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Remarks on a nonlinear Volterra equation

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Abstract. Nonnegative solutions of the nonlinear equation $u = k * g(u)$ ($g(0) = 0$) are considered. Certain necessary and sufficient conditions for the existence of a nontrivial solution are given.

1. Introduction. In the theory of nonlinear waves in tubes the following equation is considered:

$$u(x) = \int_0^x (x-s)^\alpha u^{1/\beta}(s)\,ds,$$

where $\alpha > 0$, $\beta > 1$ are physical parameters (see [2]). From a physical point of view only nonnegative solutions of this equation are interesting. But let us note that the trivial solution $u \equiv 0$ satisfies this equation. More generally, we are interested in necessary and sufficient conditions for the existence of a nonnegative continuous solution of the nonlinear Volterra equation

$$u(x) = \int_0^x k(x-s)g(u(s))\,ds. \tag{1.1}$$

The case of the kernel $k(x) = x^\alpha$, $\alpha > 0$, was solved satisfactorily by G. Gripenberg in [1]. For other classes of kernels no similar conditions are known. We present certain necessary and sufficient conditions for the existence of a nonnegative continuous solution $u \not\equiv 0$ of (1.1). Such a solution will be called shortly a *nontrivial solution* of (1.1). The class of kernels considered in this paper is larger than that in [1].

2. Assumptions and notation. Throughout this paper we assume

(2.1) k is absolutely continuous on every $[0, A)$, $A > 0$, $k(0) = 0$ and $k(x) > 0$ for $x > 0$,

(2.2) k is nondecreasing on $[0, \delta)$ for some $\delta > 0$,

(2.3) $g \in C([0, \infty))$, $g(0) = 0$,

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(2.4) $g(x)$ is nondecreasing and $g(x)/x$ is nonincreasing on $(0, \delta)$ for some $\delta > 0$,

(2.5) $g(x)/x \to \infty$ as $x \to 0+$.

In the sequel we use the following fact:

LEMMA 2.1. *Let the assumptions* (2.3)–(2.5) *be satisfied. Then* $g(x)$ *and* $x/g(x)$ *are absolutely continuous on* $[0, \delta)$ *for some* $\delta > 0$.

Proof. By (2.4) for any $x_1, x_2 \in [\delta_0, \delta)$, $x_1 < x_2$ and $\delta_0 > 0$ we have

$$0 \leqslant \frac{g(x_2)-g(x_1)}{x_2-x_1} = \frac{g(x_1)}{x_1}\,\frac{g(x_2)/g(x_1)-1}{x_2/x_1-1} \leqslant \frac{g(\delta_0)}{\delta_0}.$$

Hence g is Lipschitz continuous on every $[\delta_0, \delta)$ $(\delta_0 > 0)$. Therefore $g(x)$ is absolutely continuous on $[0, \delta)$. Now we note that $x/g(x)$ satisfies (2.3)–(2.5), so it is also absolutely continuous.

Set $K(x) = \int_0^x k(s)\,ds$ and denote by K^{-1} its inverse.

The facts collected in the following remarks are proved in [3] and [4].

Remark 2.1. (1.1) has a nontrivial solution if and only if there exists a nonnegative continuous function $F \not\equiv 0$ such that

$$F(x) \leqslant \int_0^x k(x-s)g(F(s))\,ds, \qquad x \in (0, \delta), \tag{2.6}$$

for some $\delta > 0$.

Remark 2.2. If (1.1) has a nontrivial solution, then it also has exactly one continuous solution u such that $u(x) > 0$ for $x > 0$. In the sequel, this solution is called the *maximal solution* of (1.1).

In this paper δ always denotes a small number greater than 0. We allow it to change its value from paragraph to paragraph.

3. The necessary condition. In this section we assume

$$\ln k(x) \quad \text{is concave on } (0, \delta). \tag{3.1}$$

Note that from this assumption it follows that

$$k(x) = \Phi(K(x)) \quad \text{on } (0, \delta), \text{ where } \Phi = k \circ K^{-1} \text{ is concave.} \tag{3.2}$$

Now we can prove

THEOREM 3.1. *Let the assumptions* (2.1)–(2.5) *and* (3.1) *be satisfied. Then the condition*

$$\int_0^\delta \frac{1}{g(s)\Phi(s/g(s))}\,ds < \infty \qquad \text{on } (0, \delta) \tag{3.3}$$

is necessary for the existence of a nontrivial solution of (1.1).

Proof. By (2.1) the maximal solution u of (1.1) is absolutely continuous. Therefore, in view of Lemma 2.1, $g(u(x))$ is also absolutely continuous. Moreover, we have

$$u'(x) = \int_0^x k(x-s)[g(u(s))]'\,ds. \tag{3.4}$$

Now using (3.2) and applying the Jensen inequality we get

$$\frac{1}{g(u(x))}\int_0^x \Phi(K(x-s))[g(u(s))]'\,ds \tag{3.5}$$
$$\leqslant \Phi\left(\frac{1}{g(u(x))}\int_0^x K(x-s)[g(u(s))]'\,ds\right) = \Phi\left(\frac{1}{g(u(x))}\int_0^x k(x-s)g(u(s))\,ds\right).$$

Combining (3.4), (3.5) and (1.1) we obtain $u'(x) \leqslant \Phi[u(x)/g(u(x))]\,g(u(x))$, from which by substitution $s = u(x)$ we get (3.3).

Remark 3.1. Since $\Phi = 1/(K^{-1})'$ and

$$\frac{1}{g(x)}(K^{-1})'(x/g(x)) = \left(K^{-1}(x/g(x))\right)' + \frac{xg'(x)}{g^2(x)}(K^{-1})'(x/g(x)),$$

taking $s = g(\tau)$ we see that (3.3) is equivalent to

$$\int_0^\delta \frac{1}{s}\frac{g^{-1}(s)}{s}(K^{-1})'\left(\frac{g^{-1}(s)}{s}\right)ds < \infty. \tag{3.6}$$

Now we present an example of application of Theorem 3.1.

Example 3.1. Let $k(x) = \beta x^{-1-\beta}\exp(-1/x^\beta)$, $\beta > 0$, and $g(x) = x\exp(A\ln^\gamma 1/x)$, $A > 0$, $0 < \gamma \leqslant \beta/(\beta+1)$. Then the assumptions of Theorem 3.1 are satisfied. By easy calculation we get

$$\frac{1}{g(x)\Phi(x/g(x))} \geqslant C\frac{1}{x(-\ln x)} \quad \text{for } x \in (0, \delta),$$

where $C > 0$ is some constant. Thus (1.1) has no nontrivial solution in this case.

4. The sufficient condition. In this section we prove that the condition

$$\int_0^\delta \frac{g'(s)}{g(s)}K^{-1}(s/g(s))\,ds < \infty \quad \text{for some } \delta > 0 \tag{4.1}$$

is sufficient for the existence of a nontrivial solution of (1.1).

Assume (4.1) and set

$$F^{-1}(x) = K^{-1}(x/g(x)) + \int_0^x \frac{g'(s)}{g(s)}K^{-1}(s/g(s))\,ds.$$

In view of Lemma 2.1, (2.1) and (2.5), $F^{-1}(x)$ is absolutely continuous on $[0, \delta)$.

LEMMA 4.1. *For any* $y \in (0, \delta)$ *we have*

$$\int_0^y K(F^{-1}(y) - F^{-1}(\tau))g'(\tau)\,d\tau \geqslant y. \tag{4.2}$$

Proof. Since, by (2.2), K is convex, we can use the Jensen inequality to obtain

$$\begin{aligned}\frac{1}{g(y)}\int_0^y K(F^{-1}(y) - F^{-1}(\tau))g'(\tau)\,d\tau \\ \geqslant K\left(\frac{1}{g(y)}\int_0^y (F^{-1}(y) - F^{-1}(\tau))g'(\tau)\,d\tau\right) = K\left(\frac{1}{g(y)}\int_0^y (F^{-1})'(\tau)g(\tau)\,d\tau\right).\end{aligned} \tag{4.3}$$

Note that

$$(F^{-1})'(\tau)g(\tau) = [g(\tau)K^{-1}(\tau/g(\tau))]'. \tag{4.4}$$

Since in view of Lemma 2.1, (2.1) and (2.5), $g(\tau)K^{-1}(\tau/g(\tau))$ is absolutely continuous on $[0, \delta)$, by (4.3) and (4.4) we have

$$\int_0^y K(F^{-1}(y) - F^{-1}(\tau))g'(\tau)\,d\tau \geqslant K(K^{-1}(y/g(y)))g(y) = y,$$

which completes the proof.

Now we can prove

THEOREM 4.1. *Let the assumptions* (2.1)–(2.5) *and* (4.1) *be satisfied. Then the equation* (1.1) *has a nontrivial solution.*

Proof. Let F be the inverse function to F^{-1}. By substitution $s = F^{-1}(\tau)$ we get

$$\int_0^x k(x-s)g(F(s))\,ds = -\int_0^{F(x)} \frac{d}{d\tau}(K(x - F^{-1}(\tau)))g(\tau)\,d\tau.$$

Integrating by parts on the right-hand side and taking $y = F(x)$ in Lemma 4.1 we see that $\int_0^x k(x-s)g(F(s))\,ds \geqslant F(x)$ on $(0, \delta)$. By Remark 2.2 this completes the proof.

Remark 4.1. Taking $\tau = g(s)$ we see that (4.1) is equivalent to

$$\int_0^\delta \frac{1}{s}K^{-1}(g^{-1}(s)/s)\,ds < \infty \quad \text{for some } \delta > 0. \tag{4.5}$$

Now we present two examples of application of Theorem 4.1.

EXAMPLE 4.1. Let k satisfy (2.1), (2.2) and $g^{-1}(x) = xK(x)$. Then $g(x)$ satisfies (2.3)–(2.5) and by (4.5) it follows that (1.1) has a nontrivial solution.

EXAMPLE 4.2. Let $k(x) = \beta x^{-1-\beta}\exp(-1/x^\beta)$, $\beta > 0$, and $g^{-1}(x) = x\exp(-A\ln^\gamma 1/x)$, $\gamma > \beta$, $A > 0$. Then the assumptions (2.1)–(2.5) are satisfied. Since

$$\frac{1}{x}K^{-1}(g^{-1}(x)/x) = A^{-1/\beta}\frac{1}{x\ln^{\lambda}1/x}, \qquad \lambda = \gamma/\beta,$$

(4.5) is satisfied and (1.1) has a nontrivial solution.

5. Comments. The case $k(x) = x^{\alpha}$, $\alpha > 0$, was considered by G. Gripenberg in [1]. His main result can be formulated as follows:

THEOREM 5.1. *Let g satisfy* (2.3)–(2.5) *and*

(5.1) *for each $p > 0$, $x(g(x)/x)^p$ is nondecreasing on $[0, \delta_p]$ $(\delta_p > 0)$.*

Then the equation (1.1) *with $k(x) = x^{\alpha}$, $\alpha > 0$, has a nontrivial solution if and only if*

$$\int_0^{\delta}\frac{1}{s}[s/g(s)]^{1/(\alpha+1)}ds < \infty. \tag{5.2}$$

It is easy to see that $k(x) = x^{\alpha}$, $\alpha > 0$, satisfies (2.1), (2.2) and (3.1). Moreover, in this case (5.2) and (3.3) are the same. Noting that

$$\frac{\lambda}{s}[s/g(s)]^{\lambda} = \frac{d}{ds}[s/g(s)]^{\lambda} + \lambda\frac{g'(s)}{g(s)}[s/g(s)]^{\lambda}, \qquad \lambda\in\mathbf{R},$$

we see that (5.2) and (4.1) are equivalent.

Therefore Theorem 5.1 follows from Theorems 3.1 and 4.1 and our proof is much simpler than that presented in [1]. Our results are also more general than Theorem 5.1. For example, we did not apply the assumption (5.1).

EXAMPLE 5.1. Let $k(x) = x^{\alpha}$, $\alpha > 0$. Verifying (5.2) we see that (1.1) has a nontrivial solution for $g(x) = x^{\beta}$, $0 < \beta < 1$, and in the case of $g(x) = x\ln^{\beta}1/x$, $\beta > 1$, it has a nontrivial solution if and only if $\beta > \alpha+1$.

EXAMPLE 5.2. For $k(x) = x^{\alpha}$, $\alpha > 0$, and $g(x) = x\exp(A\ln^{\gamma}1/x)$, $0 < \gamma < 1$, we have

$$\frac{1}{s}[s/g(s)]^{1/(\alpha+1)} = \frac{1}{s}\exp(-C\ln^{\gamma}1/s), \qquad C = A/(\alpha+1).$$

By substitution $z = -\ln s$ we get

$$\int_0^{\delta}\frac{1}{s}\exp(-C\ln^{\gamma}1/s)ds = \int_{-\ln\delta}^{\infty}\exp(-Cz^{\gamma})dz < \infty,$$

for γ, $C > 0$. Therefore (1.1) has a nontrivial solution in this case.

It is clear that under the assumptions (2.1)–(2.5) and (3.1) the condition (5.2) implies (4.1). This can also be seen by the comparison of (3.6) with (4.5). Namely, by (2.1), K^{-1} is concave and $K^{-1}(0) = 0$. Hence we have

$$\frac{g^{-1}(x)}{x}(K^{-1})'\left(\frac{g^{-1}(x)}{x}\right) \leqslant K^{-1}\left(\frac{g^{-1}(x)}{x}\right), \qquad x\in(0, \delta). \tag{5.3}$$

Unfortunately, in general, this implication cannot be replaced by equivalence, as the following example shows.

EXAMPLE 5.3. Let $k(x) = x^{-2}\exp(-1/x)$, $g(x) = x^{\gamma}$, $0 < \gamma < 1$. Then

$$\frac{\lambda}{x}\frac{g^{-1}(x)}{x}(K^{-1})'\left(\frac{g^{-1}(x)}{x}\right) = \frac{d}{dx}(K^{-1}(x^{\lambda}))$$

and

$$\frac{1}{x}K^{-1}\left(\frac{g^{-1}(x)}{x}\right) = \frac{1}{\lambda x(-\ln x)'}$$

where $\lambda = 1/\gamma - 1$. Hence (3.6) is satisfied and (4.5) is not.

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