f)}{W_r(\sigma, f)} \geqslant \frac{r}{(e^{r\sigma} - 1)} + 2 \frac{\mu'(\sigma, f)}{\mu(\sigma, f)}.$$

Hence

$$\frac{W_r'(\sigma, f)}{W_r(\sigma, f)} > 2 \frac{\mu'(\sigma, f)}{\mu(\sigma, f)}, \tag{1.19}$$

since $\dfrac{r}{(e^{r\sigma} - 1)} > 0$. From (1.17) and (1.19) we get, for any $\varepsilon \in R_+$ and for almost all $\sigma > \sigma_0(\varepsilon, f)$,

$$\log W_r(\sigma, f) < \left(1 - \frac{\lambda}{\varrho} + \varepsilon\right) \sigma \frac{W_r'(\sigma, f)}{W_r(\sigma, f)},$$

which is nothing else than (1.14).

The proof of (1.15) is similar to that of (1.14) except that instead of (1.16) we have to start with the following result of ([2], p. 311):

$$\limsup_{\sigma \to +\infty} \frac{\log W_r(\sigma, f)}{\lambda_{N(\sigma, f)} \log \lambda_{N(\sigma, f)}} \leqslant 2 \left(\frac{1}{\lambda} - \frac{1}{\varrho}\right).$$

COROLLARY 1. *If $f \in E$ is an entire function of Ritt order ϱ and lower order λ, then, for any $\varepsilon \in R_+$ and for almost all $\sigma > \sigma_0(\varepsilon, f)$,*

$$W_r'(\sigma, f) > \frac{W_r(\sigma, f) \log W_r(\sigma, f)}{\left(\frac{1}{\lambda} - \frac{1}{\varrho} + \varepsilon\right)(\varrho + \varepsilon)\sigma}. \tag{1.20}$$

This follows from (1.15) and the following result of ([5], p. 240):

$$\limsup_{\sigma\to+\infty}\frac{\log\lambda_{N(\sigma,f)}}{\sigma}=\varrho.$$

COROLLARY 2. *If $f\in E$ is an entire function of regular growth and Ritt order ϱ, then, for any $\varepsilon\in R_+$ and for almost all $\sigma>\sigma_0(\varepsilon,f)$,*

$$W_r'(\sigma,f)>\frac{W_r(\sigma,f)\log W_r(\sigma,f)}{\varepsilon\sigma}$$

and

$$W_r'(\sigma,f)>\frac{W_r(\sigma,f)\log W_r(\sigma,f)}{(\varrho+\varepsilon)\varepsilon\sigma}.$$

These follow immediately from (1.14) and (1.20), respectively, on putting $\varrho=\lambda$.

THEOREM 5. *If $f\in E$ is an entire function of Ritt order ϱ and lower order λ, such that $\lambda\geqslant\delta>0$, then for any $\varepsilon\in R_+$ and for almost all $\sigma>\sigma_0(\varepsilon,f)$,*

$$W'_{r,m}(\sigma,f^{(m)})>W_r'(\sigma,f)\left(\frac{\log W_r'(\sigma,f)}{4(1+\varepsilon)\sigma}\right)^m, \tag{1.21}$$

$$W'_{r,m}(\sigma,f^{(m)})>W_r'(\sigma,f)\left(\frac{\log W_r'(\sigma,f)}{4\left(1-\frac{\lambda}{\varrho}+\varepsilon\right)\sigma}\right)^m, \tag{1.22}$$

and

$$W'_{r,m}(\sigma,f^{(m)})>W_r'(\sigma,f)\left(\frac{\log W_r'(\sigma,f)}{4\left(\frac{1}{\lambda}-\frac{1}{\varrho}+\varepsilon\right)\log\lambda_{N(\sigma,f^{(m)})}}\right)^m. \tag{1.23}$$

Proof. Writing (1.13) for $f^{(1)}$, we get, for any $\varepsilon\in R_+$ and for almost all $\sigma>\sigma_1=\sigma_1(\varepsilon,f^{(1)})$,

$$W'_{r,1}(\sigma,f^{(1)})>\frac{W_{r,1}(\sigma,f^{(1)})\log W_{r,1}(\sigma,f^{(1)})}{(1+\varepsilon)\sigma}$$

or, since, by (1.5), $W_{r,1}(\sigma,f^{(1)})>\frac{1}{4}W_r'(\sigma,f)$,

$$\frac{W'_{r,1}(\sigma,f^{(1)})}{W_r'(\sigma,f)}>\frac{\log W_r'(\sigma,f)}{4(1+\varepsilon)\sigma}. \tag{1.24}$$

Now, writing (1.24) for $f^{(p)}$, we get

$$\frac{W'_{r,p}(\sigma,f^{(p)})}{W'_{r,p-1}(\sigma,f^{(p-1)})}>\frac{\log W'_{r,p-1}(\sigma,f^{(p-1)})}{4(1+\varepsilon)\sigma},$$

for any $\varepsilon\in R_+$ and for almost all $\sigma>\sigma_p=\sigma_p(\sigma,f^{(p)})$.

Putting $p = 1, 2, \ldots, m$, and multiplying the m inequalities thus obtained, we get

$$(1.25) \qquad W'_{r,m}(\sigma, f^{(m)}) > \frac{W'_r(\sigma,f) \prod\limits_{1\leqslant p\leqslant m} \log W'_{r,p-1}(\sigma, f^{(p-1)})}{\big(4(1+\varepsilon)\sigma\big)^m},$$

for any $\varepsilon \in R_+$ and for almost all $\sigma > \sigma_0$, where

$$\sigma_0 = \sup\{\sigma_1, \sigma_2, \ldots, \sigma_m\}.$$

But for $\lambda \geqslant \delta > 0$ and for sufficiently large σ, it follows, from Theorem 2 and the fact ([2], Theorem 1) that W_r is an increasing function of σ, that

$$(126) \qquad \log W'_{r,m}(\sigma, f^{(m)}) > \log W'_r(\sigma, f), \qquad \forall m \in Z_+.$$

Making use of (1.26) in (1.25) we get (1.21).

To prove (1.22) and (1.23), we start with (1.14) and (1.15), respectively, and proceed as above.

2. THEOREM 6. *If $f \in E$ is an entire function of Ritt order ϱ,*

$$(2.1) \qquad \lim_{\sigma\to+\infty} \frac{\sup}{\inf} \frac{\log W_r(\sigma,f)}{e^{\varrho\sigma}} = \frac{T}{t}, \qquad t, T \in R_+,$$

and

$$2.2) \qquad \frac{W'_r(\sigma,f)}{W_r(\sigma,f)} \sim a e^{\varrho\sigma},$$

for large values of σ, where a is a positive constant, then

(i) *f is of regular growth,*

(ii) *$\varrho T = \varrho t = a$,*

and

(iii) $\lim\limits_{\sigma\to+\infty} \dfrac{W'_r(\sigma,f)}{W_r(\sigma,f)\log W_r(\sigma,f)} = \varrho.$

Proof. (i) From (2.2) we have

$$\log\big(W'_r(\sigma,f)/W_r(\sigma,f)\big) \sim \varrho\sigma + \log a.$$

Therefore

$$\lim_{\sigma\to+\infty} \frac{\log\big(W'_r(\sigma,f)/W_r(\sigma,f)\big)}{\sigma} = \varrho,$$

and hence f is of regular growth, in view of (1.3).

(ii) From (1.10) and (2.2) we get,

$$\log W_r(\sigma,f) - \log W_r(\sigma_0,f) \sim a \int_{\sigma_0}^{\sigma} e^{\varrho x}\,dx = \frac{a}{\varrho}(e^{\varrho\sigma} - e^{\varrho\sigma_0}).$$

Dividing throughout by $e^{\varrho\sigma}$ and proceeding to limit, we get

$$\lim_{\sigma\to+\infty} \frac{\log W_r(\sigma,f)}{e^{\varrho\sigma}} = \frac{\alpha}{\varrho}. \tag{2.3}$$

(ii) now follows from (2.1) and (2.3).

(iii) From (2.3) we also get

$$\lim_{\sigma\to+\infty} \frac{\alpha e^{\varrho\sigma}}{\log W_r(\sigma,f)} = \varrho,$$

which, on using (2.2), gives (iii).

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UNIVERSITY OF JAMMU
JAMMU, INDIA

Reçu par la Rédaction le 6. 10. 1971