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RELATIVE GENERATORS FOR THE ACTION OF A COUNTABLE ABELIAN GROUP ON A LEBESGUE SPACE

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0. Introduction

In the recent years increasing interest in the relative ergodic theory can be observed.

The Thouvenot relative isomorphism theory is an important part of it (cf. [12]). For applications of Thouvenot's theory and related results see [1], [8], [9] and [12].

The theory of measure preserving transformations with relative discrete spectrum is another well-known part of the relative ergodic theory (cf. [2] and [13]). The characterization of these transformations in terms of relative sequence entropy is given in [4].

Using the relative Rohlin–Sinai theory of invariant partitions for measure preserving automorphisms the author of this note has obtained certain results concerning such partitions for Z^d-actions, $d \geqslant 2$ (cf. [5]). The investigation of perfect partitions for Z^2-actions with finite entropy has led to the question of the existence of relative generators for single automorphisms. It is proved in [6] that this question has a positive answer. In order to describe the construction of perfect partitions for Z^d-actions, we need in fact a theorem on the existence of relative generators also for Z^d-actions, $d \geqslant 2$. To avoid technical difficulties we prove this theorem for actions of abelian countable groups. The proof of our main result runs in the same way as in the absolute case and its idea is the same as that of Rohlin [11].

1. Auxiliary results

Let G be a countable abelian group and let $\mathscr{F}(G)$ be a family of all nonempty finite subsets of G. For $A \in \mathscr{F}(G)$, we denote by $|A|$ the cardinality of A.

A set $A \subset G$ is said to be *tiling* (cf. [10]) if $A \in \mathscr{F}(G)$ and there exists a set $C \subset G$ such that the sets $\{A \cdot g;\ g \in C\}$ form a partition of G.

LEMMA 1 ([10]). *For every $\delta > 0$ and every $K \in \mathscr{F}(G)$ there exists a tiling set A with*

$$|\{g \in A;\ K \cdot g \subset A\}| > (1-\delta)|A|.$$

A sequence $(A_n) \subset \mathscr{F}(G)$ is said to be a *Fϕlner sequence* if for every $g \in G$

$$\lim_{n \to \infty} \frac{|g \cdot A_n \cap A_n|}{|A_n|} = 1.$$

It follows from [3] that in every countable abelian group there exists a Fϕlner sequence (A_n) such that $A_n \subset A_{n+1}$, $n \geqslant 1$ and $\bigcup_{n=1}^{\infty} A_n = G$.

Remark. In every countable abelian group there exists a Fϕlner sequence (A_n) such that A_n is a tiling set $n \geqslant 1$ and $\lim_{n \to \infty} |A_n| = \infty$.

Proof. Let $(K_n) \subset \mathscr{F}(G)$ be such that $K_n \subset K_{n+1}$, $n \geqslant 1$ and $\bigcup_{n=1}^{\infty} K_n = G$. It follows from Lemma 1 that for every $n \geqslant 1$ there exists a tiling set A_n such that

$$\Big|\bigcap_{h \in K_n} A_n \cap h^{-1} A_n\Big| = |\{g \in A_n;\ K_n \cdot g \subset A_n\}| > \left(1 - \frac{1}{n+1}\right)|A_n|.$$

Let $g \in G$ be arbitrary and n_0 be such that $g \in K_n$ for $n \geqslant n_0$. Hence

$$|gA_n \cap A_n| \geqslant \Big|\bigcap_{h \in K_n} A_n \cap h^{-1} A_n\Big| > \left(1 - \frac{1}{n+1}\right)|A_n|, \qquad n \geqslant n_0,$$

i.e. (A_n) is a Fϕlner sequence.

It is clear that for every $n \geqslant 1$ there exists $g \in A_n$ with $K_n \cdot g \subset A_n$, i.e. $|K_n| \leqslant |A_n|$. Therefore, $\lim_{n \to \infty} |A_n| = \infty$.

Now, let $(X, \mathscr{B}, \mu)$ be a Lebesgue probability space, $\mathscr{M}$ be the set of all measurable partitions of X and let $\mathscr{Z}$ be the subset of $\mathscr{M}$ consisting of partitions with finite entropy. We consider in $\mathscr{Z}$ the metric d given by the formula

$$d(P, Q) = H(P|Q) + H(Q|P), \qquad P, Q \in \mathscr{Z}.$$

We denote by ε the measurable partition of X into single points.

LEMMA 2 ([11]). *For every $P, Q \in \mathscr{M}$ such that $P \geqslant Q$ and $H(P|Q) < \infty$ there exists $R \in \mathscr{Z}$ with $P = Q \vee R$ and $H(R) < H(P|Q) + 3\sqrt{H(P|Q)}$.*

If $P = (P_i, i \geqslant 1) \in \mathcal{M}$, $B \in \mathcal{B}$ then $P \cap B$ denotes the partition

$$P \cap B = \{P_i \cap B,\ X \setminus B,\ i \geqslant 1\}.$$

Let $T\colon G \times X \to X$ be a measure preserving action of G on X and let $T_g(\cdot) = T(g, \cdot)$, $g \in G$.

The action T is said to be *free* if $T_g x \neq x$ for μ a.e. $x \in X$, and for any g different from the unit of G.

LEMMA 3 ([10]). *If T is a free action then for every $\delta > 0$ and every tiling set $A \subset G$ there exists a set $F \in \mathcal{B}$ such that*

(i) *the sets $\{T_g; F, g \in A\}$ are pairwise disjoint,*
(ii) $\mu\left(\bigcup_{g \in A} T_g F\right) > 1 - \delta$.

Now, let $P \in \mathcal{M}$ and $A \subset G$. We put $P(A) = \bigvee_{g \in A} T_g P$ and $P_T = P(G)$.

A partition $P \in \mathcal{M}$ is said to be a *σ-relative generator* of T if $P_T \vee \sigma = \varepsilon$.

Let $\sigma \in \mathcal{M}$ be G-invariant, i.e. $T_g \sigma = \sigma$, $g \in G$. It is possible to show, in a similar way to that used in [7] that for every $P \in \mathcal{Z}$ and every Følner sequence (A_n) the limit

$$\lim_{n \to \infty} \frac{1}{|A_n|} H\left(P(A_n) | \sigma\right)$$

exists and does not depend on the choice of (A_n).

We denote this limit by $h(P, T|\sigma)$.

The entropy $h(T|\sigma)$ of the action T relative to σ is defined by the formula

$$h(T|\sigma) = \sup\{h(P, T|\sigma);\ P \in \mathcal{Z}\}.$$

LEMMA 4. *For every $P, Q \in \mathcal{Z}$ such that $P \leqslant Q_T \vee \sigma$ we have $h(P, T|\sigma) \leqslant h(Q, T|\sigma)$.*

Proof. Let (A_n) be an arbitrary Følner sequence in G such that $A_n \subset A_{n+1}$, $n \geqslant 1$ and $\bigcup_{n=1}^{\infty} A_n = G$. Since $Q(A_n) \vee \sigma \nearrow Q_T \vee \sigma$, it suffices to prove our inequality for $P \in \mathcal{Z}$ such that $P \leqslant Q(A_m) \vee \sigma$ where $m \geqslant 1$ is a positive integer. For any such P we have

$$P(A_n) \leqslant \bigvee_{g \in A_n} T_g Q(A_m) \vee \sigma = Q(A_n \cdot A_m) \vee \sigma$$

and so

$$H\left(P(A_n)|\sigma\right) \leqslant H\left(Q(A_n \cdot A_m)|\sigma\right), \qquad n \geqslant 1.$$

Since $(A_n \cdot A_m;\ n \geqslant 1)$ is a Følner sequence we obtain the desired inequality.

COROLLARY (Relative Kolmogorov–Sinai Theorem). *If $P \in \mathscr{Z}$ is a σ-relative generator of T then $h(T|\sigma) = h(P, T|\sigma)$.*

Proof. Let $Q \in \mathscr{Z}$ be arbitrary. By our assumption $Q \leqslant P_T \vee \sigma$ and so Lemma 4 implies

$$h(Q, T|\sigma) \leqslant h(P, T|\sigma), \quad Q \in \mathscr{Z}.$$

The result is an easy consequence of this inequality.

2. Main result

APPROXIMATION LEMMA. *If T is a free action with $h(T|\sigma) < \infty$ then for every $P, Q \in \mathscr{Z}$ and $\delta > 0$ there exists $R \in \mathscr{Z}$ such that*

$$P \leqslant R_T, \quad H(R|Q_T \vee \sigma) \leqslant h(T|\sigma) - h(Q, T|\sigma) + \delta.$$

Proof. It follows from the previous Remark that there exists a Følner sequence (A_n) such that every A_n, $n \geqslant 1$ is a tiling set and $\lim_{n\to\infty} |A_n| = \infty$. We may suppose that the unit of G belongs to A_n, $n \geqslant 1$. Indeed, let $(A_n) \subset \mathscr{F}(G)$ be an arbitrary sequence satisfying the two above properties. For every $n \geqslant 1$ there exists $g_n \in G$ such that the unit of G belongs to $g_n \cdot A_n$. It is easy to check that the sequence $(\tilde{A}_n)$ defined by $\tilde{A}_n = g_n \cdot A_n$, $n \geqslant 1$ has properties as desired.

Let $P, Q \in \mathscr{Z}$ and $\delta > 0$ be arbitrary. We put $\alpha = P \vee Q$. There exists a positive integer $n \geqslant 1$ with

$$\frac{1}{|A_n|} H(\alpha(A_n)|\sigma) - h(\alpha, T|\sigma) < \frac{\delta}{4}, \tag{1}$$

$$\frac{1}{|A_n|} H(Q(A_n)|\sigma) - h(Q, T|\sigma) > -\frac{\delta}{4}, \tag{2}$$

$$\text{if } 0 < t < \frac{1}{|A_n|} \text{ then } -t \log t - (1-t)\log(1-t) < \frac{\delta}{4}. \tag{3}$$

We choose $\lambda > 0$ so that

$$\text{if } B \in \mathscr{B} \text{ and } \mu(B) < \lambda \text{ then } H(P \cap B) < \frac{\delta}{4}. \tag{4}$$

Let $B_n = A_n^{-1}$, $n \geqslant 1$. It is clear that B_n is a tiling set, $n \geqslant 1$. Lemma 3 implies that there exists a set $F \in \mathscr{B}$ such that the sets $\{T_g F, g \in B_n\}$ are pairwise disjoint and $\mu(D) < \lambda$ where $D = X \setminus \bigcup_{g \in B_n} T_g F$. Let $\beta = \{T_g F, D;\ g \in B_n\}$ and (σ_m) be a sequence of finite measurable partitions such that $\sigma_m \nearrow \sigma$. Since $\alpha \geqslant Q$, one can check, using the definition and simple properties of conditional entropy, that

$$(5)\quad \sum_{g\in B_n} H\big(\alpha(A_n)\cap T_g F\,|\,Q(A_n)\cap T_g F\vee\sigma_m\big)$$
$$= \sum_{g\in B_n} H\big(\alpha(A_n)\cap T_g F\,|\,\sigma_m\big)-\sum_{g\in B_n} H\big(Q(A_n)\cap T_g F\,|\,\sigma_m\big)$$
$$= H\big(\alpha(A_n)\vee\beta\,|\,\sigma_m\big)-H\big(\alpha(A_n)\cap D\,|\,\sigma_m\big)$$
$$-H\big(Q(A_n)\vee\beta\,|\,\sigma_m\big)+H\big(Q(A_n)\cap D\,|\,\sigma_m\big),\qquad m\geqslant 1.$$

Taking in (5) the limit as $m\to\infty$ we see that (5) remains valid for σ in place of σ_m. Therefore, there exists $g_0\in B_n$ with

$$(6)\quad H\big(\alpha(A_n)\cap T_{g_0}F\,|\,Q(A_n)\cap T_{g_0}F\vee\sigma\big)$$
$$\leqslant \frac{1}{|B_n|}\Big(H\big(\alpha(A_n)\vee\beta\,|\,\sigma\big)-H\big(\alpha(A_n)\cap D\,|\,\sigma\big)\Big)$$
$$-H\big(Q(A_n)\vee\beta\,|\,\sigma\big)+H\big(Q(A_n)\cap D\,|\,\sigma\big)\big)$$
$$\cdot\leqslant \frac{1}{|B_n|}\Big(H\big(\alpha(A_n)\vee\beta\,|\,\sigma\big)-H\big(Q(A_n)\vee\beta\,|\,\sigma\big)\Big)$$
$$\leqslant \frac{1}{|B_n|}\Big(H\big(\alpha(A_n)\,|\,\sigma\big)-H\big(Q(A_n)\,|\,\sigma\big)\Big).$$

Applying (1), (2) and the equality $|B_n|=|A_n|$ we get

$$(7)\quad H\big(\alpha(A_n)\cap T_{g_0}F\,|\,Q(A_n)\cap T_{g_0}F\vee\sigma\big)$$
$$\leqslant h(\alpha,\,T|\sigma)-h(Q,\,T|\sigma)+\frac{\delta}{2}\leqslant h(T|\sigma)-h(Q,\,T|\sigma)+\frac{\delta}{2}.$$

Now we consider the partition $\gamma=\{T_{g_0}F,\,X\setminus T_{g_0}F\}$. It is easy to verify that

$$Q(A_n)\cap T_{g_0}F\leqslant Q(A_n)\vee\gamma.$$

The obvious inequality $\mu(F)\leqslant\dfrac{1}{|A_n|}$ together with (3) imply that $H(\gamma)<\dfrac{\delta}{4}$. Therefore using (7) we obtain

$$(8)\quad H\big(P(A_n)\cap T_{g_0}F\,|\,Q_T\vee\sigma\big)$$
$$\leqslant H\big(P(A_n)\cap T_{g_0}F\vee\gamma\,|\,Q_T\vee\sigma\big)\leqslant H\big(\alpha(A_n)\cap T_{g_0}F\vee\gamma\,|\,Q(A_n)\vee\sigma\big)$$
$$= H\big(\gamma\,|\,Q(A_n)\vee\sigma\big)+H\big(\alpha(A_n)\cap T_{g_0}F\,|\,Q(A_n)\vee\gamma\vee\sigma\big)$$
$$\leqslant H(\gamma)+H\big(\alpha(A_n)\cap T_{g_0}F\,|\,Q(A_n)\cap T_{g_0}P\vee\sigma\big)$$
$$\leqslant h(T|\sigma)-h(Q,\,T|\sigma)+\frac{3}{4}\delta.$$

Let us observe that the partition

$$R = P(A_n) \cap T_{g_0} F \vee P \cap T_{g_0} D$$

satisfies all requirements. Indeed, it follows from (4) and (8) that

$$H(R|Q_t \vee \sigma) \leqslant H(P(A_n) \cap T_{g_0} F|Q_t \vee \sigma) + H(P \cap T_{g_0} D) < \delta.$$

Since B_n contains the unit of G we have

$$\begin{aligned} P \leqslant P \vee T_{g_0} \beta &= P \cap T_{g_0} D \vee \bigvee_{g \in B_n} P \cap T_g(T_{g_0} F) \\ &= P \cap T_{g_0} D \vee \bigvee_{g \in B_n} T_g(T_{g^{-1}} P \cap T_{g_0} F) \\ &\leqslant \bigvee_{g \in B_n} T_g(P \cap T_{g_0} D) \vee \bigvee_{g \in B_n} T_g(P(A_n) \cap T_{g_0} F) = \bigvee_{g \in B_n} T_g R \leqslant R_T, \end{aligned}$$

and this completes the proof.

Let now $\sigma, \tau \in \mathscr{M}$ be G-invariant and $\sigma \leqslant \tau$.

THEOREM. *If the factor action T_τ of G on $(X/\tau, \mathscr{B}/\tau, \mu_\tau)$ is free with $h(T_\tau|\sigma) < \infty$ then there exists a σ-relative generator $P \in \mathscr{Z}$ of T_τ. Moreover, the set of all these generators is dense in $\{Q \in \mathscr{Z};\ Q \leqslant \tau,\ h(Q, T|\sigma) = h(T_\tau|\sigma)\}$.*

Proof. We may suppose $\tau = \varepsilon$. Let $\delta > 0$ be arbitrary and $Q \in \mathscr{Z}$ be such that

$$h(T|\sigma) - h(Q, T|\sigma) < \frac{\delta^2}{2^7}. \tag{9}$$

There exists a sequence $(Q_n) \subset \mathscr{Z}$ with $Q_0 = Q$, $Q_n \uparrow \varepsilon$ and

$$h(T|\sigma) - h(Q_k, T|\sigma) = \frac{\delta^2}{2^{2k+7}}, \quad k \geqslant 0. \tag{10}$$

It follows from (10) and the Approximation Lemma that there exists a sequence $(R_k) \in \mathscr{Z}$ such that $(R_k)_T \geqslant (Q_k)_T$ and

$$\begin{aligned} &H((Q_{k-1})_T \vee R_k \vee \sigma|(Q_{k-1})_T \vee \sigma) \\ &\qquad = H(R_k|(Q_{k-1})_T \vee \sigma) < \frac{\delta^2}{2^{2k+4}}, \quad k \geqslant 1. \end{aligned}$$

Therefore, using Lemma 2, we get a sequence $(P_k) \subset \mathscr{Z}$ with

$$(Q_{k-1})_T \vee R_k \vee \sigma = (Q_{k-1})_T \vee P_k \vee \sigma \quad \text{and} \quad H(P_k) < \frac{\delta}{2^k}, \quad k \geqslant 1. \tag{11}$$

Since $(R_k)_T \geqslant (Q_k)_T$ we obtain from (10)

$$(Q_{k-1})_T \vee (P_k)_T \vee \sigma \geqslant (Q_{k-1})_T \vee (Q_k)_T \vee \sigma = (Q_k)_T \vee \sigma, \quad k \geqslant 1,$$

and so

$$(12)\qquad (Q \vee \bigvee_{k=1}^{n} P_k)_T \vee \sigma \geqslant (Q_n)_T \vee \sigma, \qquad n \geqslant 1.$$

We put $P = Q \vee \bigvee_{k=1}^{\infty} P_k$. The fact that $Q_n \uparrow \varepsilon$ and (12) imply $P_T \vee \sigma = \varepsilon$.

Using the inequality from (11), we get $H(P) \leqslant H(Q) + \delta < \infty$ and

$$(13)\qquad d(P, Q) \leqslant \sum_{k=1}^{\infty} H(P_k) < \delta.$$

Thus the first part of the theorem is proved.

The Relative Kolmogorov–Sinai Theorem ensures that the set of σ-relative generators of T with finite entropy is contained in the set $\{Q \in \mathscr{Z};\ h(T|\sigma) = h(Q, T|\sigma)\}$. Let $\delta > 0$ and $Q \in \mathscr{Z}$ with $h(T|\sigma) = h(Q, T|\sigma)$ be arbitrary. Starting in (9) with such a Q we get, by the above procedure, a σ-relative generator $P \in \mathscr{Z}$ of T satisfying (13), which completes the proof.

It is possible to prove the existence of a finite relative generator for ergodic actions with finite entropy.

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