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A solution of the problem of four limits

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1. The formulation of the problem. The ecart and the capacity of the set. F. Leja posed in 1938 in Annales de la Soc. Pol. de Math. 17 (1938), p. 130, the following problem (1).

Let R be the complex plane and $p^{(n)} = \{p_0, p_1, \dots, p_n\}$ an arbitrary system of $n+1$ different points of a closed and bounded set E and $u_{jk} = u_{jk}(z, p^{(n)})$ the function defined by the formulas:

$$u_{jk} = \frac{z-p_k}{p_j-p_k} \quad \text{for} \quad j \neq k, \quad u_{jj} = 1.$$

Let us denote by $A_n(z)$, $B_n(z)$, $C_n(z)$, $D_n(z)$, the greatest lower bounds of the products

$$A_n(z; p^{(n)}) = \prod_{j=0}^{n}\prod_{k=0}^{n} |u_{jk}|, \qquad B_n(z; p^{(n)}) = \max_{(j)} \prod_{k=0}^{n} |u_{jk}u_{kj}|,$$

$$C_n(z; p^{(n)}) = \max_{(j)} \prod_{k=0}^{n} |u_{jk}|, \qquad D_n(z; p^{(n)}) = \max_{(j)} \prod_{k=0}^{n} |u_{kj}|$$

respectively, when the system $p^{(n)}$ changes in E.

It is known [1], [2], [3] that, if the ecart $v(E, |p-q|)$ of the set E is positive (2), then the sequences $A_n(z)^{1/n(n+1)}$, $B_n(z)^{1/2n}$, $C_n(z)^{1/n}$, $D_n(z)^{1/n}$ are convergent to $A(z)$, $B(z)$, $C(z)$, $D(z)$ respectively and that $A(z) \equiv B(z)$. Prove that $A(z) \equiv B(z) \equiv C(z) \equiv D(z)$ or that at least one of the equalities is false.

The object of the present paper is to give a solution of the above problem.

To begin with we shall introduce some definitions.

Let $\omega(p, q)$ be an arbitrary generating function, i.e. a function defined and continuous for every pair of points p and q of the complex plane and satisfing the conditions

$$\omega(p, q) \geqslant 0, \quad \omega(p, q) = \omega(q, p)$$

(1) See also Colloquium Mathematicum 7 (1959), p. 151.

(2) The ecart will be defined later.

and let E be a closed and bounded set in the complex plane. Let us consider a system $p^{(n)} = \{p_0, p_1, \dots, p_n\}$ of $n+1$ points of E. Let us denote by $\vee(p^{(n)})$ the product $\prod_{0\leqslant j<k\leqslant n} \omega(p_j, p_k)$ and by $V_n(E,\omega)$ the supremum of the expression $[\prod_{0\leqslant j<k\leqslant n} \omega(p_j, p_k)]^{2/n(n+1)}$ when $p^{(n)}$ changes arbitrarily in E. Since E is closed and bounded and $\omega(p, q)$ is continuous, there exists a system

$$q^{(n)} = \{q_0^{(n)}, q_1^{(n)}, \dots, q_n^{(n)}\}, \tag{1.1}$$

which will also be denoted by $\{q_0, q_1, \dots, q_n\}$, such that

$$V_n(E, \omega) = [\vee(q^{(n)})]^{2/n(n+1)}.$$

It is known [4] that there exists a finite limit

$$\lim_{n\to\infty} V_n(E, \omega) = V(E, \omega). \tag{1.2}$$

System (1.1) will be called the *n-th extremal system of E with respect to* ω and the limit (1.2) will be called the *ecart of E with respect to* ω.

Let us denote by M the class of all non-negative Radon measures $\mu(e)$ defined on the subsets of E and satisfing the condition $\mu(E) = 1$.

Let δ be an arbitrary positive number. Let us denote by $\omega_\delta(p, q)$ the function defined as follows:

$$\omega_\delta(p, q) = \begin{cases} \omega(p, q) & \text{for} \quad \omega(p, q) \geqslant \delta, \\ \delta & \text{for} \quad \omega(p, q) < \delta. \end{cases}$$

For fixed δ the function $\log \dfrac{1}{\omega_\delta(p, q)}$ is continuous with respect to the pair of points p and q and non-decreasing with respect to δ. Then the integrals

$$\int_E \log \frac{1}{\omega_\delta(p, q)}\, d\mu(q), \qquad \int_E \Big[\int_E \log \frac{1}{\omega_\delta(p, q)}\, d\mu(q)\Big]\, d\mu(p)$$

and their limits (finite or not) as $\delta \to 0$ exist. The limit of the first integral will be denoted by

$$\int_E \log \frac{1}{\omega(p, q)}\, d\mu(q)$$

and the limit of the second integral by

$$\int_E \Big[\int_E \log \frac{1}{\omega(p, q)}\, d\mu(q)\Big]\, d\mu(p),$$

or shortly by $I(\mu)$.

Let $K = \max\left(1, \sup_{p,q \in E} \omega(p,q)\right)$. Since E is a closed and bounded set and $\omega(p,q)$ is continuous, K is a finite number $\geqslant 1$. Since $\log \frac{K}{\omega(p,q)} \geqslant 0$ (in the case where $\omega(p,q) = 0$ we put $\log \frac{K}{\omega(p,q)} = \infty$) and since

$$\int_E \left[\int_E \log \frac{1}{\omega(p,q)}\, d\mu(q)\right] d\mu(p) - \log\frac{1}{K} = \int_E \left[\int_E \log \frac{K}{\omega(p,q)}\, d\mu(q)\right] d\mu(p)$$

we have

$$\int_E \left[\int_E \log \frac{1}{\omega(p,q)}\, d\mu(q)\right] d\mu(p) \geqslant \log \frac{1}{K}. \tag{1.3}$$

We shall consider two cases:

Case 1. There exists a $\mu \in M$ such that $I(\mu) < \infty$. In this case we can prove by the method as in [1] that there exists an $\eta \in M$ such that

$$I(\eta) = \inf_{\mu \in M} I(\mu).$$

By (1.3) we have $I(\eta) > -\infty$. The measure $\eta(e)$ will be called the *equilibrium measure with respect to the function* ω. The function $U(p)$ defined in the whole plane by the formula

$$U(p) = \int_E \log \frac{1}{\omega(p,q)}\, d\eta(q)$$

will be called the *equilibrium potential with respect to* ω and the number

$$C(E,\omega) = e^{-I(\eta)}$$

the *capacity of the set* E *with respect* ω.

Case 2. $I(\mu) = \infty$ for every $\mu \in M$. In this case we put $C(E,\omega) = 0$. In this case we do not define the equilibrium measure and the equilibrium potential.

2. The equality of the ecart and the capacity of a set.

LEMMA 1. *For an arbitrary closed and bounded set* E *in the plane and for an arbitrary generating function* $\omega(p,q)$ *we have*

$$C(E,\omega) = V(E,\omega).$$

Proof. Let δ be an arbitrary positive number and $\omega_\delta(p,q)$ a function defined on page 58. Let $V_n(E,\omega_\delta)$, $V(E,\omega_\delta)$ and $C(E,\omega_\delta)$ be defined analogically to $V_n(E,\omega)$, $V(E,\omega)$ and $C(E,\omega)$. It is sufficient to prove that

$$V(E,\omega_\delta) = C(E,\omega_\delta), \tag{2.1}$$

$$V(E,\omega_\delta) \to V(E,\omega) \quad \text{as} \quad \delta \to 0, \tag{2.2}$$

$$C(E,\omega_\delta) \to C(E,\omega) \quad \text{as} \quad \delta \to 0. \tag{2.3}$$

We shall first prove (2.1). Let $K = \sup_{p,q\in E} \omega_\delta(p,q)$. Since E is closed and bounded and $\omega_\delta(p,q)$ is continuous, K is a finite number $\geqslant \delta$. Let $\omega_\delta^1(p,q) = \omega_\delta(p,q)/K$. It is easy to see that $\omega_\delta^1(p,q) \leqslant 1$ for $p, q \in E$ and $V(E, \omega_\delta^1) = V(E, \omega_\delta)/K$, $C(E, \omega_\delta^1) = C(E, \omega_\delta)/K$. In order to prove (2.1) it is sufficient to prove that $C(E, \omega_\delta^1) = V(E, \omega_\delta^1)$.

In the sequel we shall write $\omega_\delta(p,q)$ instead of $\omega_\delta^1(p,q)$. Hence we have $\omega_\delta(p,q) \leqslant 1$ for $p, q \in E$.

Let $\mu(e)$ be a measure belonging to M (see p. 58). Let us denote by $I^\delta(\mu)$ the integral defined by the formula

$$I^\delta(\mu) \overset{\mathrm{df}}{=} \int_E \Big[\int_E \log \frac{1}{\omega_\delta(p,q)}\, d\mu(q) \Big]\, d\mu(p)$$

and by $\eta_\delta(e)$ a measure realising the greatest lower bound of the integrals $I^\delta(\mu)$ when $\mu \in M$:

$$I^\delta(\eta_\delta) = \inf_{\mu\in M} I(\mu)\,.$$

Without any loss of generality we can assume that there exists a number $a > 1$ such that E is contained in the rectangle

$$0 \leqslant x \leqslant a-1\,, \quad 0 \leqslant y \leqslant a-1\,.$$

Let us divide the square $0 \leqslant x < a$, $0 \leqslant y < a$ into n^2 partially open squares

$$\frac{j-1}{n} a \leqslant x < \frac{j}{n} a\,, \quad \frac{k-1}{n} a \leqslant y < \frac{k}{n} a\,, \quad j, k = 1, 2, \ldots, n$$

and let us denote these squares by $P_i^{(n)}$, $i = 1, 2, \ldots, n^2$. Let us denote the sets $E \cap P_i^{(n)}$ by $E_i^{(n)}$ for $i = 1, 2, \ldots, n^2$. Hence we have

$$\sum_{i=1}^{n^2} E_i^{(n)} = E\,.$$

Let $m(n)$ be a natural number and $l_i(n)$, $i = 1, 2, \ldots, n^2$ non-negative integers such that

$$\frac{l_i(n)}{m(n)} - \frac{1}{n^5} \leqslant \eta_\delta(E_i^{(n)}) \leqslant \frac{l_i(n)}{m(n)}\,, \quad i = 1, 2, \ldots, n^2\,. \tag{2.4}$$

If the set $E_i^{(n)}$ is empty we put $l_i(n) = 0$. We can assume (taking suitable $m(n)$) that

$$\sum_{i=1}^{n^2} l_i(n) \to \infty\,, \quad \text{as} \quad n \to \infty\,.$$

Let us take $l_i(n)$ arbitrary points (different or not) in every set $E_i^{(n)}$ $(i = 1, 2, ..., n^2)$. We shall have the system of the $l(n)$ points

$$(2.5) \qquad \{p_0, p_1, ..., p_{l(n)}\} \quad \text{where} \quad l(n) = \sum_{i=1}^{n^2} l_i(n) - 1 .$$

Next, for brevity we shall denote the sets $E_i^{(n)}$ by E_i and the numbers $l(n), l_i(n), m(n)$ by l, l_i, m.

Let

$$(2.6) \qquad \{q_0, q_1, ..., q_l\}$$

be the lth extremal system of the set E with respect to the function ω_δ. Let us denote by $\mu_l(e)$ and $\mu_l'(e)$ the measures defined on the Borel subsets e of E as follows:

$$\mu_l(e) = \frac{k}{l+1}, \qquad \mu_l'(e) = \frac{k'}{l+1},$$

where k and k' denote the numbers of the points of system (2.5) and of the extremal system (2.6) contained in e. It is easy to see that the measures μ_l and μ_l' belong to the class M (see p. 58). From the definition of the extremal system (2.6) we have

$$(2.7) \qquad \log\frac{1}{V_l(E, \omega_\delta)} = \frac{2}{l(l+1)} \sum_{0 \leqslant i < k \leqslant l} \log\frac{1}{\omega_\delta(q_i, q_k)}$$

$$= \frac{l+1}{l}\left[\frac{1}{(l+1)^2}\sum_{i,k=0}^{l} \log\frac{1}{\omega_\delta(q_i, q_k)} - \frac{1}{(l+1)^2}\sum_{i=0}^{l} \log\frac{1}{\omega_\delta(q_i, q_i)}\right].$$

Since

$$\frac{1}{(l+1)^2}\sum_{i,k=0}^{l} \log\frac{1}{\omega_\delta(q_i, q_k)} = \int_E\left[\int_E \log\frac{1}{\omega_\delta(p, q)}\, d\mu_l(q)\right] d\mu_l(p) \overset{\text{df}}{=} I^\delta(\mu_l)$$

and

$$\log\frac{1}{\omega_\delta(p, q)} \leqslant \log\frac{1}{\delta},$$

we have from (2.7) and from the definition of μ_l and η_δ

$$(2.8) \quad \log\frac{1}{V_l(E, \omega_\delta)} \geqslant \frac{l+1}{l}\left[I^\delta(\mu_l) - \frac{1}{l+1}\log\frac{1}{\delta}\right] \geqslant \frac{l+1}{l}\left[I^\delta(\eta_\delta) - \frac{1}{l+1}\log\frac{1}{\delta}\right].$$

Since $V_l(E, \omega_\delta) \to V(E, \omega_\delta)$ as $l \to \infty$ (p. 58), we shall obtain from (2.8) the inequality

$$\log\frac{1}{V(E, \omega_\delta)} \geqslant I^\delta(\eta_\delta)$$

and hence

$$V(E,\omega_\delta) \leqslant \exp[-I^\delta(\eta_\delta)] = C(E,\omega_\delta)\,.$$

In order to prove (2.1) it is sufficient to prove that $V(E,\omega_\delta) \geqslant C(E,\omega_\delta)$.

Since system (2.6) is extremal, we have

$$\log\frac{1}{V_l(E,\omega_\delta)} = \frac{2}{l(l+1)}\sum_{0\leqslant i<k\leqslant l}\log\frac{1}{\omega_\delta(q_i,q_k)} \leqslant \frac{2}{l(l+1)}\sum_{0\leqslant i<k\leqslant l}\log\frac{1}{\omega_\delta(p_i,p_k)}$$

(2.9)

$$= \frac{l+1}{l}\left[\frac{1}{(l+1)^2}\sum_{i,k=0}^{l}\log\frac{1}{\omega_\delta(p_i,p_k)} - \frac{1}{(l+1)^2}\sum_{i=0}^{l}\log\frac{1}{\omega_\delta(p_i,p_i)}\right].$$

Let $M_{ik} = \sup_{p\in E_i, q\in E_k}\log\frac{1}{\omega_\delta(p,q)}$. Since

$$\frac{1}{(l+1)^2}\sum_{i,k=0}^{l}\log\frac{1}{\omega_\delta(p_i,p_k)} = I^\delta(\mu_l)\,,$$

we shall obtain from (2.9)

$$\log\frac{1}{V_l(E,\omega_\delta)} \leqslant \frac{l+1}{l}\left(I^\delta(\mu_l) - \frac{\log(1/M)}{l+1}\right)$$

$$\leqslant \frac{l+1}{l}\left(\sum_{i,k=1}^{n^2} M_{ik}\mu_l(E_i)\mu_l(E_k) - \frac{\log(1/M)}{l+1}\right),$$

where $M = \sup_{p,q\in E}\log\frac{1}{\omega_\delta(p,q)}$. From inequality (2.4) and from the definition of μ_l we have

$$\sum_{i,k=1}^{n^2} M_{ik}\mu_l(E_i)\mu_l(E_k) = \sum_{i,k=1}^{n^2} M_{ik}\frac{l_i}{m}\cdot\frac{l_k}{m} \leqslant \sum_{i,k=1}^{n^2} M_{ik}\left[\eta_\delta(E_i)+\frac{1}{n^5}\right]\left[\eta_\delta(E_k)+\frac{1}{n^5}\right]$$

and hence

$$\log\frac{1}{V_l(E,\omega_\delta)} \leqslant \frac{l+1}{l}\Bigg[\sum_{i,k=1}^{n^2} M_{ik}\eta_\delta(E_i)\eta_\delta(E_k) + \frac{1}{n^{10}}\sum_{i,k=1}^{n^2} M_{ik} +$$

$$+\frac{1}{n^5}\sum_{i,k=1}^{n^2} M_{ik}\eta_\delta(E_i) + \frac{1}{n^5}\sum_{i,k=1}^{n^2} M_{ik}\eta_\delta(E_k) - \frac{\log(1/M)}{l+1}\Bigg].$$

Since $M_{ik} \leqslant \log(1/\delta)$ and $0 \leqslant \eta_\delta(E_i) \leqslant 1$, we have from the last inequality

$$(2.10) \quad \log\frac{1}{V_l(E,\omega)} \leqslant \frac{l+1}{l}\Big[\sum_{i,k=1}^{n^2} M_{ik}\eta_\delta(E_i)\eta_\delta(E_k) + \frac{\log(1/\delta)}{n^6} + \frac{2\log(1/\delta)}{n} - \frac{\log(1/M)}{l+1}\Big].$$

The function $\log\dfrac{1}{\omega_\delta(p,q)}$ is uniformly continuous in the set $E\times E$; so for any $\varepsilon > 0$ there exists a $N(\varepsilon)$ such that

$$M_{ik} - \log\frac{1}{\omega_\delta(p,q)} < \varepsilon \quad \text{for} \quad p\in E_i,\ q\in E_k,\ n \geqslant N(\varepsilon).$$

Hence

$$\begin{aligned}\sum_{i,k=1}^{n^2} M_{ik}\eta_\delta(E_i)\eta_\delta(E_k) &= \sum_{i,k=1}^{n^2}\int_{E_i}\Big[\int_{E_k} M_{ik}\,d\eta_\delta(q)\Big]d\eta_\delta(p)\\ &\leqslant \sum_{i,k=1}^{n^2}\int_{E_i}\Big[\int_{E_k}\log\frac{1}{\omega_\delta(p,q)}\,d\eta_\delta(q)\Big]d\eta_\delta(p) +\\ &\quad + \varepsilon\sum_{i,k=1}^{n^2}\int_{E_i}\Big[\int_{E_k} d\eta_\delta(q)\Big]d\eta_\delta(p) = I^\delta(\eta_\delta) + \varepsilon\end{aligned}$$

and from (2.10) we have

$$\log\frac{1}{V_l(E,\omega_\delta)} \leqslant I^\delta(\eta_\delta) + \varepsilon + \frac{\log(1/\delta)}{n^6} + \frac{2\log(1/\delta)}{n} - \frac{\log(1/M)}{l+1}.$$

Since $l = l(n)\to\infty$ as $n\to\infty$, we have from the last inequality

$$\log\frac{1}{V(E,\omega_\delta)} \leqslant I^\delta(\eta_\delta) + \varepsilon.$$

Since ε is an arbitrary positive number, we shall hence obtain

$$\log\frac{1}{V(E,\omega_\delta)} \leqslant I^\delta(\eta_\delta)$$

or

$$V(E,\omega_\delta) \geqslant \exp[-I^\delta(\eta_\delta)] = C(E,\omega_\delta).$$

Hence and from the opposite inequality (see p. 62) we have equality (2.1).

Proof of (2.2). It is known [5] that

$$(2.11) \qquad V_{n+1}(E,\omega) \leqslant V_n(E,\omega) \quad \text{for} \quad n = 1, 2, \ldots$$

From the definition of ω_δ it follows that

$$V_n(E, \omega_\delta) \geqslant V_n(E, \omega), \quad n = 1, 2, \dots$$

Hence and from formula (1.2) (see p. 58) follows

$$V(E, \omega_\delta) \geqslant V(E, \omega). \tag{2.12}$$

In order to prove (2.2) it is sufficient to prove that for any $\varepsilon > 0$ there exists a $\delta_0(\varepsilon)$ such that

$$V(E, \omega_\delta) < V(E, \omega) + \varepsilon \quad \text{for} \quad \delta < \delta_0 = \delta_0(\varepsilon). \tag{2.13}$$

Since $V_n(E, \omega) \to V(E, \omega)$ there exists a natural number $N(\varepsilon)$ such that

$$V_n(E, \omega) \leqslant V(E, \omega) + \varepsilon/2 \quad \text{for} \quad n \geqslant N(\varepsilon). \tag{2.14}$$

Next we shall prove that for $N = N(\varepsilon)$ there exists a δ_0 such that

$$V_N(E, \omega^\delta) < V_N(E, \omega) + \varepsilon/2 \quad \text{for} \quad \delta < \delta_0. \tag{2.15}$$

Let $q_0, q_1, \dots, q_N$ be the Nth extremal system of E with respect to ω_δ. If $V_N(E, \omega) > 0$ we have the equality

$$\omega_\delta(q_i, q_k) = \omega(q_i, q_k)$$

for

$$\delta < \delta_0 = \min\left(K, \frac{[V_N(E, \omega)]^{\frac{N(N+1)}{2}}}{K^{\frac{N(N+1)}{2}-1}}\right), \qquad K = \sup_{p,q \in E} \omega_\delta(p, q)$$

because in the contrary case there would exist a positive number $\delta < \delta_0$ and a pair of points q_{i_0}, q_{k_0}, $0 \leqslant i_0 \leqslant N$, $0 \leqslant k_0 \leqslant N$ such that $\omega_\delta(q_{i_0}, q_{k_0}) = \delta < \delta_0$. From the definition of K and δ we should have

$$V_N(E, \omega_\delta) = \Big[\prod_{0 \leqslant i < k \leqslant N} \omega_\delta(q_i, q_k)\Big]^{\frac{2}{N(N+1)}} \leqslant \Big[\delta K^{\frac{N(N+1)}{2}-1}\Big]^{\frac{2}{N(N+1)}} \leqslant V_N(E, \omega),$$

which is absurd. Hence

$$V_N(E, \omega_\delta) = \Big[\prod_{0 \leqslant i < k \leqslant N} \omega_\delta(q_i, q_k)\Big]^{\frac{2}{N(N+1)}} = \Big[\prod_{0 \leqslant i < k \leqslant N} \omega(q_i, q_k)\Big]^{\frac{2}{N(N+1)}}$$
$$< V_N(E, \omega) + \varepsilon/2,$$

i.e. we have inequality (2.15).

If $V_N(E, \omega) = 0$ then for $\delta < \dfrac{\varepsilon^{\frac{N(N+1)}{2}}}{K^{1-\frac{2}{N(N+1)}}}$ there exists a pair of points q_{i_1}, q_{k_1} of the extremal system such that $\omega_\delta(q_{i_1}, q_{k_1}) = \delta$; so

$$V_N(E, \omega_\delta) \leqslant \left[\delta K^{\frac{N(N+1)}{2}-1}\right]^{\frac{2}{N(N+1)}} < \varepsilon/2 = V_N(E, \omega) + \varepsilon/2$$

and inequality (2.15) is also proved.

From (2.14) and (2.15) it follows that

$$V_N(E, \omega_\delta) \leqslant V(E, \omega) + \varepsilon \quad \text{for} \quad \delta < \delta_0 \tag{2.16}$$

and from (2.11) and (2.16) we have

$$V(E, \omega_\delta) < V(E, \omega) + \varepsilon \quad \text{for} \quad \delta < \delta_0 ,$$

i.e. inequality (2.13).

Since ε is an arbitrary positive number, formula (2.2) follows hence.

Proof of (2.3). We shall consider two cases.

Case 1. $V(E, \omega) > 0$. In this case from (2.1) and (2.2) it follows that

$$C(E, \omega_\delta) \to V(E, \omega) \quad \text{as} \quad \delta \to 0 .$$

Hence it follows that $I^\delta(\eta_\delta)$ converges to a finite limit I as $\delta \to 0$:

$$\lim_{\delta \to 0} I^\delta(\eta_\delta) = I . \tag{2.17}$$

From the definition of η (see p. 59) and η_δ (see p. 60) we have

$$I^\delta(\eta_\delta) \leqslant I^\delta(\eta) . \tag{2.18}$$

From (2.17), (2.18) and the definition of the integral $I(\eta)$ we obtain the inequality

$$I \leqslant I(\eta) . \tag{2.19}$$

Now we shall prove that $I = I(\eta)$. In the opposite case there would exist a number $\varepsilon > 0$ such that

$$I < I(\eta) - \varepsilon . \tag{2.20}$$

Let $\{\delta_n\}$ be a sequence of positive numbers such that $\delta_n \to 0$ and $I^{\delta_n}(\eta_{\delta_n}) \to I$. We can suppose that the sequence of the measures $\eta_{\delta_n}(e)$ is convergent to a measure $\eta_0(e)$ (because from the sequence of such measures we can take a convergent subsequence, see [1]). Let ε' be an arbitrary positive number. It is easy to see that $I^{\delta_n}(\eta_{\delta_n}) \geqslant I^{\varepsilon'}(\eta_{\delta_n})$ for $\delta_n < \varepsilon'$ and hence if $n \to \infty$, we shall obtain $I \geqslant I^{\varepsilon'}(\eta_0)$. Now if $\varepsilon' \to 0$ we shall have $I \geqslant I(\eta_0)$, which together with inequality (2.20) gives the inequality $I(\eta_0) < I(\eta) - \varepsilon$ in contradiction to the definition of η. Therefore $I = I(\eta)$.

Hence and from equality (2.1) we have

$$C(E, \omega_\delta) = e^{-I^\delta(\eta_\delta)} \to e^{-I} = e^{-I(\eta)} = C(E, \omega) \quad \text{as} \quad \delta \to 0 .$$

Case 2. $V(E, \omega) = 0$. In this case from (2.1) and (2.2) it follows that $I^\delta(\eta_\delta) \to \infty$ as $\delta \to 0$. Hence and from the definition of η_δ it follows that $I^\delta(\mu) \to \infty$ as $\delta \to 0$ for $\mu \in M$ and hence $I(\mu) = \infty$ for $\mu \in M$. By the definition of capacity we have in this case $C(E, \omega) = 0 = V(E, \omega)$. Formula (2.3) is thus proved.

From (2.1), (2.2) and (2.3) it follows that $C(E, \omega) = V(E, \omega)$, q.e.d.

3. The properties of the extremal measure and its connection with the Green function. Let E be a closed set on the z-plane and z be a point not belonging to E. Suppose $C(E, |p-q|) > 0$. Since (see p. 57)

$$A_n(z; p^{(n)}) = \prod_{\substack{j,k=0\\ j\neq k}}^{n} \frac{\sqrt{|z-p_j|\,|z-p_k|}}{|p_j-p_k|} \quad \text{and} \quad A_n(z) = \inf_{p^{(n)}\in E} A_n(z; p^{(n)}) ,$$

it follows that

$$\log A_n(z) = \inf_{p^{(n)}} \log A_n(z) = 2 \sum_{0\leq j<k\leq n} \log \frac{1}{\omega(q_i, q_k; z)} ,$$

where $\omega(p, q; z) = \dfrac{|p-q|}{\sqrt{|z-p|\,|z-q|}}$ and $q_0, q_1, \ldots, q_n$ is the nth extremal system of E with respect to $\omega(p, q; z)$. Since

$$\log A(z) = \lim \frac{1}{n(n+1)} \log A_n(z) = \lim \frac{2}{n(n+1)} \sum_{0\leq j<k\leq n} \log \frac{1}{\omega(q_i, q_k; z)}$$

$$= \log \frac{1}{V\big(E, \omega(p, q; z)\big)}$$

it follows from lemma 1 that $\log A(z) = I(\mu_z)$ where μ_z is the equilibrium measure with respect to $\omega(p, q; z)$ (see p. 59).

Let

$$u_z(p) \overset{\text{df}}{=} \int_E \log \frac{1}{\omega(p, q; z)}\, d\mu_z(q) .$$

Then

$$u_z(p) = \int_{E^*} \log \frac{1}{\omega(p, q; z)}\, d\mu_z(q)$$

where E^* is the kernel of μ_z, i.e. the set of the points of E such that, for every $z_0 \in E^*$ and every open set U_0 containing z_0, $\mu_z(E \cap U_0) > 0$.

It is easy to see that

$$(3.1)\qquad \lim_{p\to z} u_z(p) = -\infty\,, \quad \lim_{p\to\infty} u_z(p) = -\infty$$

and

$$(3.2)\qquad u_z(p) = \int_{E^*} \log\frac{1}{|p-q|}\,d\mu_z(q) - \frac{1}{2}\log\frac{1}{|z-p|} - \frac{1}{2}\int_{E^*}\log\frac{1}{|z-q|}\,d\mu_z(q)\,.$$

From (3.1) and (3.2) it follows that $u_z(p)$ is a function of p, subharmonic on the outside of E. Hence we can prove by the same method as in [1] or in [6] that $u_z(p)$ satisfies the maximum principle, i.e. if $u_z(p) \leqslant K$ for $p \in E$, where K is a constant, then $u_z(p) \leqslant K$ for every p. Hence

$$(3.3)\qquad u_z(p) \leqslant \sup_{q\in E^*} u_z(q)$$

for every p.

LEMMA 2. *For every $p \in E$ except a set whose capacity with respect to $|p-q|$ equals zero we have the inequality*

$$(3.4)\qquad u_z(p) \geqslant \sup_{q\in E^*} u_z(q)\,.$$

Proof. In the opposite case there would exist a set $F\subset E$ such that $C(F, |p-q|) > 0$ and

$$(3.5)\qquad u_z(p) < \sup_{q\in E^*} u_z(q) \quad \text{for} \quad p\in F\,.$$

Hence there would exist a positive number ε and a closed subset F_0 of F with a positive capacity such that

$$(3.5')\qquad u_z(p) < \sup_{q\in E^*} u_z(q) - 2\varepsilon \quad \text{for} \quad p\in F_0\,.$$

Since z does not belong to E, it is easy to prove that also $C\big(F_0, \omega(p, q; z)\big) > 0$. It is evident that there exists a point p_0 belonging to E^* such that

$$(3.6)\qquad u_z(p_0) > \sup_{q\in E^*} u_z(q) - \varepsilon\,.$$

The potential $u_z(p)$ for a fixed z is the lower semi-continuous function of p. Hence and from (3.6) it follows that there exists a neighbourhood U_0 of the point p_0 such that

$$(3.6')\qquad u_z(p) + \varepsilon > \sup_{q\in E^*} u_z(q) \quad \text{for} \quad p\in U_0\,.$$

Diminishing U_0 if necessary, we can suppose that the distance of F_0 and U_0 is positive. Because $p_0 \in E^*$, $\mu_z(U_0) > 0$. Let $m \overset{\text{df}}{=} \mu_z(U_0)$. Since $C\big(F_0, \omega(p, q; z)\big) > 0$, there exists (see p. 59) a non-negative measure $\sigma'(e)$

defined on the Borel subsets of F_0 such that

$$(3.7)\qquad \int\limits_{F_0}\Big[\int\limits_{F_0}\log\frac{1}{\omega(p,q;z)}\,d\sigma'(q)\Big]d\sigma'(p)<\infty\,,\qquad \sigma'(F_0)=1\,.$$

Let $\sigma''(e)=m\sigma'(e)$. Since

$$\int\limits_{F_0}\Big[\int\limits_{F_0}\log\frac{1}{\omega(p,q;z)}\,d\sigma''(q)\Big]d\sigma''(p)=m^2\int\limits_{F_0}\Big[\int\limits_{F_0}\log\frac{1}{\omega(p,q;z)}\,d\sigma'(q)\Big]\,d\sigma'(p)\,,$$

we have by (3.7)

$$\int\limits_{F_0}\Big[\int\limits_{F_0}\log\frac{1}{\omega(p,q;z)}\,d\sigma''(q)\Big]\,d\sigma''(p)<\infty\,,\qquad \sigma''(F_0)=m\,.$$

Let us consider the measure $\sigma(e)$ defined as follows:

$$\sigma(e)=\begin{cases}-\mu_z(e) & \text{for the Borel sets } e\subset U_0\,,\\ \sigma''(e) & \text{for the Borel sets } e\subset F_0\,,\\ 0 & \text{for the Borel sets lying in the outside } U_0 \text{ and } F_0\,.\end{cases}$$

Let ϱ be a positive number smaller than 1. It is evident that the measure $\mu_z+\varrho\sigma$ belongs to M. By the same method as in [1] and [6] we can prove that

$$(3.8)\qquad I(\sigma)<\infty$$

and

$$(3.9)\qquad I(\mu_z+\varrho\sigma)-I(\mu_z)\leqslant-\varrho[2m\varepsilon-\varrho I(\sigma)]\,.$$

For sufficiently small ϱ the right side of (3.9) is, by (3.8), smaller than zero; so μ_z would not realise the infimum of the integrals $I(\mu)$, which is a contradiction of the definition of μ_z. The lemma is thus proved.

From (3.3) and (3.4) we have the equality

$$(3.10)\qquad u_z(p)=\sup_{q\in E^*}u_z(q)$$

for $p\in E$ except a set of capacity zero (for example, with respect to $|p-q|$).

We shall introduce the following notations:

$$h(p,z)\overset{\text{df}}{=}\int\limits_{E}\log\frac{1}{|p-q|}\,d\mu_z(q)-\frac{1}{2}\log\frac{1}{|p-z|}\,,$$

$$\gamma(z)=\sup_{p\in E^*}u_z(p)+\frac{1}{2}\int\limits_{E^*}\log\frac{1}{|q-z|}\,d\mu_z(q)\,.$$

From (3.2) and (3.10) it follows that

$$(3.11)\qquad h(p,z)=\gamma(z)$$

for $p \in E$ except a set of capacity zero. The function $h(p, z)$ considered as a function of p on E is continuous except a set of capacity zero. So we can prove by the same method as in [1] and [6] that the function $h(p, z)$ is continuous in the whole plane except a set of capacity zero.

Let CE denote the complementary set of E. CE consists of an at most enumerably infinite number of domains. Let D_∞ be that one of those domains which contains $z = \infty$ and let F_∞ be its boundary. Let $C(E)$ be the capacity of E with respect to $|p-q|$.

We shall prove

LEMMA 3. *If $z_0 \in E$ and z_0 is a regular point for D_∞ with respect to the Dirichlet problem, then*

$$\gamma(z) \to \frac{1}{2}\log\frac{1}{C(E)} \quad \text{as} \quad z \to z_0 \quad (z \in D_\infty)\,.$$

Proof. Let μ be an equilibrium measure with respect to the generating function $|p-q|$. It is known (see [1], [6]) that for every $z \in E$ except a set of capacity zero we have the equality

$$u(z) = \int_E \log\frac{1}{|p-z|}\,d\mu(p) = \log\frac{1}{C(E)} \tag{3.12}$$

and for every z

$$u(z) \leqslant \log\frac{1}{C(E)}\,. \tag{3.13}$$

In [1] and [6] it is proved that if $\int_E \log\frac{1}{|p-z|}\,d\mu(z) < \infty$ and $C(E_0) = 0$ then $\mu(E_0) = 0$. Hence and by the Fubini theorem we have

$$\begin{aligned}\gamma(z) &= \int_E \gamma(z)d\mu(p) = \int_E h(p, z)d\mu(p) \\ &= \int_E \Big[\int_E \log\frac{1}{|p-q|}\,d\mu_z(q) - \frac{1}{2}\log\frac{1}{|p-z|}\Big]\,d\mu(p) \\ &= \int_E \Big[\int_E \log\frac{1}{|p-q|}\,d\mu(p)\Big]\,d\mu_z(q) - \frac{1}{2}\int_E \log\frac{1}{|p-z|}\,d\mu(p)\,,\end{aligned}$$

and so by (3.12) we have

$$\gamma(z) = \log\frac{1}{C(E)} - \frac{1}{2}u(z)\,. \tag{3.14}$$

Since (see [1], [6]) $u(z) \to \log\frac{1}{C(E)}$, as $z \to z_0$, it follows from (3.13) that $\gamma(z) \to \frac{1}{2}\log\frac{1}{C(E)}$ as $z \to z_0$, q.e.d.

By (3.11), (3.12), (3.13) and (3.14) it is easy to prove that the function

$$(3.15)\quad G(p,z) = -2h(p,z)+2\gamma(z)+u(p)-\log\frac{1}{C(E)}$$
$$= \log\frac{1}{|p-z|}-2\int_E \log\frac{1}{|p-q|}\,d\mu_z(q)+\log\frac{1}{C(E)}-u(z)+u(p)$$

is the Green function for the domain D_∞ with the pole at z.

4. A certain property of the Green function. Let $G(p,z)$ be the Green function for D_∞ (defined in Chapter 3) with the pole at z. We shall always assume that $z \neq \infty$.

It is known that there exists a finite limit

$$(4.1)\qquad \lim_{p\to z,\, p\neq z}\left[G(p,z)-\log\frac{1}{|p-z|}\right]$$

and the function $U(p,z)$ defined for $z, p \in D_\infty$ by the formulas

$$(4.2)\qquad U(p,z) = \begin{cases} G(p,z)-\log\dfrac{1}{|p-z|} & \text{for}\quad p\neq z, \\ \lim\limits_{p\to z,\, p\neq z}\left[G(p,z)-\log\dfrac{1}{|p-z|}\right] & \text{for}\quad p=z \end{cases}$$

is the solution of the Dirichlet problem for D_∞ with the boundary value equal to $-\log\dfrac{1}{|p-z|}$ on F_∞.

About the function $U(p,z)$ we shall prove the following

LEMMA 4. *If $z_0 \in F_\infty$ and z_0 is a regular point for the Dirichlet problem, then* $\lim\limits_{z\to z_0,\, z\in D_\infty} U(z,z) = -\infty$.

Proof. Let K be an arbitrary positive number. Let

$$(4.3)\qquad \log_K x = \begin{cases} \ln x & \text{for}\quad 0<x\leqslant e^K, \\ K & \text{for}\quad x>e^K \end{cases}$$

and let $U_K(p,z)$ be the solution of the Dirichlet problem for D_∞ with the bonndary value equal to $-\log_K\dfrac{1}{|p-z|}$. We put $\log_K\dfrac{1}{|p-z|}\overset{\mathrm{df}}{=}K$ for $p=z$. It is easy to see that

$$\lim_{z\to z_0,\, z\in D_\infty}\log_K\frac{1}{|p-z|}=\log_K\frac{1}{|p-z_0|}\qquad (z_0\in F_\infty)$$

and the convergence is uniform with respect to $p \in D_\infty$. Hence there exists a $\delta > 0$ such that for every $p \in F_\infty$

$$\left|\log_K\frac{1}{|p-z|}-\log_K\frac{1}{|p-z_0|}\right|<1,\qquad \text{when}\quad |z-z_0|<\delta,\ z\in D_\infty.$$

From the last inequality and from the maximum principle for harmonic functions we have

$$|U_K(p,z)-U_K(p,z_0)|<1, \quad \text{as} \quad |z-z_0|<\delta,\ p,z\in D_\infty .$$

Putting $p=z$ we shall obtain

$$U_K(z,z)<1+U_K(z,z_0), \quad \text{as} \quad |z-z_0|<\delta,\ z\in D_\infty \tag{4.4}$$

and hence

$$U(z,z)<1+U_K(z,z_0) \quad \text{for} \quad |z-z_0|<\delta,\ z\in D_\infty \tag{4.5}$$

(because $U(p,z)\leqslant U_K(p,z)$). Since z_0 is a regular point, we have

$$\lim_{z\to z_0,\, z\in D_\infty} U_K(z,z_0)=U_K(z_0,z_0)=-K .$$

Hence and from (4.5) it follows that $\overline{\lim}_{z\to z_0,\, z\in D_\infty} U(z,z)\leqslant 1-K$ and, since K is arbitrary positive number, we have $\lim_{z\to z_0,\, z\in D_\infty} U(z,z)=-\infty$, q.e.d.

5. Proof of the inequality $A(z)<C(z)$. Let D_∞ be the domain defined in Chapter 4 and $z\in D_\infty$.

THEOREM 1. *If the capacity $C(E)$ of E is positive then, for every $z\in D_\infty$, $z\neq\infty$, $A(z)<C(z)$* (¹).

Proof. From the definition of $A(z)$ and $B(z)$ it follows that $A(\infty)=B(\infty)$ and for every z

$$A(z)\leqslant C(z) . \tag{5.1}$$

The function $\frac{1}{n(n+1)}\log A_n(z)$ is the lower envelope of the harmonic functions $\frac{1}{n(n+1)}\sum_{j\neq k}\log\left|\frac{z-p_k}{p_j-p_k}\right|$. Hence (see p. 66) $\log A(z)$ is a superharmonic function in $D_\infty-\{\infty\}$. Since the function $\log C(z)$ (see [5]) is the Green function for D_∞ with the pole at infinity, it follows from (5.1) that $\log A(z)-\log C(z)$ is a non-positive superharmonic function in $D_\infty-\{\infty\}$; thus in order to prove our theorem it is sufficient to prove that there exists one point $z'\in D_\infty$ such that $A(z')<C(z')$.

Let z_0 be a regular point for D_∞. Since

$$G(p,z)=\log\frac{1}{|p-z|}-2\int_E\log\frac{1}{|p-q|}\,d\mu_z(q)+\log\frac{1}{C(E)}-u(z)+u(p)$$

(¹) Editor's remark: J. Siciak has observed that there exist sets E such that the inequality $A(z)<C(z)$ holds not for all points $z\in D_\infty$ ($z\neq\infty$).

is the Green function for D_∞ with the pole in z (see (3.14)) we have (see (4.2))

$$U(z, z) = -2 \int_E \log \frac{1}{|z-q|} d\mu_z(q) + \log \frac{1}{C(E)} .$$

Hence and from lemma 4 it follows that

$$(5.2) \qquad \int_E \log \frac{1}{|q-z|} d\mu_z(q) \to \infty , \quad \text{as} \quad z \to z_0 \ (z \in D_\infty) .$$

Since

$$\log A(z) = I(\mu_z) = \int_E h(p, z) d\mu_z(p) - \frac{1}{2} \int_E \log \frac{1}{|p-z|} d\mu_z(p)$$

(see p. 66 and p. 68), from (3.11) using the same method as in the proof of lemma 3 we shall obtain

$$(5.3) \qquad \log A(z) = \gamma(z) - \frac{1}{2} \int_E \log \frac{1}{|p-z|} d\mu_z(p) .$$

From (5.2), (5.3) and lemma 3 it follows that $\log A(z) \to \infty$ as $z \to z_0$ $(z \in D_\infty)$. Since $\log C(z)$ is the Green function it follows that $\log C(z) \to 0$ as $z \to z_0$ $(z \in D_\infty)$. Hence there exists a $z' \in D_\infty$ such that $A(z') < C(z')$, q.e.d.

Example. From (3.12) it follows that $\log \dfrac{1}{C(E)} - u(z)$ is the Green function for D_∞ with the pole $z = \infty$, and so $\log C(z) = \log 1/C(E) - u(z)$.

Let E be the circle $|z| = 1$. In this case (see [5], pp. 270 and 281) $C(E) = C(E, |p-q|) = V(E, |p-q|) = 1$, $\log C(z) = \log|z|$ for $z \in D_\infty$ and $\log\left|\dfrac{p\bar{z}-1}{p-z}\right|$ is the Green function for D_∞ with the pole at z. Hence and from (3.14) it follows that

$$\log\left|\frac{p\bar{z}-1}{p-z}\right| = \log \frac{1}{|p-z|} - 2 \int_E \log \frac{1}{|p-q|} d\mu_z(q) - \log \frac{1}{|z|} + \log \frac{1}{|p|}$$

and hence

$$(5.4) \qquad \log|p\bar{z}-1| = -2 \int_E \log \frac{1}{|p-q|} d\mu_z(q) + \log|z| - \log|p| .$$

If $p \to z$ then from (5.4) we shall obtain

$$(5.5) \qquad \log\big||z|^2-1\big| = -2 \int_E \log \frac{1}{|z-q|} d\mu_z(q) .$$

Since (see (5.3) and (3.13))

$$\log A(z) = \gamma(z) - \frac{1}{2}\log\frac{1}{|z-q|}\,d\mu_z(q)$$

$$= \log\frac{1}{C(E)} - \frac{1}{2}u(z) - \frac{1}{2}\int_E \log\frac{1}{|z-q|}\,d\mu_z(q)$$

$$= \frac{1}{2}\log|z| - \frac{1}{2}\int_E \log\frac{1}{|z-q|}\,d\mu_z(q)\,,$$

we have by (5.5)

$$\log A(z) = \tfrac{1}{2}\log|z| + \tfrac{1}{4}\log\big||z|^2-1\big| = \log\sqrt[4]{|z|^2\big||z|^2-1\big|}\,,$$

i.e. $A(z) = \sqrt[4]{|z|^2\big||z|^2-1\big|}$ for $z \in D_\infty$.

In this case it is easy to see that $A(z) = \sqrt[4]{|z|^2\big||z|^2-1\big|} < |z| = C(z)$ for $|z| \geqslant 1$, i.e. for $z \in D_\infty$.

Remark 1. It is known that if the set E is the circle $|z| = 1$ then $C(z) = 1$ for $|z| \leqslant 1$. It is also easy to prove that $A(z) = \sqrt[4]{\big||z|^2-1\big|}$ for $|z| \leqslant 1$.

Remark 2. From the definition of $A(z)$ and $C(z)$ it follows that $\log A(z) - \log C(z) \to 0$ as $z \to \infty$. Hence and from the inequality $A(z) < C(z)$ for $z \in D_\infty$, $z \neq \infty$ it follows that $\log A(z)$ cannot be a harmonic function.

6. Proof of the equality $C(z) = D(z)$.

THEOREM 2. *If $V(E, |p-q|) > 0$ then $C(z) = D(z)$ for every z which does not belong to E.*

Proof. At first we shall prove that

$$\log D(z) \leqslant \log C(z) \quad \text{for} \quad z \in D_\infty\,. \tag{6.1}$$

Let

$$q^n = \{q_0, q_1, \ldots, q_n\} \tag{6.2}$$

be the nth extremal system with respect to the generating function $\omega'(p, q; z) = \frac{|p-q|}{|z-p|\,|z-q|}$. Let us denote by $\mu_z^n(e)$ the measure defined on the Borel subsets of E as follows:

$$\mu_z^n(e) = \frac{k}{n}\,, \tag{6.3}$$

where k is the number of the points of the nth extremal system (6.2) contained in e. From the sequence (6.3) we can choose (see [1]) a subsequence $\mu_z^{n_k}(e)$ convergent to a measure $\mu_z'(e)$:

$$\mu_z^{n_k}(e) \to \mu_z'(e) \quad \text{as} \quad k \to \infty\,. \tag{6.4}$$

From the proof of lemma 1 it is easy to deduce that the measure $\mu_z'(e)$ realises the infimum of the integrals

$$I(\mu) = \int_E \Big[\int_E \log \frac{1}{\omega'(p, q; z)} \, d\mu(q) \Big] \, d\mu(p)$$

for $\mu \in M$ (see p. 58).

Let us introduce the notation

$$u_z'(p) \overset{\text{df}}{=} \int_E \log \frac{1}{\omega'(p, q; z)} \, d\mu_z'(q) , \tag{6.5}$$

$$h'(p, z) \overset{\text{df}}{=} \int_E \log \frac{1}{|p-q|} \, d\mu_z'(q) - \log \frac{1}{|p-z|} , \qquad \gamma'(z) \overset{\text{df}}{=} \sup_{p \in E^*} h'(p, z) ,$$

where E^* is the kernel of E. It can be proved by the same method as in Chapter 3 that

$$I(\mu_z') = \gamma'(z) = - \int_E \log \frac{1}{|p-q|} \, d\mu_z'(q) \tag{6.6}$$

and

$$\gamma'(z) = \log \frac{1}{C(E)} - u(z) , \tag{6.7}$$

where $C(E) = C(E, |p-q|)$ is the capacity and $u(z)$ is the equilibrium potential with respect to the generating function $|p-q|$. By the same method as in [1] and [6] we can prove also that

$$u_z(p) \leqslant I(\mu_z') \tag{6.8}$$

for every p. From (3.12) and (3.13) it follows that $\gamma'(z)$ is the Green function for D_∞ with the pole at ∞. In order to prove (6.1) it is sufficient to prove the inequality

$$\log D(z) \leqslant \gamma'(z) \qquad \text{for} \qquad z \in D_\infty . \tag{6.9}$$

Let us write

$$D_n^{(j)}(z) \overset{\text{df}}{=} \prod_{\substack{k=0 \\ k \neq j}}^{n} \left| \frac{z - q_j}{q_k - q_j} \right| , \qquad j = 0, 1, \ldots, n .$$

Without any loss of generality we can assume that $D_n^{(0)}(z) \geqslant D_n^{(1)}(z) \geqslant \ldots$ $\ldots \geqslant D_n^{(n)}(z)$. It is evident that

$$D_n(z) \leqslant D_n^{(0)}(z) , \tag{6.10}$$

where $D_n(z)$ is defined on page 57. Let

$$\Delta_n^{(0)}(z) \overset{\mathrm{df}}{=} \prod_{k=1}^{n} \omega'(q_0, q_k; z) = \frac{1}{D_n^{(0)}(z)} \prod_{k=1}^{n} |z-q_k| .$$

Since system (6.2) is the extremal system, we can prove by the same method as in [5], p. 267 that

$$\Delta_n^{(0)}(z) = \prod_{k=1}^{n} \omega'(q_0, q_k; z) = \sup_{p\in E} \prod_{k=1}^{n} \omega'(p, q_k; z) . \tag{6.11}$$

Hence

$$\frac{1}{n}\sum_{k=1}^{n} \log\frac{1}{\omega'(p, q_k; z)} \geqslant \log\frac{1}{\sqrt[n]{\Delta_n^{(0)}(z)}} \qquad \text{for every } p \in E .$$

Integrating the last inequality with respect to $\mu_z'(e)$ we shall obtain (see p. 61)

$$\frac{1}{n}\sum_{k=1}^{n} u_z'(q) = \frac{1}{n}\sum_{k=1}^{n} \int_E \log\frac{1}{\omega'(p, q_k; z)}\, d\mu_z'(p) \geqslant \log\frac{1}{\sqrt[n]{\Delta_n^{(0)}(z)}} .$$

Hence and from inequality (6.8) we shall obtain

$$I(\mu_z') \geqslant \log\frac{1}{\sqrt[n]{\Delta_n^{(0)}(z)}} . \tag{6.12}$$

By (6.10), (6.11) and (6.12) we have

$$I(\mu_z') \geqslant \frac{1}{n}\log D_n(z) - \frac{1}{n}\sum_{k=1}^{n} \log\frac{1}{|z-q_k|} .$$

In particular

$$I(\mu_z') \geqslant \frac{1}{n_k}\log D_{n_k}(z) - \frac{1}{n_k}\sum_{l=1}^{n_k} \log\frac{1}{|z-q_l|} \tag{6.13}$$

(see p. 74). Hence we have

$$I(\mu_z') \geqslant \log D(z) - \int_E \log\frac{1}{|p-z|}\, d\mu_z'(p) \qquad \text{or} \qquad \gamma'(z) \geqslant \log D(z) ,$$

i.e. inequality (6.1).

In order to prove that $\log C(z) = \log D(z)$ for $z \in D_\infty$ it is sufficient to prove that

$$\gamma'(z) \leqslant \log D(z) \qquad \text{for} \qquad z \in D_\infty . \tag{6.14}$$

Let $r^{(n)} = \{r_0, r_1, \ldots, r_n\}$ be the nth system of points realising the infimum of $D_n(z; p^{(n)})$ and let μ_z'' be a measure defined similarly to the measure μ_z' (see p. 73). It is easy to prove (see the definition of $D(z)$) that for every $p \in E$

$$\log D(z) \geqslant \int_E \log\frac{1}{|p-q|}\, d\mu_z''(q) - \log\frac{1}{|p-z|}\,.$$

Let μ be nth extremal measure with respect to $|p-q|$. By the same method as in the proof of lemma 3 (see p. 69) we can prove that

$$\log D(z) = \int_E \log D(z)\, d\mu(p) \geqslant \int_E \Big[\int_E \log\frac{1}{|p-q|}\, d\mu_z''(q) - \log\frac{1}{|p-z|}\Big]\, d\mu(p)$$
$$= \log\frac{1}{C(E)} - u(z) = \gamma'(z)\,,$$

i.e. inequality (6.14).

In the case where z does not belong to $D_\infty \cup F_\infty$ we have (see [5], p. 278) $\log C(z) = 0$ and (see [1], [6]) $u(z) = \log\dfrac{1}{C(E)}$. We can also prove by the same method as in the case of $z \in D_\infty$ that $\log D(z) = 0$.

Remark. It is easy to prove that if $z \in F_\infty$ and z is a regular point for the Dirichlet problem for D_∞ then $D(z) = C(z) = 1$. If z is a irregular point then $D(z)$ can be greater than $C(z)$.

Example. Let E be the set consisting of the circle $|z| = 1$ and the point $z = 5$. In this case $C(5) = 1$ (see [5], p. 278) and

$$D_n(5) = \inf_{p^{(n)}}\Big\{\max_{(j)} \prod_{\substack{k=0\\k\neq j}}^{n} \frac{|5-p_j|}{|p_k-p_j|} \geqslant \frac{4^n}{2^n\cdot 6}\Big\}\,.$$

Hence $D(5) \geqslant 2 > 1 = C(5)$.

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