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A generalization of a formalized theory of fields of sets on non-classical logics

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Introduction

This paper[1] deals with the classical predicate calculus $\mathscr{S}^*$ of the second order which contains as primitive signs individual variables, set variables, the sign for the universal set, the sign for the empty set, the signs for set-theoretical operations on sets, the sign for the identity of individuals, the sign for the equality between sets and the sign for inclusion, propositional connectives and quantifiers binding individual and set variables. The set of axioms is composed of logical axioms and of axioms which characterize: the universal and the empty set, the operations on sets by means of propositional connectives, the identity relation between individuals, the equality relation between sets and the relation of inclusion. Usual rules of inference are admitted.

It is shown that $\mathscr{S}^*$ is the theory of fields of sets, i.e. a formula is a theorem in $\mathscr{S}^*$ if and only if it is true in every field of sets, when set variables are interpreted as variables ranging over a fixed field of sets and quantifiers binding set variables are restricted to sets belonging to the same field of sets. Moreover, a formula of $\mathscr{S}^*$ which does not contain any individual variables and quantifiers binding individual variables is a theorem in $\mathscr{S}^*$ if and only if it is a theorem in the elementary formalized theory of Boolean algebras.

Let $\mathscr{S}$ be a propositional non-classical calculus which contains the disjunction sign, the conjunction sign, the implication sign and perhaps some other propositional connectives. We assume that all theorems of positive logic are theorems of $\mathscr{S}$. The system $\mathscr{S}$ determines a class of abstract algebras (called $\mathscr{S}$-algebras) which are matrices of $\mathscr{S}$. The system $\mathscr{S}$ determines also a corresponding system $\mathscr{S}^*$ of the second order predicate calculus analogous to the classical system described above. Under some additional hypotheses concerning the propositional calculus $\mathscr{S}$, an intepretation of the system $\mathscr{S}^*$ in question is suggested and the completeness theorem with regard to this interpretation is proved. Moreover, it is shown that a formula of the form $A = B$, or $A \subset B$ is a theorem in $\mathscr{S}^*$ if and only if it is a theorem in the elementary formalized theory of $\mathscr{S}$-algebras based on $\mathscr{S}$-logic.

(1) For a summary covering a part of the results given in this paper, see [3].

It remains an open question whether this statement is true for arbitrary formulas without individual variables and quantifiers binding individual variables (p. 27).

Note, that as the propositional calculus $\mathscr{S}$ we can admit, for instance, the positive, minimal, intuitionistic or modal S4 propositional calculus.

§ 1. System $\mathscr{S}$ of a propositional calculus

We shall consider a fixed system $\mathscr{S}$ (see [5]) of a propositional calculus described briefly as follows:

The primitive symbols of $\mathscr{S}$ are propositional variables $a_1, a_2, \ldots,$ parentheses, and the following propositional operators

(a) the disjunction sign $\cup$, the conjunction sign $\cap$, the implication sign $\Rightarrow$;

(b) some other binary propositional operators $o_1, \ldots, o_r$;

(c) some unary propositional operators $o^1, \ldots, o^s$.

The sets of operators mentioned in (b) and (c) may be empty.

The set $\mathscr{F}$ of all formulas in $\mathscr{S}$ is the least set such that

(i) a_j are in $\mathscr{F}$ $(j = 1, 2, \ldots)$,

(ii) if α, β are in $\mathscr{F}$, then so are $(\alpha\cup\beta)$, $(\alpha\cap\beta)$, $(\alpha \Rightarrow \beta)$, $(\alpha o_i \beta)$ $(i = 1, \ldots, r)$, $(o^i\alpha)$ $(i = 1, \ldots, s)$.

Instead of $\big((\alpha \Rightarrow \beta)\cap(\beta \Rightarrow \alpha)\big)$ we shall write for brevity $(\alpha \Leftrightarrow \beta)$.

In writing formulas we shall practice the omission of the parentheses, the rule being that

1. *each of the operators* $\cap$, $\cup$, $\Rightarrow$ binds an expression less *strongly than the previous one*;

2. *each of the operators* o^i $(i = 1, \ldots, s)$ *binds an expression more strongly than any binary operator.*

In the set $\mathscr{F}$ of all formulas we distinguish a subset $\mathscr{T} \subset \mathscr{F}$ of all theorems. We shall assume that the set $\mathscr{T}$ fulfils the following conditions:

(t_1) *if* α *and* $\alpha \Rightarrow \beta$ *are in* $\mathscr{T}$, *then* β *is in* $\mathscr{T}$ (*modus ponens*);

(t_2) *if* γ *is a part of* α, *if* $\gamma \Rightarrow \delta$ *and* $\delta \Rightarrow \gamma$ *are in* $\mathscr{T}$, *and if* β *is obtained from* α *by the replacement of the part* γ *by* δ, *then formulas* $\alpha \Rightarrow \beta$ *and* $\beta \Rightarrow \alpha$ *are both in* $\mathscr{T}$ (*the rule of replacement*);

(t_3) *if* α, β, γ *are in* $\mathscr{F}$, *then each of the formulas* T_1-T_9 *given below is in* $\mathscr{T}$:

T_1 $\quad (\alpha \Rightarrow \beta) \Rightarrow \big((\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma)\big)$,

T_2 $\quad \alpha \Rightarrow \alpha \cup \beta$,

T_3 $\beta \Rightarrow \alpha\cup\beta$,
T_4 $(\alpha \Rightarrow \gamma) \Rightarrow \big((\beta \Rightarrow \gamma) \Rightarrow (\alpha\cup\beta \Rightarrow \gamma)\big)$,
T_5 $\alpha\cap\beta \Rightarrow \alpha$,
T_6 $\alpha\cap\beta \Rightarrow \beta$,
T_7 $(\gamma \Rightarrow \alpha) \Rightarrow \big((\gamma \Rightarrow \beta) \Rightarrow (\gamma \Rightarrow \alpha\cap\beta)\big)$,
T_8 $\big(\alpha \Rightarrow (\beta \Rightarrow \gamma)\big) \Rightarrow (\alpha\cap\beta \Rightarrow \gamma)$,
T_9 $(\alpha\cap\beta \Rightarrow \gamma) \Rightarrow \big(\alpha \Rightarrow (\beta \Rightarrow \gamma)\big)$;
(t_4) *the set* $\mathscr{F}$-$\mathscr{T}$ *is non-empty.*

The formulas T_1-T_9 are the axioms of positive logic (see [1]). Consequently all formulas which are substitutions of theorems of the positive propositional calculus are in $\mathscr{T}$. In particular the following formulas are in $\mathscr{T}$:

T'_{10} $\alpha \Rightarrow \alpha$,
T'_{11} $\alpha \Rightarrow (\beta \Rightarrow \alpha)$,
T'_{12} $\big(\alpha \Rightarrow (\beta \Rightarrow \gamma)\big) \Rightarrow \big(\beta \Rightarrow (\alpha \Rightarrow \gamma)\big)$,
T'_{13} $\alpha \Rightarrow \big(\beta \Rightarrow (\alpha\cap\beta)\big)$.

Note that the condition (t_2) implies the extensionality of all propositional operators mentioned in (a), (b), (c).

If among unary propositional operators there appears an unary operator $-$ for which the following condition is fulfilled:

(1) $-(\beta \Rightarrow \beta) \Rightarrow \alpha$ *is in* $\mathscr{T}$ *for any* α, β *in* $\mathscr{F}$,

then it will be called a *negation sign* and $\mathscr{S}$ will be called a *propositional calculus with negation.* (See [5], p. 74).

The propositional calculus $\mathscr{S}$ will be said to be *axiomatizable* if there exists a recursive set $\mathscr{Z}$ of formulas in $\mathscr{F}$ such that the set $\mathscr{T}$ is the least set of formulas satisfying the conditions (t_1)-(t_4) and moreover the following one:

(t_5) $\mathscr{T}$ *contains every formula obtained from formulas in* $\mathscr{Z}$ *by an arbitrary substitution for propositional variables of any formulas in* $\mathscr{F}$.

§ 2. System $\mathscr{S}^*$

With the system $\mathscr{S}$ of a propositional calculus we shall associate a non-elementary system $\mathscr{S}^*$ which is a generalization of the classical calculus of classes.

The primitive signs of $\mathscr{S}^*$ are the parentheses and

(a) the free individual variables denoted by $x_1, x_2, \ldots$;

(a′) the bound individual variables denoted by $\xi_1, \xi_2, \ldots$;

(b) the free set variables denoted by $F_1, F_2, \ldots$;

(b') the bound set variables denoted by $\varphi_1, \varphi_2, \ldots$;

(c) the symbol V for the universal set;

(d) the symbol Λ for the empty set (if $\mathscr{S}$ is a propositional calculus with negation);

(e) the symbol $=$ for the identity relation between individuals;

(f) the symbols $\cup, \cap, \rightarrow, o_1, \ldots, o_r$ for binary operations on sets;

(g) the symbols $o^1, \ldots, o^s$ for unary operations on sets;

(h) the symbols for binary relations between sets: the sign $\subset$ for inclusion and the sign $=$ for equality;

(i) the propositional operators in $\mathscr{S}$: $\cup, \cap, \Rightarrow, o_1, \ldots, o_r, o^1, \ldots, o^s$;

(j) quantifiers $\bigcup_{\xi_k}, \bigcap_{\xi_k}, \bigcup_{\varphi_k}, \bigcap_{\varphi_k}$ $(k = 1, 2, \ldots)$.

From these signs we form expressions of two kinds: set designations and formulas.

The set $\mathscr{D}^*$ of all set designations is the least set containing F_k $(k = 1, 2, \ldots)$, V and Λ (if Λ appears among the primitive signs of $\mathscr{S}^*$) and such that if A, B are in $\mathscr{D}^*$, then $(A \cup B), (A \cap B), (A \rightarrow B), (A o_i B)$ where $i = 1, \ldots, r$, $(o^i A)$ for $i = 1, \ldots, s$ are also in $\mathscr{D}^*$.

In writing set designations we shall practice the omission of the parentheses, the rule being analogous to those mentioned in § 1 for formulas in $\mathscr{S}$.

The set $\mathscr{F}^*$ of all formulas is the least set such that

1) if A is in $\mathscr{D}^*$, then $A(x_k)$ is in $\mathscr{F}^*$ $(k = 1, 2, \ldots)$;

2) if A, B are in $\mathscr{D}^*$, then $(A = B)$ and $(A \subset B)$ are in $\mathscr{F}^*$;

3) $(x_k = x_m)$ is in $\mathscr{F}^*$ $(k, m = 1, 2, \ldots)$;

4) if α, β are in $\mathscr{F}^*$ then $(\alpha \cup \beta)$, $(\alpha \cap \beta)$, $(\alpha \Rightarrow \beta)$, $(\alpha o_k \beta)$ for $k = 1, \ldots, r$, $(o^k \alpha)$ for $k = 1, \ldots, s$ are in $\mathscr{F}^*$;

5) if $\alpha(x_k)$ is in $\mathscr{F}^*$ and neither $\bigcap_{\xi_j}$ nor $\bigcup_{\xi_j}$ appears in $\alpha(x_k)$, then $\bigcup_{\xi_j} \alpha(x_k/\xi_j)$ and $\bigcap_{\xi_j} \alpha(x_k/\xi_j)$ are in $\mathscr{F}^*$, where $\alpha(x_k/\xi_j)$ is the expression obtained from $\alpha(x_k)$ by the substitution of ξ_j for x_k;

6) if $\alpha(F_k)$ is in $\mathscr{F}^*$ and neither $\bigcup_{\varphi_j}$ nor $\bigcap_{\varphi_j}$ appears in α, then $\bigcup_{\varphi_j} \alpha(F_k/\varphi_j)$ and $\bigcap_{\varphi_j} \alpha(F_k/\varphi_j)$ are in $\mathscr{F}^*$;

We shall sometimes write $(\alpha \equiv \beta)$ instead of $((\alpha \Rightarrow \beta) \cap (\beta \Rightarrow \alpha))$. We shall also practice the omission of parentheses, the rules being the same as those adopted for the system $\mathscr{S}$ and additionally the following rule.

3. *The quantifiers bind more strongly than any operators mentioned in* (i) *above.*

The axioms of $\mathscr{S}^*$ are all formulas obtained from the axioms of $\mathscr{S}$ by the substitution for propositional variables of arbitrary formulas in $\mathscr{F}^*$ and, moreover, the following formulas:

(c_1) $\bigcap_{\xi_1} \vee(\xi_1)$,

(c_2) $\bigcap_{\xi_1} (\wedge(\xi_1) \Rightarrow -\vee(\xi_1))$ (if $\mathscr{S}$ is a propositional calculus with negation),

(c_3) $\bigcap_{\xi_1} \big((F_1 \cup F_2)(\xi_1) \equiv (F_1(\xi_1) \cup F_2(\xi_1))\big)$,

(c_4) $\bigcap_{\xi_1} \big((F_1 \cap F_2)(\xi_1) \equiv (F_1(\xi_1) \cap F_2(\xi_1))\big)$,

(c_5) $\bigcap_{\xi_1} \big((F_1 \dot{\rightarrow} F_2)(\xi_1) \equiv (F_1(\xi_1) \Rightarrow F_2(\xi_1))\big)$,

(c_6) $\bigcap_{\xi_1} \big((F_1 o_k F_2)(\xi_1) \equiv (F_1(\xi_1) o_k F_2(\xi_1))\big)$ $(k = 1, \ldots, r)$,

(c_7) $\bigcap_{\xi_1} \big((o^k F_1)(\xi_1) \equiv (o^k(F_1(\xi_1)))\big)$ $(k = 1, \ldots, s)$,

(c_8) $(F_1 \subset F_2) \equiv \bigcap_{\xi_1} (F_1(\xi_1) \Rightarrow F_2(\xi_1))$,

(c_9) $(F_1 = F_2) \equiv (F_1 \subset F_2) \cap (F_2 \subset F_1)$,

(c_{10}) $\bigcap_{\xi_1} (\xi_1 = \xi_1)$,

(c_{11}) $\bigcap_{\xi_1} \bigcap_{\xi_2} \bigcap_{\xi_3} \big((\xi_1 = \xi_2) \Rightarrow ((\xi_3 = \xi_2) \Rightarrow (\xi_1 = \xi_3))\big)$,

(c_{12}) $\bigcap_{\xi_1} \bigcap_{\xi_2} (F_1(\xi_1) \cap (\xi_1 = \xi_2) \Rightarrow F_1(\xi_2))$.

The set $\mathscr{T}^*$ of all theorems of $\mathscr{S}^*$ is the least set of formulas containing all axioms and fulfilling the conditions (t_1), (t_2) mentioned in § 1 and the following ones:

(t_3^*) *if* $\alpha(x_k)$ *is in* $\mathscr{T}^*$, *then* $\alpha(x_k/x_m)$ *is in* $\mathscr{T}^*$,

(t_4^*) *if* $\alpha(F_k)$ *is in* $\mathscr{T}^*$, *and* A *is in* $\mathscr{D}^*$, *then* $\alpha(F_k/A)$ *is in* $\mathscr{T}^*$;

(t_5^*) *if* $\alpha \Rightarrow \bigcap_{\xi_k} \beta(x_m/\xi_k)$ *is in* $\mathscr{T}^*$, *then* $\alpha \Rightarrow \beta(x_m)$ *is in* $\mathscr{T}^*$; *if* $\alpha \Rightarrow \bigcap_{\varphi_k} \beta(F_m/\varphi_k)$ *is in* $\mathscr{T}^*$, *then* $\alpha \Rightarrow \beta(F_m)$ *is in* $\mathscr{T}^*$;

(t_6^*) *if* $\bigcup_{\xi_k} \alpha(x_m/\xi_k) \Rightarrow \beta$ *is in* $\mathscr{T}^*$, *then* $\alpha(x_m) \Rightarrow \beta$ *is in* $\mathscr{T}^*$; *if* $\bigcup_{\varphi_k} \alpha(F_m/\varphi_k) \Rightarrow \beta$ *is in* $\mathscr{T}^*$, *then* $\alpha(F_m) \Rightarrow \beta$ *is in* $\mathscr{T}^*$;

(t_7^*) *if* $\alpha \Rightarrow \beta(x_m)$ *is in* $\mathscr{T}^*$ *and neither* $\bigcup_{\xi_k}$ nor $\bigcap_{\xi_k}$ *appears in* $\beta(x_k)$ *and, moreover, there is no occurrence of* x_m *in* α, *then* $\alpha \Rightarrow \bigcap_{\xi_k} \beta(x_m/\xi_k)$ *is in* $\mathscr{T}^*$; *if* $\alpha \Rightarrow \beta(F_m)$ *is in* $\mathscr{T}^*$, *neither* $\bigcap_{\varphi_k}$ *nor* $\bigcup_{\varphi_k}$ *appears in* $\beta(F_m)$ *and there is no occurrence of* F_m *in* α, *then* $\alpha \Rightarrow \bigcap_{\varphi_k} \beta(F_m/\varphi_k)$ *is in* $\mathscr{T}^*$;

(t_8^*) *if* $\alpha(x_m) \Rightarrow \beta$ *is in* $\mathscr{T}^*$, *neither* $\bigcap_{\xi_k}$ *nor* $\bigcup_{\xi_k}$ *appears in* $\alpha(x_m)$ *and there is no occurrence of* x_m *in* β, *then* $\bigcup_{\xi_k} \alpha(x_m/\xi_k) \Rightarrow \beta$ *is in* $\mathscr{T}^*$; *if* $\alpha(F_m) \Rightarrow \beta$ *is in* $\mathscr{T}^*$, *neither* $\bigcap_{\varphi_k}$ *nor* $\bigcup_{\varphi_k}$ *appears in* $\alpha(F_m)$ *and there is no occurrence of* F_m *in* β, *then* $\bigcup_{\varphi_k} \alpha(F_m/\varphi_k) \Rightarrow \beta$ *is in* $\mathscr{T}^*$.

(t_9^*) *for any* $A, B \in \mathscr{D}^*$, *if* $A = B$ *is in* $\mathscr{T}^*$, *then the formulas* $o^i A = o^i B$ *for* $i = 1, 2, \ldots, s$ *are in* $\mathscr{T}^*$,

(t_{10}^*) *for any* $A, B, C, D \in \mathscr{D}^*$ *if* $A = B$ *and* $C = D$ *are in* $\mathscr{T}^*$ *then* $A o_i C = B o_i D$ *for* $i = 1, \ldots, r$ *are also in* $\mathscr{T}^*$.

It is easy to show that also the following condition is satisfied:

(t_{11}^*) *if* α *is the result of a substitution for propositional variables of any formulas of* $\mathscr{F}^*$ *in a theorem of* $\mathscr{S}$, *then* α *is a theorem of* $\mathscr{S}^*$;

(t_{12}^*) *if* $\alpha(x_m)$ *is in* $\mathscr{T}^*$ *and neither* $\bigcap_{\xi_k}$ *nor* $\bigcup_{\xi_k}$ *appears in* $\alpha(x_m)$, *then* $\bigcap_{\xi_k}\alpha(x_m/\xi_k)$ *is in* $\mathscr{T}^*$; *if* $\alpha(F_m)$ *is in* $\mathscr{T}^*$ *and neither* $\bigcap_{\varphi_k}$ *nor* $\bigcup_{\varphi_k}$ *appears in* $\alpha(F_m)$, *then* $\bigcap_{\varphi_k}\alpha(F_m/\varphi_k)$ *is in* $\mathscr{T}^*$.

It is also easy to prove that the following formulas are in $\mathscr{T}^*$:

(T_1^*) $F_1 = F_1$,

(T_2^*) $(F_1 = F_2) \Rightarrow (F_2 = F_1)$,

(T_3^*) $(F_1 = F_2) \Rightarrow \big((F_2 = F_3) \Rightarrow (F_1 = F_3)\big)$,

(T_4^*) $(F_1 = F_2) \Rightarrow \big((F_3 = F_4) \Rightarrow \big((F_1 \cup F_3) = (F_2 \cup F_4)\big)\big)$,

(T_5^*) $(F_1 = F_2) \Rightarrow \big((F_3 = F_4) \Rightarrow \big((F_1 \cap F_3) = (F_2 \cap F_4)\big)\big)$,

(T_6^*) $(F_1 = F_2) \Rightarrow \big((F_3 = F_4) \Rightarrow \big((F_1 \rightarrow F_3) = (F_2 \rightarrow F_4)\big)\big)$,

(T_7^*) $(F_1 \cup F_2) = (F_2 \cup F_1)$,

(T_8^*) $(F_1 \cap F_2) = (F_2 \cap F_1)$,

(T_9^*) $\big(F_1 \cup (F_2 \cup F_3)\big) = \big((F_1 \cup F_2) \cup F_3\big)$,

(T_{10}^*) $\big(F_1 \cap (F_2 \cap F_3)\big) = \big((F_1 \cap F_2) \cap F_3\big)$,

(T_{11}^*) $\big(F_1 \cap (F_1 \cup F_2)\big) = F_1$,

(T_{12}^*) $\big((F_1 \cap F_2) \cup F_2\big) = F_2$,

(T_{13}^*) $\big((F_1 \cap F_2) \subset F_3\big) \equiv \big(F_1 \subset (F_2 \rightarrow F_3)\big)$,

(T_{14}^*) $(F_1 \subset F_2) \equiv \big((F_1 \cap F_2) = F_1\big)$,

(T_{15}^*) $(F_1 \subset F_2) \equiv \big((F_1 \rightarrow F_2) = \vee\big)$,

(T_{16}^*) $F_1 \subset \vee$,

(T_{17}^*) $\wedge \subset F_1$ (if $\mathscr{S}$ is a propositional calculus with negation),

(T_{18}^*) $F_1 \rightarrow F_1 = \vee$,

(T_{19}^*) $-(F_1 \rightarrow F_1) = \wedge$ (if $\mathscr{S}$ is a propositional calculus with negation).

Let $\alpha(a_{i_1}, \ldots, a_{i_n})$ be a formula of $\mathscr{S}$ and let $a_{i_1}, \ldots, a_{i_n}$ be all propositional variables occurring in α. Substituting for every a_{i_j} $(j = 1, \ldots, n)$ in α a formula $F_{i_j}(x_k)$ where x_k is an arbitrary free individual variable, we obtain the formula

$$\alpha\big(F_{i_1}(x_k), \ldots, F_{i_n}(x_k)\big)$$

of $\mathscr{S}^*$. On the other hand, replacing in $a(a_{i_1}, \ldots, a_{i_n})$ every occurrence of the propositional variables a_{i_j} $(j = 1, \ldots, n)$ by the set variable F_{i_j}, and every propositional operator $\cup, \cap, \Rightarrow, o_k$ $(k = 1, \ldots, r)$, o^k $(k = 1, \ldots, s)$ by the set operator $\mathbf{\cup}, \mathbf{\cap}, \rightarrow, \boldsymbol{o}_k$ $(k = 1, \ldots, r)$, $\boldsymbol{o}^k$ $(k = 1, \ldots, s)$, respectively, we obtain the set designation

$$D_a(F_{i_1}, \ldots, F_{i_n}) \quad \text{in } \mathscr{D}^*.$$

2.1. *The formula* $a\big(F_{i_1}(x_k), \ldots, F_{i_n}(x_k)\big) \equiv D_a(F_{i_1}, \ldots, F_{i_n})(x_k)$ *is in the set* $\mathscr{T}^*$.

The easy proof by induction on the length of a is omitted.

2.2. *If* $a(a_{i_1}, \ldots, a_{i_n})$ *is a theorem of* $\mathscr{S}$, *then the formula* $D_a(F_{i_1}, \ldots, F_{i_n}) = \vee$ *is a theorem of* $\mathscr{S}^*$.

Indeed, the formula $a\big(F_{i_1}(x_k), \ldots, F_{i_n}(x_k)\big)$ is an axiom of $\mathscr{S}^*$ and consequently is a theorem of $\mathscr{S}^*$. Hence, by 2.1, T_5 and (t_1), $D_a(F_{i_1}, \ldots, F_{i_n})(x_k)$ is a theorem of $\mathscr{S}^*$. Consequently, on account of T'_{11}, (t^*_{11}) and (t_1) the formula $\vee(x_k) \Rightarrow D_a(F_{i_1}, \ldots, F_{i_n})(x_k)$ is a theorem of $\mathscr{S}^*$. On the other hand, by (c_1), T'_{11} and (t^*_5) $D_a(F_{i_1}, \ldots, F_{i_n})(x_k) \Rightarrow \vee(x_k)$ is also a theorem of $\mathscr{S}^*$. Thus, by (t^*_{12}), $\bigcap_{\xi_1}\big(D_a(F_{i_1}, \ldots, F_{i_n})(\xi_1) \Rightarrow \vee(\xi_1)\big)$ is in $\mathscr{T}^*$ and $\bigcap_{\xi_1}\big(\vee(\xi_1) \Rightarrow D_a(F_{i_1}, \ldots, F_{i_n})(\xi_1)\big)$ is in $\mathscr{T}^*$. Hence, by (c_8), T'_{13} and (c_0), 2.2 holds.

§ 3. $\mathscr{S}$-algebras (2)

With the system $\mathscr{S}$ of a propositional calculus we shall associate a type of abstract algebras. Each algebra of the type under consideration is an ordered set $\langle \mathscr{A}; \vee, \cup, \cap, \Rightarrow, o_1, \ldots, o_r, o^1, \ldots, o^s\rangle$ where

(a) $\vee$ is a distinguished element of $\mathscr{A}$,

(b) $\cup, \cap, \Rightarrow, o_1, \ldots, o_r$ are binary operations defined over $\mathscr{A}$ and class-closing on $\mathscr{A}$;

(c) $o^1, \ldots, o^s$ are unary operations on $\mathscr{A}$ and class-closing on $\mathscr{A}$.

For convenience, we shall denote such an algebra by the same letter as the set of its elements, i.e. we shall write "the algebra $\mathscr{A}$" instead of "the algebra $\langle \mathscr{A}; \vee, \cup, \ldots, o^s\rangle$".

Any mapping $v\colon P \to \mathscr{A}$, where P is the set of all propositional variables of $\mathscr{S}$ will be called a *valuation* of propositional variables in an algebra $\mathscr{A}$. Any formula a in $\mathscr{F}$ determines uniquely an operation

$$a_{\mathscr{A}}\colon \mathscr{A}^P \to \mathscr{A}$$

(2) See [5], p. 67.

defined inductively as follows [3]:

$$a_{i\mathscr{A}}(v) = v(a_i) \text{ for any } i = 1, 2, \ldots \text{ and every } v\colon P \to \mathscr{A},$$
$$(\beta\cup\gamma)_{\mathscr{A}}(v) = \beta_{\mathscr{A}}(v)\cup\gamma_{\mathscr{A}}(v),$$
$$(\beta\cap\gamma)_{\mathscr{A}}(v) = \beta_{\mathscr{A}}(v)\cap\gamma_{\mathscr{A}}(v),$$
$$(\beta \Rightarrow \gamma)_{\mathscr{A}}(v) = \beta_{\mathscr{A}}(v) \Rightarrow \gamma_{\mathscr{A}}(v),$$
$$(\beta\, o_k \gamma)_{\mathscr{A}}(v) = \beta_{\mathscr{A}}(v)\, o_k \gamma_{\mathscr{A}}(v), \quad k = 1, \ldots, r,$$
$$(o^k\beta)_{\mathscr{A}}(v) = o^k\big(\beta_{\mathscr{A}}(v)\big), \quad k = 1, \ldots, s.$$

The algebra $\mathscr{A}$ is said to be an *$\mathscr{S}$-algebra* if the following conditions are fulfilled:

(i) $a \Rightarrow b = \vee$ and $b \Rightarrow a = \vee$ imply $a = b$,

(ii) if $\vee \Rightarrow a = \vee$, then $a = \vee$,

(iii) if a is in $\mathscr{T}$ then $a_{\mathscr{A}}(v) = \vee$ for every $v\colon P \to \mathscr{A}$.

An $\mathscr{S}$-algebra is said to be *non-degenerated* if it contains at least two elements.

In the rest of this section the letter $\mathscr{A}$ will always denote an $\mathscr{S}$-algebra. Let $a, b \in \mathscr{A}$. We shall write $a \subset b$ whenever $a \Rightarrow b = \vee$. The following theorem is known (see [5], 3.1 and 7.1).

3.1. *Every $\mathscr{S}$-algebra $\mathscr{A}$ is a relatively pseudocomplemented lattice with respect to the operations "$\cup$" (join), "$\cap$" (meet) and "$\Rightarrow$" (relatively pseudocomplementation). The relation $a \subset b$ is the ordering relation in the lattice $\mathscr{A}$ (i.e. $a \subset b$ if and only if $a\cup b = b$). The element $\vee$ is the unit of the lattice $\mathscr{A}$. Moreover, if $\mathscr{S}$ is a propositional calculus with negation $-$, then the element $-\vee = \wedge$ is the zero element of $\mathscr{A}$, i.e. $\wedge = -\vee \subset a$ for every a in $\mathscr{A}$.*

We will show that the following theorem holds

3.2. *If $\mathscr{S}$ is an axiomatizable propositional calculus fulfilling* (t_1)-(t_4), *then the class of all $\mathscr{S}$-algebras is equationally definable, i.e. there exists a recursive set of axioms for $\mathscr{S}$-algebras such that each axiom has the form of an equation.*

Let us suppose that there exists a recursive set $\mathscr{X}$ of formulas in $\mathscr{F}$ such that the set $\mathscr{T}$ is the least set of formulas satisfying (t_1)-(t_5) (see § 1). It is easy to prove that the set of equations composed of axioms for relatively pseudocomplemented lattices [4]

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[3] By this definition, $a_{\mathscr{A}}$ is a mapping from $\mathscr{A}^P$ into $\mathscr{A}$. On the other hand, if P_a is the set of all propositional variables appearing in a, $a_{\mathscr{A}}$ can also be considered as a mapping from $\mathscr{A}^{P_a}$ into $\mathscr{A}$, since in reality it depends only on those variables which appear in a.

[4] These axioms are dual to the axioms given in [2].

$$a \cup b = b \cup a, \quad a \cap b = b \cap a,$$
$$(a \cup b) \cup c = a \cup (b \cup c), \quad (a \cap b) \cap c = a \cap (b \cap c),$$
$$(a \cap b) \cup b = b, \quad a \cap (a \cup b) = a,$$
$$\big(a \cap (a \Rightarrow b)\big) \cup b = b, \quad a \cap \big(b \Rightarrow (a \cap b)\big) = a, \quad \big(a \Rightarrow (b \cap c)\big) \cup (a \Rightarrow c) = a \Rightarrow c$$

and of the equations

(1) $$a_{\mathscr{A}} = V \quad \text{for every } a \text{ in } \mathscr{L},$$

is the required set of axioms for the class of all $\mathscr{S}$-algebras. In (1) $a_{\mathscr{A}}$ is treated as a mapping from $\mathscr{A}^{I_a}$ into $\mathscr{A}$ (see footnote 4, p. 10).

If an $\mathscr{S}$-algebra $\mathscr{A}$ is a complete lattice, then $\mathscr{A}$ is said to be an *$\mathscr{S}$*-algebra* (cf. [5], p. 68).

A mapping h of an $\mathscr{S}$-algebra $\mathscr{A}$ into an $\mathscr{S}$-algebra $\mathscr{A}'$ is said to be an *$\mathscr{S}$-homomorphism*, if it preserves all finite algebraic operations, i.e. if $h(aob) = h(a)oh(b)$ where o is one of the signs $\cup$, $\cap$, $\Rightarrow$, $o_1, \ldots, o_r$, and $h(oa) = oh(a)$ where o is one of the signs $o^1, \ldots, o^s$. An $\mathscr{S}$-homomorphism is called an *$\mathscr{S}$-isomorphism* if it is one-to-one. An $\mathscr{S}$-homomorphism h is said *to preserve the infinite join* (*meet*)

$$a = \bigcup_{u \in U} a_u \qquad \Big(a = \bigcap_{u \in U} a_u\Big)$$

if

$$h(a) = \bigcup_{u \in U} h(a_u) \qquad \Big(h(a) = \bigcap_{u \in U} a_u\Big).$$

The system $\mathscr{S}$ is said to have the property (E) (cf. [5], p. 69) if, for every $\mathscr{S}$-algebra $\mathscr{A}$ and for arbitrary enumerable sequences of equations

(*) $$a_n = \bigcup_{u \in U_n} a_{nu}, \qquad b_n = \bigcap_{w \in W_n} b_{nw},$$

there is an $\mathscr{S}$-isomorphism h of $\mathscr{A}$ into an $\mathscr{S}^*$-algebra $\mathscr{A}'$ which preserves all the joins and meets (*).

§ 4. The algebra of set designations of $\mathscr{S}^*$

Let us consider the set $\mathscr{D}^*$ of all set designations of $\mathscr{S}^*$ as an abstract algebra $\langle \mathscr{D}^*, V, \cup, \cap, \rightarrow, o_1, \ldots, o_r, o^1, \ldots, o^s \rangle$. We will prove that

4.1. *The relation $\simeq$ defined for any A, B in $\mathscr{D}^*$ by the equivalence:*

$$A \simeq B \quad \textit{if and only if} \quad (A = B) \textit{ is in } \mathscr{T}^*,$$

is a congruence relation in $\mathscr{D}^*$.

In fact, it is an equivalence relation by (T_1^*)-(T_3^*). By (T_4^*)-(T_6^*) and (t_9^*), (t_{10}^*) this relation preserves all algebraic operations in $\mathscr{D}^*$. Consequently it is a congruence relation in $\mathscr{D}^*$.

Consider the quotient algebra $\mathscr{D}^*/\simeq$. It is formed of all equivalence classes $[A]$, $A \in \mathscr{D}^*$ of the relation $\simeq$. The class $[A]$ is composed of all designations B such that $A \simeq B$. We recall that the algebraic operations in $\mathscr{D}^*/\simeq$ are defined as follows:

$$\begin{gathered} [A]\cup[B] = [A\cup B], \qquad [A]\cap[B] = [A\cap B], \\ [A]\rightarrow[B] = [A\rightarrow B], \qquad [A]o_k[B] = [Ao_kB], \qquad k = 1, \ldots, r, \\ o^k[A] = [o^kA], \qquad k = 1, \ldots, s. \end{gathered} \tag{1}$$

For any $[A]$, $[B]$, let us set

$$[A] \leqslant [B] \quad \text{if and only if} \quad (A \subset B) \text{ is in } \mathscr{T}^*.$$

It is easy to show that if $(A \subset B)$ is in $\mathscr{T}^*$, $A \simeq A_1$ and $B \simeq B_1$, then $(A_1 \subset B_1)$ is also in $\mathscr{T}^*$. Consequently, the definition adopted above is correct.

4.2. $[A] \leqslant [B]$ *if and only if* $[A]\rightarrow[B] = [\mathsf{V}]$.

This is an immediate consequence of (T_{15}^*).

4.3. $[A] \leqslant [B]$ *if and only if* $[A]\cap[B] = [A]$.

This follows directly from (T_{14}^*).

4.4. *The algebra* $\langle D^*/\simeq, [\mathsf{V}], \cup, \cap, \rightarrow, o_1, \ldots, o_r, o^1, \ldots, o^s\rangle$ *is an* $\mathscr{S}$*-algebra with the set of generators* $[F_k]$, $k = 1, 2, \ldots$

By (T_7^*)-(T_{12}^*), it is a lattice. It follows from 4.3 that the relation $\leqslant$ is the lattice partial ordering relation. Thus, by (T_{16}^*) the class $[\mathsf{V}]$ is the unit of this lattice. By (T_{13}^*) the algebra $\langle \mathscr{D}^*/\simeq, [\mathsf{V}], \cup, \cap, \rightarrow\rangle$ is a relatively pseudocomplemented lattice with the unit $[\mathsf{V}]$.

It follows from 4.2 and the fact that the relation $\leqslant$ is the partial ordering in the lattice in question that condition (i) for $\mathscr{S}$-algebras is fulfilled. On account of 4.2, since $[\mathsf{V}]$ is the unit of the lattice $\mathscr{D}^*/\simeq$, we infer that also condition (ii) for $\mathscr{S}$-algebras is satisfied. To prove that condition (iii) for $\mathscr{S}$-algebras is fulfilled, suppose that a formula $\alpha(a_{i_1}, \ldots, a_{i_n})$ of the propositional calculus $\mathscr{S}$ is in $\mathscr{T}$. Hence, by 2.2, $D_\alpha(F_{i_1}, \ldots, F_{i_n}) = \mathsf{V}$ is in $\mathscr{T}^*$. Let v be an arbitrary valuation of the propositional variables of $\mathscr{S}$ in the algebra $\mathscr{D}^*/\simeq$. Suppose that $v(a_{i_1}) = [A_1], \ldots, v(a_{i_n}) = [A_n]$, where $A_1, \ldots, A_n$ are in $\mathscr{D}^*$. By (t_4^*) the formula $D_\alpha(A_1, \ldots, A_n) = \mathsf{V}$ is in $\mathscr{T}^*$. Consequently $[D_\alpha(A_1, \ldots, A_n)] = [\mathsf{V}]$. By (1) and the definition of $\alpha_{D^*/\simeq}(v)$ we get $\alpha_{D^*/\simeq}(v) = [\mathsf{V}]$, which completes the proof.

We shall prove in § 5 that the algebra $\langle \mathscr{D}^*/\simeq, [\mathsf{V}], \cup, \cap, \rightarrow, o_1, \ldots, o_r, o^1, \ldots, o^s\rangle$ is a free $\mathscr{S}$-algebra.

§ 5. Models of the system $\mathscr{S}^*$

We shall assume in the sequel that the system $\mathscr{S}$ of a propositional calculus described in § 1 is axiomatizable and that it has the property (E) defined in § 3. Let I be a non-empty set and let

$$\langle \mathscr{A};\ \vee,\ \cup,\ \cap,\ \Rightarrow,\ o_1, \ldots, o_r, o^1, \ldots, o^s \rangle$$

be an $\mathscr{S}^*$-algebra. The set $\mathscr{A}$ can be conceived as the set of logical values. Consider the product of algebras $\mathscr{A}^I$. Since, by 3.2, the class of $\mathscr{S}$-algebras is equationally definable, $\mathscr{A}^I$ is an $\mathscr{S}$-algebra. Let us fix a subalgebra $\mathscr{B}$ of $\mathscr{A}^I$. It is also an $\mathscr{S}$-algebra. The elements of $\mathscr{B}$ are functions $f: I \to \mathscr{A}$, i.e. functions associating with every element j of I an element $f(j)$ of the algebra $\mathscr{A}$. These functions can be treated as a generalization of characteristic functions of sets and will be called *$\mathscr{A}$-characteristic functions on I*. In particular, if $\mathscr{S}$ is the classical propositional calculus and the $\mathscr{S}^*$-algebra $\mathscr{A}$ is the two-element Boolean algebra, then $\mathscr{A}$-characteristic functions on I are characteristic functions of subsets of I, in the usual sense.

Every $\mathscr{A}$-characteristic function f on I determines an *$\mathscr{A}$-subset I_f* of I in the following sense: for every j in I the sentence "j belongs to I_f" has the logical value $f(j) \in \mathscr{A}$. In particular, if $f(j) = \vee$, then this sentence is true. Of course, $\mathscr{A}$-subsets of I, where $\mathscr{A}$ is the two-element Boolean algebra, are subsets of I defined by the characteristic functions $f: I \to \mathscr{A}$, and a subalgebra $\mathscr{B}$ of $\mathscr{A}^I$ is then isomorphic with a field of subsets of I. The mapping h defined by the equation $h(f) = I_f$ is the required isomorphism.

It seems therefore that subalgebras of $\mathscr{A}^I$ can be treated as a generalization of fields of sets and will be called *$\mathscr{A}$-fields*.

We are going to describe interpretations of the system $\mathscr{S}^*$ in a set I, in an $\mathscr{S}^*$-algebra $\mathscr{A}$ and in a subalgebra $\mathscr{B}$ of $\mathscr{A}^I$. Every function V from the set F of all free set variables into $\mathscr{B}$ will be called a *$\mathscr{B}$-valuation* (clearly, $V(F_k) \in \mathscr{B}$). It can be extended to the set $\mathscr{D}^*$ of all set designations as follows. Let us set

(1) $V(\vee) = \vee_B$ where $\vee_B$ is the unit element in $\mathscr{B}$;

(2) $V(\wedge) = \wedge_B$ where $\wedge_B$ is the zero element in $\mathscr{B}$, in the case where $\mathscr{S}$ is a propositional calculus with negation;

(3) for any A, B in $\mathscr{D}^*$

$$\begin{aligned} V(A \cup B) &= V(A) \cup V(B), \\ V(A \cap B) &= V(A) \cap V(B), \\ V(A \to B) &= V(A) \Rightarrow V(B), \\ V(A \boldsymbol{o}_k B) &= V(A) o_k V(B), \qquad k = 1, \ldots, r, \\ V(\boldsymbol{o}^k A) &= o^k V(A), \qquad\quad\ \ k = 1, \ldots, s. \end{aligned}$$

Thus, for every set designation A, $V(A)$ is an element of $\mathscr{B}$ and consequently of $\mathscr{A}^I$, i.e. an $\mathscr{A}$-characteristic function on I. Every function v from the set X of all free individual variables into I is said to be an *I-valuation*. Thus $v \in I^X$.

Let $\mathfrak{M}$ be a mapping which associates with the primitive signs $=$, $\subset$, $=$ in $\mathscr{S}^*$ a two-argument function $=_{\mathfrak{M}}$ from I into $\mathscr{A}$, a two-argument function $\subset_{\mathfrak{M}}$ from $\mathscr{B}$ into $\mathscr{A}$ and a two-argument function $=_{\mathfrak{M}}$ from $\mathscr{B}$ into $\mathscr{A}$, respectively. Then with every formula α in $\mathscr{S}^*$ we can associate a functional $\alpha_{\mathfrak{M}}$ on $\mathscr{B}^F \times I^X$, where F is the set of all free set variables and X is the set of all free individual variables, with values in $\mathscr{A}$. This functional is defined by induction on the length of α as follows.

(4)
$$\begin{aligned}
&(x_k = x_l)_{\mathfrak{M}}(V, v) = \big(v(x_k) =_{\mathfrak{M}} v(x_l)\big), \quad k, l = 1, 2, \ldots, \\
&(A \subset B)_{\mathfrak{M}}(V, v) = \big(V(A) \subset_{\mathfrak{M}} V(B)\big) \text{ for any } A, B \text{ in } \mathscr{D}^*, \\
&(A = B)_{\mathfrak{M}}(V, v) = \big(V(A) =_{\mathfrak{M}} V(B)\big) \text{ for any } A, B \text{ in } \mathscr{D}^*, \\
&\big(A(x_k)\big)_{\mathfrak{M}}(V, v) = V(A)\big(v(x_k)\big) \text{ for any } A \text{ in } \mathscr{D}^* \text{ and } k = 1, 2, \ldots, \\
&(\alpha o \beta)_{\mathfrak{M}}(V, v) = \alpha_{\mathfrak{M}}(V, v) o \beta_{\mathfrak{M}}(V, v)
\end{aligned}$$

where o on the left side of this equation is any binary propositional operators in $\mathscr{S}^*$ and o on the right side is the corresponding operation in the algebra $\mathscr{A}$.

(5)
$$(o\alpha)_{\mathfrak{M}}(V, v) = o\big(\alpha_{\mathfrak{M}}(V, v)\big)$$

where o on the left side of this equation is any unary propositional operator in $\mathscr{S}^*$ and on the right side is the corresponding operation in the algebra $\mathscr{A}$.

(6)
$$\begin{aligned}
&\big(\textstyle\bigcup_{\xi_k} \alpha(\xi_k)\big)_{\mathfrak{M}}(V, v) = \bigcup_{j \in I} \alpha_{\mathfrak{M}}(j)(V, v), \\
&\big(\textstyle\bigcap_{\xi_k} \alpha(\xi_k)\big)_{\mathfrak{M}}(V, v) = \bigcap_{j \in I} \alpha_{\mathfrak{M}}(j)(V, v), \\
&\big(\textstyle\bigcup_{\varphi_k} \alpha(\varphi_k)\big)_{\mathfrak{M}}(V, v) = \bigcup_{f \in \mathscr{B}} \alpha_{\mathfrak{M}}(f)(V, v), \\
&\big(\textstyle\bigcap_{\varphi_k} \alpha(\varphi_k)\big)_{\mathfrak{M}}(V, v) = \bigcap_{f \in \mathscr{B}} \alpha_{\mathfrak{M}}(f)(V, v), \quad k = 1, 2, \ldots
\end{aligned}$$

A mapping $\mathfrak{M}$ described above will be said to be a *model of $\mathscr{S}^*$* provided for every axiom α of $\mathscr{S}^*$ the condition

(7)
$$\alpha_{\mathfrak{M}}(V, v) = \vee$$

is satisfied for every $\mathscr{B}$-valuation V and for every I-valuation v.

The following theorems are easy to prove:

5.1. *$\mathfrak{M}$ is a model of $\mathscr{S}^*$ if and only if condition* (7) *is satisfied for the axioms* (c_8), (c_9), (c_{10}), (c_{11}), (c_{12}).

5.2. *If $\mathfrak{M}$ is a model of $\mathcal{S}^*$ and a formula α is in $\mathcal{T}^*$, then condition (7) is satisfied.*

5.3. *If $\mathfrak{M}$ is a model of $\mathcal{S}^*$, then for any $f, g \in \mathcal{B}$*

(i) $(f \subset_{\mathfrak{M}} g) = \vee$ *if and only if* $f \leqslant g$,

(ii) $(f =_{\mathfrak{M}} g) = \vee$ *if and only if* $f = g$.

5.4. *If $\mathfrak{M}$ is a model of $\mathcal{S}^*$ in a set I in an $\mathcal{S}^*$-algebra $\mathcal{A}$ and in $\mathcal{A}^I$, and $\mathcal{S}$ is a propositional calculus with negation, then $=_{\mathfrak{M}}$ is the characteristic function of the relation of identity.*

The equality $(j =_{\mathfrak{M}} j) = \vee$ follows immediately from the axiom (c_{10}). To show that $(j =_{\mathfrak{M}} i) = \wedge$ for any i, j in I, $i \neq j$, suppose $(j_0 =_{\mathfrak{M}} i_0) = a \in \mathcal{A}$ for some $i_0, j_0 \in I$, where $i_0 \neq j_0$ and $a \neq \wedge$. Let f be a function from I into $\mathcal{A}$ such that $f(j_0) = \vee$ and $f(i_0) = \wedge$. Of course, $f \in \mathcal{A}^I$. Let V be an $\mathcal{A}^I$-valuation such that $V(F_1) = f$ and let α be the axiom (c_{12}). Then

$$\alpha_{\mathfrak{M}}(V, v) = \bigcap_{i \in I} \bigcap_{j \in I} \big(f(i) \cap (i =_{\mathfrak{M}} j) \Rightarrow f(j)\big).$$

Since $f(j_0) \cap (j_0 =_{\mathfrak{M}} i_0) \Rightarrow f(i_0) = \vee \cap a \Rightarrow \wedge = a \Rightarrow \wedge \neq \vee$, $\alpha_{\mathfrak{M}}(V, v) \neq \vee$.

Using the notation of 2.1 and 2.2 we can establish the following theorem.

5.5. *For any formula $\alpha(a_1, \ldots, a_n)$ of $\mathcal{S}$, if the formula $D_\alpha(F_1, \ldots, F_n) = \vee$ is in $\mathcal{T}^*$, then $\alpha(a_1, \ldots, a_n)$ is a theorem in $\mathcal{S}$.*

It follows from the hypothesis and from the axioms (c_8), (c_9) that the formula $\bigcap_{\xi_1} \big(D_\alpha(F_1, \ldots, F_n)(\xi_1) \equiv \vee(\xi_1)\big)$ is in $\mathcal{T}^*$. In consequence, $\vee(x_k) \Rightarrow D_\alpha(F_1, \ldots, F_n)(x_k)$ is also in $\mathcal{T}^*$. By (c_1) and 2.1 the formula $\alpha\big(F_1(x_k), \ldots, F_n(x_k)\big)$ is in $\mathcal{T}^*$. Let us suppose that $\alpha(a_1, \ldots, a_n)$ is not any theorem of $\mathcal{S}$. Then ([5]) there exists an $\mathcal{S}$-algebra $\mathcal{A}_0$ and a valuation v of propositional variables such that $\alpha(a_1, \ldots, a_n)_{\mathcal{A}_0}(v) = b$, where $b \in \mathcal{A}_0$, $b \neq \vee$. Let $\mathcal{A}$ be a complete $\mathcal{S}$-algebra which is an extension of $\mathcal{A}_0$ and let I be an arbitrary non-empty set. For every element a in $\mathcal{A}$, let us set $f_a(j) = a$ for every j in I. Thus $f_a \in \mathcal{A}^I$. Let us set for any $f, g \in \mathcal{A}^I$

$$(f \subset_{\mathfrak{M}} g) = \begin{cases} \vee & \text{if } f \leqslant g, \\ \wedge & \text{in the opposite case,} \end{cases} \qquad (f =_{\mathfrak{M}} g) = \begin{cases} \vee & \text{if } f = g, \\ \wedge & \text{if } f \neq g. \end{cases}$$

Let $=_{\mathfrak{M}}$ be the characteristic function of the relation of identity. Then $\mathfrak{M}$ is a model of $\mathcal{S}^*$ in I, $\mathcal{A}$ and $\mathcal{A}^I$. Let V be an $\mathcal{A}^I$-valuation such that $V(F_1) = f_{v(a_1)}, \ldots, V(F_n) = f_{v(a_n)}$. Then we have

$$\alpha\big(F_1(x_k), \ldots, F_n(x_k)\big)_{\mathfrak{M}}(V, v') = b \qquad \text{for any } I\text{-valuation } v'.$$

Hence, by 5.2, $\alpha\big(F_1(x_k), \ldots, F_n(x_k)\big)$ is not any theorem in $\mathcal{S}^*$.

([5]) For instance, the Lindenbaum algebra for the propositional calculus $\mathcal{S}$ satisfies this condition.

COROLLARY. *If $\mathscr{S}$ is decidable, then the set of all formulas in $\mathscr{S}^*$ of the form $A \subset B$, $A = B$, where $A, B \in \mathscr{D}^*$ is decidable.*

This follows from 5.5, 2.2 and (T^*_{15}) § 2.

Given an arbitrary term $A(F_{k_1}, \ldots, F_{k_n})$ in $\mathscr{D}^*$ let us replace the set variables $F_{k_1}, \ldots, F_{k_n}$ by propositional variables $a_{k_1}, \ldots, a_{k_n}$, respectively, $\vee$ — by $(a_1 \Rightarrow a_1)$, $\wedge$ by $-(a_1 \Rightarrow a_1)$ and $\cup$, $\cap$, $\rightarrow$, $o_1, \ldots, o_r, o^1, \ldots, o^s$ by $\cup, \cap, \Rightarrow, o_1, \ldots, o_r, o^1, \ldots, o^s$, respectively. Then we obtain from $A(F_{k_1}, \ldots, F_{k_n})$ a formula a^A of the propositional calculus $\mathscr{S}$.

It is easy to show by inductive argument making use of (T^*_{18}), (T^*_{19}) and (t^*_9), (t^*_{10}) § 2, that

5.6. *For any term A, the formula $A = D_{a^A}$ is a theorem of $\mathscr{S}^*$.*

We shall prove the following statement. Let $\mathscr{B}$ be an arbitrary $\mathscr{S}$-algebra and let $\mathscr{A}$ be an $\mathscr{S}^*$-algebra which is an extension of $\mathscr{B}$. Then $\mathscr{B}$ can be considered as a subalgebra of $\mathscr{A}^I$, i.e. as a subalgebra formed of all functions $f_a\colon I \to \mathscr{A}$, $a \in \mathscr{B}$, where $f_a(i) = a$ for each $i \in I$.

5.7. *For any terms A, B in $\mathscr{D}^*$, if $A = B$ is a theorem of $\mathscr{S}^*$, then for every valuation V in every $\mathscr{S}$-algebra $\mathscr{B}$*

$$V(A) = V(B).$$

Suppose that $A = B$ is a theorem of $\mathscr{S}^*$. Then, by (c_9) the formulas $A \subset B$ and $B \subset A$ are theorems of $\mathscr{S}^*$, and consequently by (T^*_{15}) the formulas $A \rightarrow B = \vee$ and $B \rightarrow A = \vee$ are also theorems of $\mathscr{S}^*$. By 5.6 the formulas $D_{a^A} \rightarrow D_{a^B} = \vee$ and $D_{a^B} \rightarrow D_{a^A} = \vee$ are theorems of $\mathscr{S}^*$. By 5.5 the formulas $a^A \Rightarrow a^B$ and $a^B \Rightarrow a^A$ are theorems of the propositional calculus $\mathscr{S}$. Hence, for every valuation v of propositional variables of $\mathscr{S}$ in every $\mathscr{S}$-algebra $\mathscr{B}$,

$$a^A_{\mathscr{B}}(v) = a^B_{\mathscr{B}}(v). \tag{8}$$

Suppose that there exists a valuation V in an $\mathscr{S}$-algebra $\mathscr{B}$ such that

$$V(A) \neq V(B).$$

It is easy to see that for the valuation v defined as follows

$$v(a_k) = V(F_k), \qquad k = 1, 2, \ldots,$$

we have

$$a^A_{\mathscr{B}}(v) = V(A) \quad \text{and} \quad a^B_{\mathscr{B}}(v) = V(B),$$

which contradicts (8).

5.8. *The algebra $\langle \mathscr{D}^*/\simeq; [\vee], \cup, \cap, \rightarrow, o_1, \ldots, o_r, o^1, \ldots, o^s \rangle$ is a free $\mathscr{S}$-algebra with generators $[F_k]$, $k = 1, 2, \ldots$*

Let h be a mapping from the set of all $[F_k]$, $k = 1, 2, \ldots$, into an $\mathscr{S}$-algebra $\mathscr{B}$. It can be extended to the homomorphism of the whole algebra $\mathscr{D}^*/\simeq$ into $\mathscr{B}$ by means of the formulas

$$h([A]\cup[B]) = h([A])\cup h([B]),$$
$$h([A]\cap[B]) = h([A])\cap h([B]),$$
$$h([A]\to[B]) = h([A]) \Rightarrow h([B]),$$
$$h([A]\boldsymbol{o}_k[B]) = h([A])o_k h([B]), \qquad k = 1, \ldots, r,$$
$$h(\boldsymbol{o}_k[A]) = o_k h([A]), \qquad k = 1, \ldots, s,$$
$$h([\vee]) = \vee, \quad h([\wedge]) = \wedge.$$

This definition is correct, since by 5.7 the condition $[A] = [B]$ implies that $h([A]) = h([B])$.

§ 6. Completeness theorem

Suppose that $\mathscr{S}$ has the properties mentioned in § 5. Let us set for any formulas α, β of $\mathscr{S}^*$

(1) $\quad \alpha \sim \beta$ if and only if $(\alpha \Rightarrow \beta)\in\mathscr{T}^*$ and $(\beta \Rightarrow \alpha)\in\mathscr{T}^*$.

It is known (see [5]) that $\sim$ is a congruence relation in the set $\mathscr{F}$ of all formulas of $\mathscr{S}^*$ with respect to the logical operations $\cup$, $\cap$, $\Rightarrow$, $o_1, \ldots, o_r, o^1, \ldots, o^s$. The quotient algebra

$$\mathscr{A}^* = \langle \mathscr{F}/\sim;\ \vee,\ \cup, \cap, \Rightarrow o_1, \ldots, o_r, o^1, \ldots, o^s\rangle,$$

where $\vee = [\alpha\colon \alpha\in\mathscr{T}^*]$, is an $\mathscr{S}$-algebra. The elements of $\mathscr{A}^*$ will be denoted by $|\alpha|$, $\alpha\in\mathscr{F}$, i.e. $|\alpha| = [\beta\colon \alpha\sim\beta]$. It is known that $\{\mathscr{F}/\sim, \vee, \cup, \cap, \Rightarrow\}$ is a relatively pseudo-complemented lattice with the unit element $\vee$. Thus we have

(2) $\quad |\alpha| = \vee$ if and only if $\alpha\in\mathscr{T}^*$.

Moreover

(3) $\quad |\alpha| \leqslant |\beta|$ if and only if $(\alpha \Rightarrow \beta)\in\mathscr{T}^*$.

The following equations hold in $\mathscr{A}^*$, where X is the set of all free individual variables

(4) $\quad |\bigcap_{\xi_k} \alpha(x_m/\xi_k)| = \bigcap_{x\in X} |\alpha(x_m/x)|,$

(5) $\quad |\bigcup_{\xi_k} \alpha(x_m/\xi_k)| = \bigcup_{x\in X} |\alpha(x_m/x)|,$

(6) $\quad |\bigcap_{\varphi_k} \alpha(F_m/\varphi_k)| = \bigcap_{A\in\mathscr{D}^*} |\alpha(F_m/A)|,$

(7) $\quad |\bigcup_{\varphi_k} \alpha(F_m/\varphi_k)| = \bigcup_{A\in\mathscr{D}^*} |\alpha(F_m/A)|.$

Since $\mathscr{S}$ has the property (E), there exists an $\mathscr{S}$-isomorphism h of $\mathscr{A}^*$ into an $\mathscr{S}^*$-algebra $\mathscr{A}$ preserving all infinite joins and meets (4)-(7). Thus we have

$$(8)\qquad h|\bigcap_{\xi_k} a(x_m/\xi_k)| = \bigcap_{x\in X} h|a(x_m/x)|,$$

$$(9)\qquad h|\bigcup_{\xi_k} a(x_m/\xi_k)| = \bigcup_{x\in X} h|a(x_m/x)|,$$

$$(10)\qquad h|\bigcap_{\varphi_k} a(F_m/\varphi_k)| = \bigcap_{A\in\mathscr{D}^*} h|a(F_m/A)|,$$

$$(11)\qquad h|\bigcup_{\varphi_k} a(F_m/\varphi_k)| = \bigcup_{A\in\mathscr{D}^*} h|a(F_m/A)|.$$

We are going to define an interpretation of $\mathscr{S}^*$ which will be called a *canonical interpretation.*

Consider the product $\mathscr{A}^X$ where X is the set of all free individual variables. The elements of $\mathscr{A}^X$ are functions associating with every free individual variable x_k an element $h|a|$ of $\mathscr{A}$. Let $\mathscr{B}$ be the subset of $\mathscr{A}^X$ formed of all functions f_A, $A\in\mathscr{D}^*$, defined as follows

$$(12)\qquad f_A(x_k) = h|A(x_k)| \quad \text{for any} \quad x_k\in X.$$

It is easy to see that $\mathscr{B}$ is a subalgebra of $\mathscr{A}^X$.

We shall define the canonical interpretation $\mathfrak{M}$ in the set X, in the $\mathscr{S}^*$-algebra $\mathscr{A}$, and in the $\mathscr{A}$-field $\mathscr{B}$. Viz. let $=_{\mathfrak{M}}$, $\subset_{\mathfrak{M}}$, $\mathbf{=}_{\mathfrak{M}}$ be functions defined by means of the following equations

$$(13)\qquad (x_k =_{\mathfrak{M}} x_m) = h|x_k = x_m| \quad \text{for any} \quad x_k, x_m \text{ in } X,$$

$$(14)\qquad f_A \subset_{\mathfrak{M}} f_B = h|A \subset B| \quad \text{for any} \quad A, B \text{ in } \mathscr{D}^*,$$

$$(15)\qquad f_A \mathbf{=}_{\mathfrak{M}} f_B = h|A \mathbf{=} B| \quad \text{for any} \quad A, B \text{ in } \mathscr{D}^*.$$

Observe that any X-valuation v of free individual variables can be treated as a substitution for free individual variables. Moreover, every $\mathscr{B}$-valuation V of free set variables determines a substitution $\mathrm{sb}\,V$ for free set variables defined as follows:

$$(16)\qquad \text{if } V(F_k) = f_A, \text{ then we set } \mathrm{sb}\,V(F_k) = A,\ k = 1, 2, \ldots$$

It is easy to show that for any set designation $B\in\mathscr{D}^*$

$$(17)\qquad V(B) = f_{\mathrm{sb}\,V(B)}.$$

For any formula a, let $\mathrm{sb}\,Vva$ denote the formula obtained from a by performing the substitutions v and $\mathrm{sb}\,V$. Then

$$(18)\qquad a_{\mathfrak{M}}(V, v) = h|\mathrm{sb}\,Vva|.$$

In fact, if $A\in\mathscr{D}^*$ and a is the formula $A(x_k)$, then

$$a_{\mathfrak{M}}(V, v) = A(x_k)_{\mathfrak{M}}(V, v) = V(A)(v(x_k)) = f_{\mathrm{sb}\,V(A)}(v(x_k))$$
$$= h|\mathrm{sb}\,VvA(x_k)| = h|\mathrm{sb}\,Vva| \quad \text{by (17) and (12).}$$

If α is the formula $A \subset B$ for any $A, B \in \mathscr{D}^*$, then

$$\begin{aligned}\alpha_{\mathfrak{M}}(V, v) &= (A \subset B)_{\mathfrak{M}}(V, v) = V(A) \subset_{\mathfrak{M}} V(B) = f_{\mathrm{sb}V(A)} \subset_{\mathfrak{M}} f_{\mathrm{sb}V(B)} \\ &= h|\mathrm{sb}V(A) \subset \mathrm{sb}V(B)| = h|\mathrm{sb}V(A \subset B)| = h|\mathrm{sb}Vv(A \subset B)| \\ &= h|\mathrm{sb}Vv\alpha|, \quad \text{by (17) and (14).}\end{aligned}$$

The case where α is the formula $A = B$ can be proved analogously, by making use of (17) and (15).

If α is the formula $x_k = x_m$, then we have

$$\begin{aligned}\alpha_{\mathfrak{M}}(V, v) &= (x_k = x_m)_{\mathfrak{M}}(V, v) = \big(v(x_k) =_{\mathfrak{M}} v(x_m)\big) = h|v(x_k) = v(x_m)| \\ &= h|\mathrm{sb}Vv(x_k = x_m)| = h|\mathrm{sb}Vv\alpha|.\end{aligned}$$

Suppose α is a formula $\beta\cup\gamma$ and (18) holds for β and γ. Then

$$\begin{aligned}\alpha_{\mathfrak{M}}(V, v) &= (\beta\cup\gamma)_{\mathfrak{M}}(V, v) = \beta_{\mathfrak{M}}(V, v)\cup\gamma_{\mathfrak{M}}(V, v) \\ &= h|\mathrm{sb}Vv\beta|\cup h|\mathrm{sb}Vv\gamma| = h|\mathrm{sb}Vv\beta\cup\mathrm{sb}Vv\gamma| \\ &= h|\mathrm{sb}Vv(\beta\cup\gamma)| = h|\mathrm{sb}Vv\alpha|.\end{aligned}$$

The case where α is a formula $\beta\cap\gamma$, $\beta \Rightarrow \gamma$, $\beta o_j \gamma$ for $j = 1, \dots, r$, $o^j\beta$ for $j = 1, \dots, s$ can be proved analogously.

Suppose α is a formula $\bigcap_{\xi_k} \beta(\xi_k)$ and (18) holds for β. Then by (8) and (4)

$$\begin{aligned}\alpha_{\mathfrak{M}}(V, v) &= \textstyle\bigcap_{x \in X} \beta_{\mathfrak{M}}(x)(V, v) = \bigcap_{x \in X} h|\mathrm{sb}Vv\beta(x)| \\ &= h \textstyle\bigcap_{x\in X}|\mathrm{sb}Vv\beta(\xi_x)| = h|\bigcap_{\xi_k} \mathrm{sb}Vv\beta(\xi_k)| \\ &= h|\mathrm{sb}Vv\textstyle\bigcap_{\xi_k} \beta(\xi_k)| = h|\mathrm{sb}Vv\alpha|.\end{aligned}$$

The case where α is a formula $\bigcup_{\xi_k} \beta(\xi_k)$ can be stated analogously, by making use of (9) and (5).

Suppose α is a formula $\bigcap_{\varphi_k}\beta(\varphi_k)$ and (18) holds for β. Then we have by (10) and (6)

$$\begin{aligned}\alpha_{\mathfrak{M}}(V, v) &= \textstyle\bigcap_{f \in \mathscr{B}}\beta_{\mathfrak{M}}(f)(V, v) = \bigcap_{A \in \mathscr{D}^*}\beta_{\mathfrak{M}}(f_A)(V, v) \\ &= \textstyle\bigcap_{A \in \mathscr{D}^*}h|\mathrm{sb}Vv\beta(A)| = h\bigcap_{A \in \mathscr{D}^*}|\mathrm{sb}Vv\beta(A)| \\ &= h|\textstyle\bigcap_{\varphi_k}\mathrm{sb}Vv\beta(\varphi_k)| = h|\mathrm{sb}Vv\bigcap_{\varphi_k}\beta(\varphi_k)| = h|\mathrm{sb}Vv\alpha|.\end{aligned}$$

The case where α is a formula $\bigcup_{\varphi_k}\beta(\varphi_k)$ can be stated analogously.

It follows from equation (18) that $\mathfrak{M}$ is a model. In fact, if α is an axiom, then $\mathrm{sb}Vv\alpha$ is a theorem of $\mathscr{S}^*$. Hence, by (2), for every X-valuation v and for every $\mathscr{B}$-valuation V

$$\alpha_{\mathfrak{M}}(V, v) = h|\mathrm{sb}Vv\alpha| = h\vee = \vee.$$

On the other hand, if a is not any theorem, then $h|a| \neq \vee$. But for the valuations V_0 and v_0 defined as follows

$$V_0(F_k) = f_{F_k}, \quad v_0(x_k) = x_k, \quad k = 1, 2, \ldots,$$

$\mathrm{sb}Vva = a$. In consequence $a_{\mathfrak{M}}(V_0, v_0) = h|a| \neq \vee$.

Thus we have proved the following theorem.

6.1. *$\mathfrak{M}$ is a functionally free model for $\mathcal{S}^*$, i.e. for any formula a of $\mathcal{S}^*$, a is a theorem of $\mathcal{S}^*$ if and only if $a_{\mathfrak{M}}(V, v) = \vee$ for every X-valuation v and for every $\mathcal{B}$-valuation V.*

It follows from 5.2 and 6.1 that

6.2. *For any formula a of $\mathcal{S}^*$ the following conditions are equivalent:*

(i) *a is a theorem of $\mathcal{S}^*$,*

(ii) *a is valid in every model of $\mathcal{S}^*$,*

(iii) *a is valid in the model $\mathfrak{M}$ from 6.1.*

(iv) *a is valid in every model in an enumerable set I, in every $\mathcal{S}^*$-algebra $\mathcal{A}$ and in every subalgebra $\mathcal{B}$ of $\mathcal{A}^X$.*

§ 7. Formalized theory of fields of sets

In this section let $\mathcal{S}^*$ be the classical propositional calculus. Then the algebra $\mathcal{A}^*$ described in § 6 is a Boolean algebra. Suppose that a formula β is not any theorem of $\mathcal{S}^*$. Hence, by (2) § 6,

(1) $$|\beta| \neq \vee.$$

Since $\mathcal{A}^*$ is enumerable, it is known (see [4]) that there exists a prime filter ∇ in $\mathcal{A}^*$ which satisfies the following conditions:

(2) $$|\beta| \notin \nabla,$$

(3) if $\bigcup_{x \in X} |a(x)| = |\bigcup_{\xi_k} a(\xi_k)| \in \nabla$, then there exists $x_m \in X$, such that $|a(x_m)| \in \nabla$,

(4) if $\bigcup_{A \in \mathcal{D}^*} |a(A)| = |\bigcup_{\varphi_k} a(\varphi_k)| \in \nabla$, then there exists a set designation B such that $|a(B)| \in \nabla$.

Let h be the homomorphism of $\mathcal{A}^*$ into the two-element Boolean algebra $\mathcal{A}$ determined by ∇. Since conditions (3), (4) are satisfied, the homomorphism h preserves all infinite joins and meets (4)-(7) § 6, i.e. the equalities (8)-(11) hold. Moreover, by (1)

(5) $$h\,|\beta| = \wedge.$$

For any free individual variables x_k, x_m, let us set

$$x_k \sim x_m \quad \text{if and only if} \quad |x_k = x_m| \in \nabla.$$

By (c_{10}), (c_{11}) § 2, (2), (3) § 6 this relation is an equivalence relation in X. Moreover,

(6) if $x_k \sim x_m$, then for any set designation A, $h|A(x_k)| = h|A(x_m)|$.

In fact, by (c_{12}) § 2 and (3) § 6

$$|A(x_k) \Rightarrow A(x_m)| \in \nabla \quad \text{and} \quad |A(x_m) \Rightarrow A(x_k)| \in \nabla.$$

Hence,

$$h|A(x_k) \Rightarrow A(x_m)| = \vee \quad \text{and} \quad h|A(x_m) \Rightarrow A(x_k)| = \vee,$$

i.e.

$$h|A(x_k)| = h|A(x_m)|.$$

Let us set for any $x_k \in X$, $[x_k] = [x_m\colon x_k \sim x_m]$ and let I be the set of all cosets $[x_k]$, $k = 1, 2, \ldots$

For any set designations A, B, let us set

$$A \prec B \quad \text{if and only if} \quad |A \subset B| \in \nabla,$$

$$A \approx B \quad \text{if and only if} \quad |A = B| \in \nabla.$$

Observe that by (c_9) § 2, $A \approx B$ if and only if $A \prec B$ and $B \prec A$. Moreover, $\approx$ is the congruence relation in $\mathcal{D}^*$ with respect to the operations $\cup$, $\cap$, $\rightarrow$, $-$. In fact, $\approx$ is an equivalence relation by (T_1^*), (T_2^*), (T_3^*) § 2 and (2), (3) § 6. It is easy to show that the following formula $(F_1 = F_2) \Rightarrow (-F_1 = -F_2)$ is a theorem of $\mathcal{S}^*$. Consequently, by (2), (3) § 6 and (T_4^*), (T_5^*), (T_6^*) § 2, $\approx$ is a congruence relation.

By (c_8) § 2,

(7) if $A \prec B$, then $h|A(x_k)| \leqslant h|B(x_k)|$, $k = 1, 2, \ldots$

Consequently,

(8) if $A \approx B$, then $h|A(x_k)| = h|B(x_k)|$, $k = 1, 2, \ldots$

For every A in $\mathcal{D}^*$, let $\|A\|$ denote the equivalence class determined by A, i.e. $\|A\| = [B\colon A \approx B]$.

Consider the product $\mathcal{A}^I$. The elements of $\mathcal{A}^I$ are functions f associating with every $[x_k] \in I$ an element $h|a|$ of the two-element Boolean algebra $\mathcal{A}$.

Let us set for any $A \in \mathcal{D}^*$

(9) $$f_{\|A\|}[x_k] = h|A(x_k)|, \quad k = 1, 2, \ldots$$

This definition is correct, since by (6) and (8) equation (9) does not depend on the choice of representative elements in $[x_k]$ and $\|A\|$. The set $\mathcal{B}$ of all functions $f_{\|A\|}$, $A \in \mathcal{D}^*$ is a subalgebra of $\mathcal{A}^I$.

We shall define an interpretation $\mathfrak{M}$ of $\mathcal{S}^*$ in the set I, in the two-element Boolean algebra $\mathcal{A}$ and in the algebra $\mathcal{B}$, which is isomorphic with a field of subsets of I. Let us set

$$(10)\qquad ([x_k] =_{\mathfrak{M}} [x_m]) = h|x_k = x_m|, \qquad k, m = 1, 2, \ldots,$$

$$(11)\qquad (f_{\|A\|} \subset_{\mathfrak{M}} f_{\|B\|}) = h|A \subset B|, \qquad A, B \in \mathcal{D}^*,$$

$$(12)\qquad (f_{\|A\|} =_{\mathfrak{M}} f_{\|B\|}) = h|A = B|, \qquad A, B \in \mathcal{D}^*.$$

It is easy to verify that these definitions are correct, i.e. do not depend on the choice of representative elements in $[x_k]$, $[x_m]$, $[A]$, $[B]$.

Observe that

$$([x_k] =_{\mathfrak{M}} [x_m]) = \vee \quad \text{if and only if} \quad [x_k] = [x_m],$$
$$(f_{\|A\|} \subset_{\mathfrak{M}} f_{\|B\|}) = \vee \quad \text{if and only if} \quad f_{\|A\|} \leqslant f_{\|B\|},$$
$$(f_{\|A\|} =_{\mathfrak{M}} f_{\|B\|}) = \vee \quad \text{if and only if} \quad f_{\|A\|} = f_{\|B\|}.$$

Thus, $=_{\mathfrak{M}}$ is the characteristic function of the identity relation in I, $\subset_{\mathfrak{M}}$ is the characteristic function of the relation of inclusion in $\mathcal{B}$ and $=_{\mathfrak{M}}$ is the characteristic function of the relation of equality in $\mathcal{B}$.

With every I-valuation v we can associate a substitution $\mathrm{sb}\,v$ defined as follows:

if $v(x_k) = [x_m]$, and n is the least positive integer such that $x_n \sim x_m$, then we set $\mathrm{sb}\,v(x_k) = x_n$.

Let $A_1, A_2, \ldots$ be a sequence such that every set designation appears exactly once in this sequence and all A_n, $n = 1, 2, \ldots$, are set designations.

Then with every $\mathcal{B}$-valuation V we can associate a substitution $\mathrm{sb}\,V$ defined as follows:

if $V(F_k) = f_{\|A\|}$ and A_n is the first element in the sequence, fixed above, of set designations, such that $A_n \approx A$, then we put $\mathrm{sb}\,V(F_k) = A_n$.

For any formula α of $\mathcal{S}^*$, let $\mathrm{sb}\,Vv\alpha$ denote the formula obtained from α by performing the substitutions $\mathrm{sb}\,V$ and $\mathrm{sb}\,v$. It is easy to show by inductive argument on the length of α that

$$(13)\qquad \alpha_{\mathfrak{M}}(V, v) = h|\mathrm{sb}\,Vv\alpha|.$$

In the proof we make use of the following equation

$$V(B) = f_{\|\mathrm{sb}\,V(B)\|},$$

which holds for any $B \in \mathcal{D}^*$ and any $\mathcal{B}$-valuation V, and of the fact that the equations (8)-(11) hold in $\mathcal{A}$.

In consequence, if α is an axiom of $\mathcal{S}^*$, then $\mathrm{sb}\,Vv\alpha$ is a theorem and by (2) § 6, $h|\mathrm{sb}\,Vv\alpha| = \vee$, i.e. $\alpha_{\mathfrak{M}}(V, v) = \vee$. Thus, $\mathfrak{M}$ is a model of $\mathcal{S}^*$. On the other hand, for the valuations

$$V_0(F_k) = f_{\|F_k\|}, \quad k = 1, 2, \ldots,$$

$$v_0(x_k) = [x_k], \quad k = 1, 2, \ldots,$$

we have

$$\beta_{\mathfrak{M}}(V_0, v_0) = h\,|\beta| = \wedge .$$

We have proved the following theorem.

7.1. *A formula β is a theorem of $\mathcal{S}^*$ if and only if β is valid in the semantic model in an enumerable set I, and in every field $\mathcal{B}$ of subsets of I, i.e. when the set variables run over $\mathcal{B}$ and the quantifiers binding set variables are restricted to $\mathcal{B}$.*

Thus, in the case where $\mathcal{S}$ is a classical propositional calculus, $\mathcal{S}^*$ can be considered to be a formalized theory of fields of sets.

§ 8. Classical elementary theory of Boolean algebras

With the classical propositional calculus we shall associate the following system $\mathcal{S}'$ of the classical elementary theory of Boolean algebras.

The primitive signs of $\mathcal{S}'$ are the parentheses and

(a) the free individual variables denoted by $F_1, F_2, \ldots$,

(b) the bound individual variables denoted by $\varphi_1, \varphi_2, \ldots$,

(c) the symbol $\vee$ for the unit element,

(d) the symbol $\wedge$ for the zero element,

(e) the symbols $\cup$, $\cap$, $\rightarrow$, $-$ for Boolean operations,

(f) the symbol $\subset$ for the Boolean inclusion,

(g) the symbol $=$ for the relation of equality,

(h) the propositional connectives $\cup$, $\cap$, $\Rightarrow$, $-$,

(i) quantifiers $\bigcup_{\varphi_k}$, $\bigcap_{\varphi_k}$, $k = 1, 2, 3, \ldots$

From these signs we form expressions of two kinds: terms and formulas.

The set $\mathcal{D}$ of all terms of $\mathcal{S}'$ coincides with the set $\mathcal{D}^*$ of all set designations of the system $\mathcal{S}^*$ of the classical calculus of classes, considered in § 7.

The set $\mathcal{F}'$ of all formulas is the least set such that

1) if $A, B \in \mathcal{D}$, then $A \subset B$ and $A = B$ are in $\mathcal{F}'$,

2) if $\alpha, \beta \in \mathcal{F}'$, then $\alpha \cup \beta$, $\alpha \cap \beta$, $\alpha \Rightarrow \beta$, $-\alpha$ are in $\mathcal{F}'$,

3) if $\alpha(F_k)$ is in $\mathcal{F}'$ and neither $\bigcup_{\varphi_m}$ nor $\bigcap_{\varphi_m}$ appears in α, then $\bigcup_{\varphi_m}\alpha(F_k/\varphi_m)$ and $\bigcap_{\varphi_m}\alpha(F_k/\varphi_m)$ are in $\mathcal{F}'$.

Thus the set $\mathscr{F}'$ is contained in the set $\mathscr{F}^*$ of all formulas of the system $\mathscr{S}^*$ of the classical calculus of classes.

The axioms of $\mathscr{S}'$ are all formulas obtained from theorems of the classical propositional calculus $\mathscr{S}$ by the substitution for propositional variables of arbitrary formulas in $\mathscr{F}'$ and, moreover, the following formulas, which are axioms for Boolean algebras:

(T$'_1$) $F_1 = F_1$,

(T$'_2$) $(F_1 = F_2) \Rightarrow \big((F_3 = F_2) \Rightarrow (F_1 = F_3)\big)$,

(T$'_3$) $(F_1 = F_2) \Rightarrow (F_1 \cup F_3 = F_2 \cup F_3)$,

(T$'_4$) $(F_1 = F_2) \Rightarrow (F_1 \cap F_3 = F_2 \cap F_3)$,

(T$'_5$) $(F_1 = F_2) \Rightarrow (-F_1 = -F_2)$,

(T$'_6$) $F_1 \rightarrow F_2 = -F_1 \cup F_2$,

(T$'_7$) $F_1 \cup F_2 = F_2 \cup F_1, \quad F_1 \cap F_2 = F_2 \cap F_1$,

(T$'_8$) $F_1 \cup (F_2 \cup F_3) = (F_1 \cup F_2) \cup F_3, \quad F_1 \cap (F_2 \cap F_3) = (F_1 \cap F_2) \cap F_3$,

(T$'_9$) $(F_1 \cap F_2) \cup F_2 = F_2, \quad F_1 \cap (F_1 \cup F_2) = F_1$,

(T$'_{10}$) $F_1 \cap (F_2 \cup F_3) = (F_1 \cap F_2) \cup (F_1 \cap F_3)$,

(T$'_{11}$) $(F_1 \cap -F_1) \cup F_2 = F_2, \quad (F_1 \cup -F_1) \cap F_2 = F_2$,

(T$'_{12}$) $\vee = F_1 \cup -F_1, \quad \wedge = F_1 \cap -F_1$,

(T$'_{13}$) $\big((F_1 \subset F_2) \Rightarrow (F_1 \cap F_2 = F_1)\big) \cap \big((F_1 \cap F_2 = F_1) \Rightarrow (F_1 \subset F_2)\big)$.

The set $\mathscr{T}'$ of all theorems of $\mathscr{S}'$ is the least set of formulas containing all axioms and closed with respect to the rules of inference: *modus ponens*, the rule of substitution for free individual variables, the rules of elimination and of the introduction of existential and universal quantifiers.

It is easy to show that

8.1. *If a is a formula in $\mathscr{F}'$ and a is a theorem of $\mathscr{S}'$, then a is also a theorem of $\mathscr{S}^*$.*

In fact, all axioms of $\mathscr{S}'$ are axioms or theorems of $\mathscr{S}^*$, and all rules of inference in $\mathscr{S}'$ are also admitted in $\mathscr{S}^*$ (the rule of substitution for free individual variable in $\mathscr{S}'$ corresponds to the rule of substitution for free set variables in $\mathscr{S}^*$, the rules for quantifiers in $\mathscr{S}'$ correspond to the suitable rules for quantifiers binding set variables in $\mathscr{S}^*$).

We shall show that also the converse theorem holds.

8.2. *If a is a formula of $\mathscr{S}'$ and a is a theorem of $\mathscr{S}^*$, then a is also a theorem of $\mathscr{S}'$.*

If follows from the Gödel theorem that if a is not any theorem of $\mathscr{S}'$, then there exists a model of $\mathscr{S}'$ in which a is not valid.

More exactly, there exists a Boolean algebra $\mathscr{B}'$ for which the following conditions are satisfied.

Let $\subset_{\mathfrak{M}'}$ be the characteristic function of the relation of inclusion in $\mathscr{B}'$, i.e. for any a, b in $\mathscr{B}'$

$$(1) \qquad (a \subset_{\mathfrak{M}'} b) = \begin{cases} \vee & \text{if} \quad a \leqslant b, \\ \wedge & \text{if} \quad \text{non } a \leqslant b. \end{cases}$$

Let $=_{\mathfrak{M}'}$ be the characteristic function of the equality relation in $\mathscr{B}'$, i.e. for any a, b in $\mathscr{B}'$

$$(2) \qquad (a =_{\mathfrak{M}'} b) = \begin{cases} \vee & \text{if} \quad a = b, \\ \wedge & \text{if} \quad a \neq b. \end{cases}$$

Every valuation V in $\mathscr{B}'$, i.e. every function V on the set F of all free individual variables with values in $\mathscr{B}'$, can be extended onto the set $\mathscr{D}$ of all terms of $\mathscr{S}'$ (see § 5 (1), (2), (3)).

With every formula β in $\mathscr{F}'$ we shall associate a functional $\beta_{\mathfrak{M}'}$ on $\mathscr{B}'^F$ with values in the two-element Boolean algebra defined by induction as follows:

$$(3) \qquad \begin{aligned}
(A \subset B)_{\mathfrak{M}'}(V) &= V(A) \subset_{\mathfrak{M}'} V(B) && \text{for any } A, B \text{ in } \mathscr{D}, \\
(A = B)_{\mathfrak{M}'}(V) &= V(A) =_{\mathfrak{M}'} V(B) && \text{for any } A, B \text{ in } \mathscr{D}, \\
(\gamma \cup \delta)_{\mathfrak{M}'}(V) &= \gamma_{\mathfrak{M}'}(V) \cup \delta_{\mathfrak{M}'}(V), && \text{for any } \gamma, \delta \text{ in } \mathscr{F}', \\
(\gamma \cap \delta)_{\mathfrak{M}'}(V) &= \gamma_{\mathfrak{M}'}(V) \cup \delta_{\mathfrak{M}'}(V) && \text{for any } \gamma, \delta \text{ in } \mathscr{F}', \\
(\gamma \Rightarrow \delta)_{\mathfrak{M}'}(V) &= \gamma_{\mathfrak{M}'}(V) \Rightarrow \delta_{\mathfrak{M}'}(V) && \text{for any } \gamma, \delta \text{ in } \mathscr{F}', \\
(-\gamma)_{\mathfrak{M}'}(V) &= -\big(\gamma_{\mathfrak{M}'}(V)\big) && \text{for any } \gamma \text{ in } \mathscr{F}', \\
\big(\textstyle\bigcup_{\varphi_k} \gamma(\varphi_k)\big)_{\mathfrak{M}'}(V) &= \textstyle\bigcup_{a \in \mathscr{B}'} \gamma_{\mathfrak{M}'}(a)(V) && \text{for any } \gamma \text{ in } \mathscr{F}', \\
\big(\textstyle\bigcap_{\varphi_k} \gamma(\varphi_k)\big)_{\mathfrak{M}'}(V) &= \textstyle\bigcap_{a \in \mathscr{B}'} \gamma_{\mathfrak{M}'}(a)(V) && \text{for any } \gamma \text{ in } \mathscr{F}'.
\end{aligned}$$

The fact that $\mathfrak{M}'$ is a model in $\mathscr{B}'$ for $\mathscr{S}'$ means that for every valuation V in $\mathscr{B}'$ and for every axiom β of $\mathscr{S}'$, $\beta_{\mathfrak{M}'}(V) = \vee$, i.e. β is valid in $\mathfrak{M}'$. On the other hand, since a is not valid in $\mathfrak{M}'$, there exists a valuation V' such that

$$(4) \qquad a_{\mathfrak{M}'}(V') = \wedge .$$

By the Stone representation theorem for Boolean algebras, there exists an isomorphism h of $\mathscr{B}'$ onto a subalgebra $\mathscr{B}$ of a product $\mathscr{A}^I$ of two-element Boolean algebras $\mathscr{A}$.

Let us set

$$(5) \qquad (i =_{\mathfrak{M}} j) = \begin{cases} \vee & \text{if} \quad i = j \\ \wedge & \text{if} \quad i \neq j \end{cases} \quad \text{for any } i, j \text{ in } I;$$

$$(6) \qquad \big(h(a) \subset_{\mathfrak{M}} h(b)\big) = \begin{cases} \vee & \text{if} \quad h(a) \leqslant h(b), \text{ i.e. if } a \leqslant b, \\ \wedge & \text{if non } h(a) \leqslant h(b), \text{ i.e. if non } a \leqslant b \end{cases}$$

for any elements $h(a)$, $h(b)$ of $\mathscr{B}$;

$$(7) \qquad \big(h(a) =_{\mathfrak{M}} h(b)\big) = \begin{cases} \vee & \text{if} \quad h(a) = h(b), \text{ i.e. if } a = b, \\ \wedge & \text{if} \quad h(a) \neq h(b), \text{ i.e. if } a \neq b, \end{cases}$$

for any elements $h(a)$, $h(b)$ of $\mathscr{B}$.

Observe that $\mathfrak{M}$ is a semantic model of $\mathscr{S}^*$ in I and $\mathscr{B}$. Consider the following $\mathscr{B}$-valuation V

$$(8) \qquad V(F_k) = h\big(V'(F_k)\big), \qquad k = 1, 2, \ldots$$

It is easy to see that for any $A \in \mathscr{D}$ we have

$$(9) \qquad V(A) = h\big(V'(A)\big).$$

By inductive argument it can be proved by making use of (1)-(3), (5)-(9) that for any formula β in $\mathscr{F}' \subset \mathscr{F}^*$

$$(10) \qquad \beta_{\mathfrak{M}'}(V') = \beta_{\mathfrak{M}}(V).$$

In particular, $\alpha_{\mathfrak{M}}(V) = \alpha_{\mathfrak{M}'}(V') = \wedge$.

Thus, there exists a model of $\mathscr{S}^*$ in which α is not valid. Hence, by 7.1, α is not any theorem of $\mathscr{S}^*$, which proves 8.2.

§ 9. Elementary theories of $\mathscr{S}$-algebras based on $\mathscr{S}$-logic

With the system $\mathscr{S}$ of a propositional calculus with negation [6], satisfying the conditions mentioned in § 5, we shall associate the following system $\mathscr{S}'$ of the elementary theory of $\mathscr{S}$-algebras based on $\mathscr{S}$-logic.

The primitive signs of $\mathscr{S}'$ are the parentheses and

(a) the free individual variables denoted by $F_1, F_2, \ldots$

(b) the bound individual variables denoted by $\varphi_1, \varphi_2, \ldots$

(c) the symbol $\vee$ for the unit element,

(d) the symbol $\wedge$ for the zero element,

(e) the symbols $\cup$, $\cap$, $\rightarrow$, $o_1, \ldots, o_r, o^1, \ldots, o^s$ for operations in $\mathscr{S}$-algebras,

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[6] This hypothesis is not essential.

(f) the symbol $\subset$ for the lattice inclusion in $\mathscr{S}$-algebras,

(g) the symbol $=$ for the relation of equality,

(h) the propositional connectives $\cup$, $\cap$, $\Rightarrow$, $o_1, \ldots, o_r, o^1, \ldots, o^s$,

(i) quantifiers $\bigcup_{\varphi_k}$, $\bigcap_{\varphi_k}$, $k = 1, 2, \ldots$

From these signs we form in the familiar way expressions of two kinds: terms and formulas. The set $\mathscr{D}$ of all terms in $\mathscr{S}'$ coincides with the set $\mathscr{D}^*$ of all set-designations in $\mathscr{S}^*$ (see § 2). The set $\mathscr{F}'$ of all formulas in $\mathscr{S}'$ is contained in the set $\mathscr{F}^*$ of all formulas of $\mathscr{S}^*$.

The axioms for $\mathscr{S}'$ are all formulas obtained from theorems of the propositional calculus $\mathscr{S}$ by the substitution for propositional variables of arbitrary formulas in $\mathscr{F}'$ and, moreover, the following formulas: the axioms for equality, the axioms for $\mathscr{S}$-algebras, which by 3.2 are in the form of equations, and the axioms

$$\vee \cap F_1 = F_1, \quad \wedge \cap F_1 = \wedge,$$

$$(F_1 \subset F_2) \equiv (F_1 \rightarrow F_2 = \vee), \quad (F_1 \subset F_2) \equiv (F_1 \cap F_2 = F_1).$$

$$-\vee = \wedge.$$

By (T_1^*)-(T_{17}^*) § 2, 2.2 and by the proof of 3.2, all axioms of $\mathscr{S}'$ are theorems or axioms of $\mathscr{S}^*$. Moreover, all rules of inference admitted in $\mathscr{S}'$ are also in $\mathscr{S}^*$. In fact, *modus ponens* is admitted in $\mathscr{S}'$ and in $\mathscr{S}^*$. The rule of substitution for free individual variables in $\mathscr{S}'$ corresponds to the rule of substitution for free set variables in $\mathscr{S}^*$ and the rules for quantifiers in $\mathscr{S}'$ correspond to the rules for quantifiers binding set variables in $\mathscr{S}^*$. Consequently the following theorem holds.

9.1. *If a is a formula in $\mathscr{F}'$ and a is a theorem in $\mathscr{S}'$, then it is also a theorem in $\mathscr{S}^*$.*

The aim of this section is to show that also the converse theorem is true for formulas of the form $A = B$, $A, B \in \mathscr{D}$.

Let us consider the set $\mathscr{D}$ of all terms of $\mathscr{S}'$ as an abstract algebra $\langle \mathscr{D}; \vee, \cup, \cap, \rightarrow, o_1, \ldots, o_r, o^1, \ldots, o^s \rangle$.

It is easy to see that

9.2. *The relation $\approx$ defined for any A, B in $\mathscr{D}$ by the equivalence*

$$A \approx B \quad \textit{if and only if} \quad (A = B) \textit{ is in } \mathscr{T}',$$

is a congruence relation in $\mathscr{D}$.

In fact, it is a congruence relation, since the axioms for equality occur among axioms for $\mathscr{S}'$ and the rule of substitution for free individual variables and *modus ponens* are admitted in $\mathscr{S}'$.

9.3. *The quotient algebra* $\langle \mathscr{D}/\approx; [\vee], \cup, \cap, \rightarrow, o_1, \ldots, o_r, o^1, \ldots, o^s\rangle$ *is an* $\mathscr{S}$*-algebra with the unit element* $[\vee]$ *and the zero element* $[\wedge]$. *For any elements* $[A], [B]$ *of this algebra*

$$[A] \subset [B] \quad \textit{if and only if} \quad [A] \rightarrow [B] = [\vee] \textit{ and}$$

$$[A] \subset [B] \quad \textit{if and only if} \quad [A] \cap [B] = [A],$$

i.e. $\subset$ *is the lattice inclusion in this algebra. The elements* $[F_k]$, $k = 1, 2, \ldots$, *are free generators.*

This follows from the axioms of $\mathscr{S}'$.

Let v be a valuation of free individual variables F_k, $k = 1, 2, \ldots$, in an $\mathscr{S}$-algebra $\mathfrak{B}$. In the familiar way v can be extended on the set of all terms.

9.4. *For any terms* A, B *in* $\mathscr{D}$ *if the formula* $A = B$ *is a theorem of* $\mathscr{S}'$, *then* $v(A) = v(B)$ *for every valuation* v *in every* $\mathscr{S}$*-algebra* $\mathfrak{B}$.

This follows directly from 9.1 and 5.7.

9.5. *The algebra* $\langle \mathscr{D}/\approx, [\vee], \cup, \cap, \rightarrow, o_1, \ldots, o_r, o^1, \ldots, o^s\rangle$ *is a free* $\mathscr{S}$*-algebra with generators* $[F_k]$, $k = 1, 2, \ldots$

The proof, similar to that of 5.8 based on 9.4, is omitted.

It follows from 9.5 and 5.8 that the algebras of set designations of $\mathscr{S}^*$ considered in 5.8 and the algebra of terms considered in 9.5 are isomorphic. This isomorphism is given by the formula

$$h([A]^*) = [A] \quad \text{for any} \quad A \in \mathscr{D},$$

where $[A]^*$ denotes the elements of $\mathscr{D}^*/\simeq$ and $[A]$ the element of $\mathscr{D}/\approx$. In consequence

9.6. *For any terms* A, B *in* $\mathscr{D}$, *the formula* $A = B$ *is a theorem of* $\mathscr{S}'$ *if and only if it is a theorem of* $\mathscr{S}^*$.

This condition is necessary by 9.1. If $A = B$ is a theorem of $\mathscr{S}^*$, then $[A]^* = [B]^*$. Hence $h([A]^*) = h([B]^*)$, i.e. $[A] = [B]$. Thus $A = B$ is a theorem of $\mathscr{S}'$.

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